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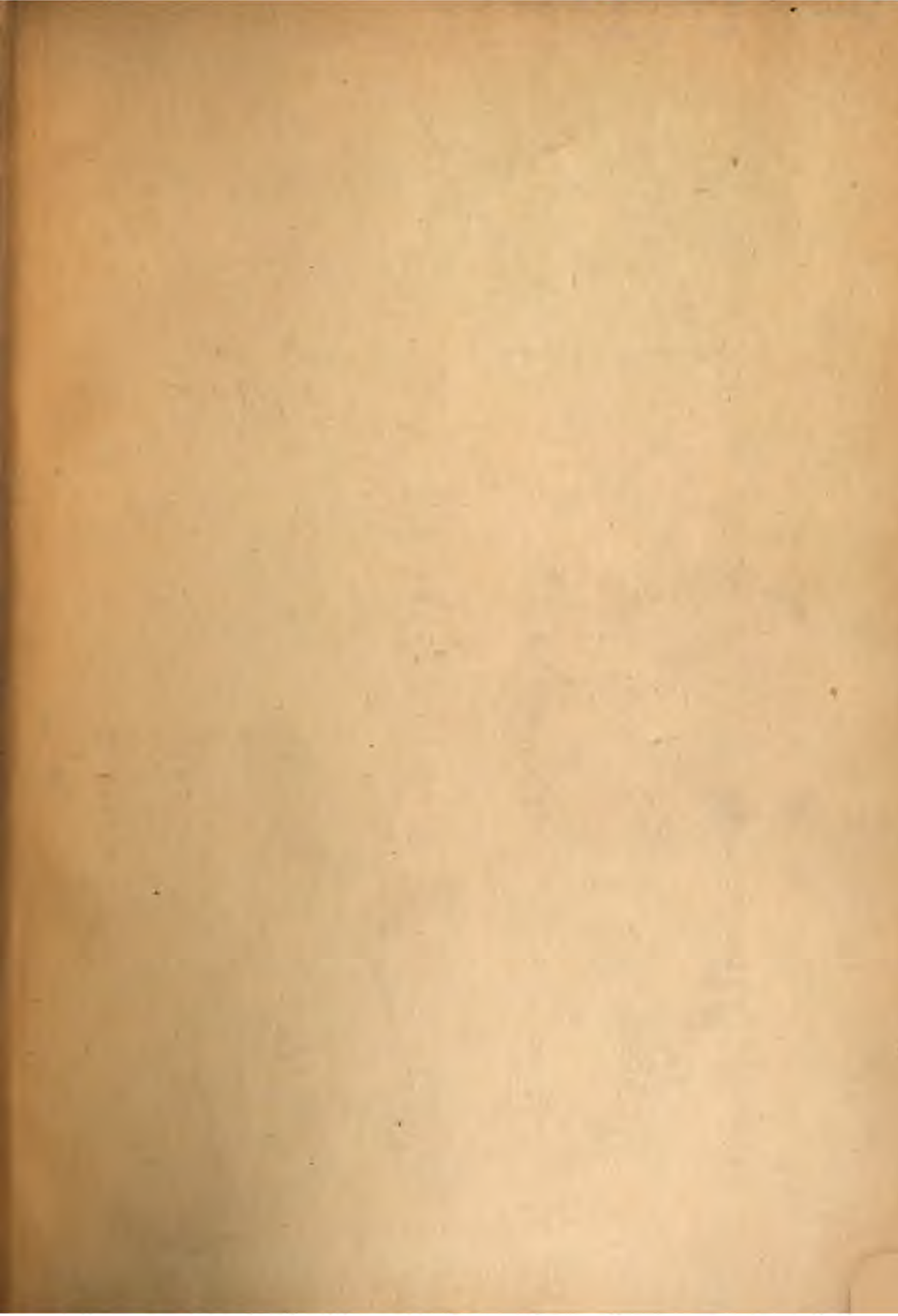
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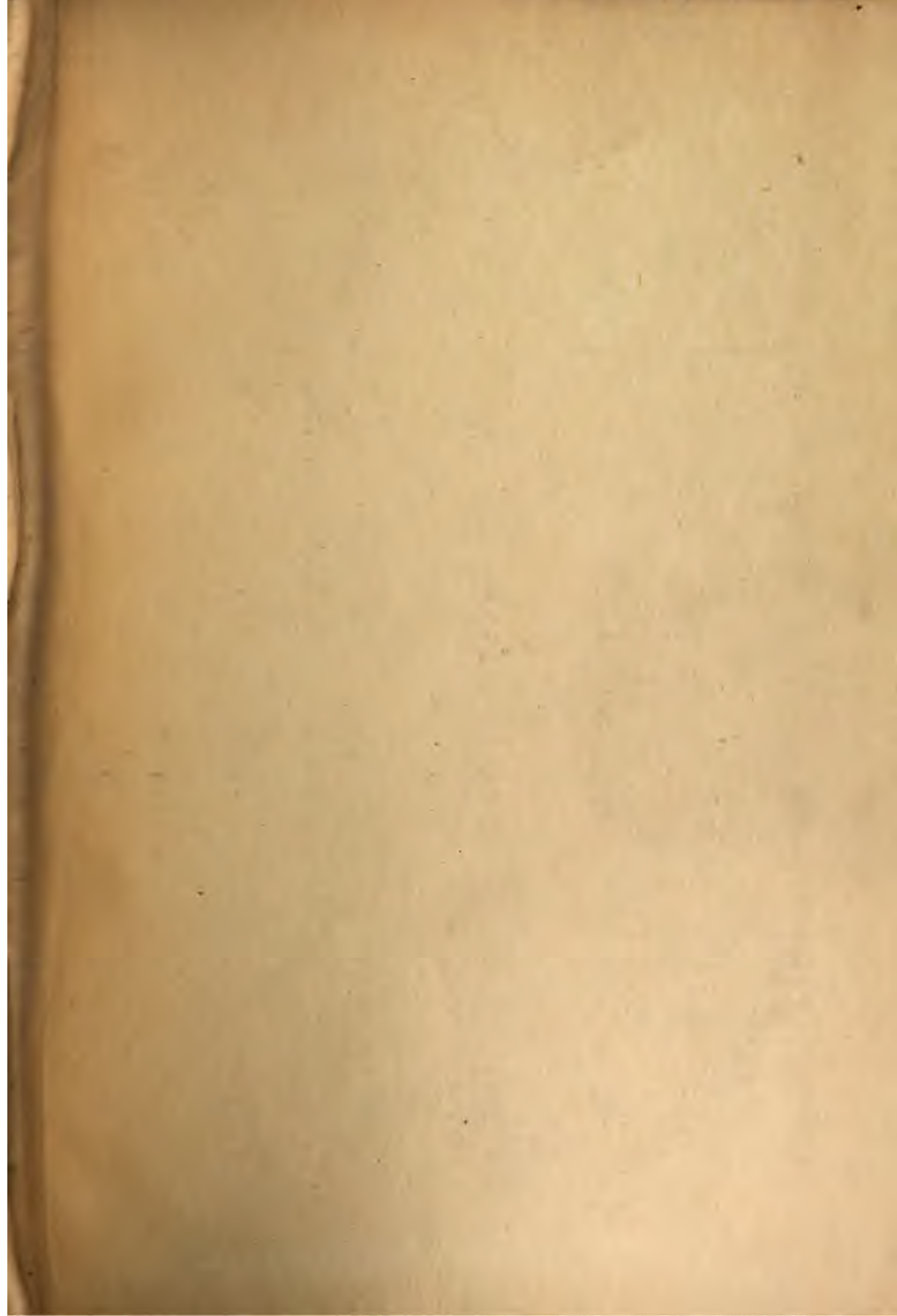
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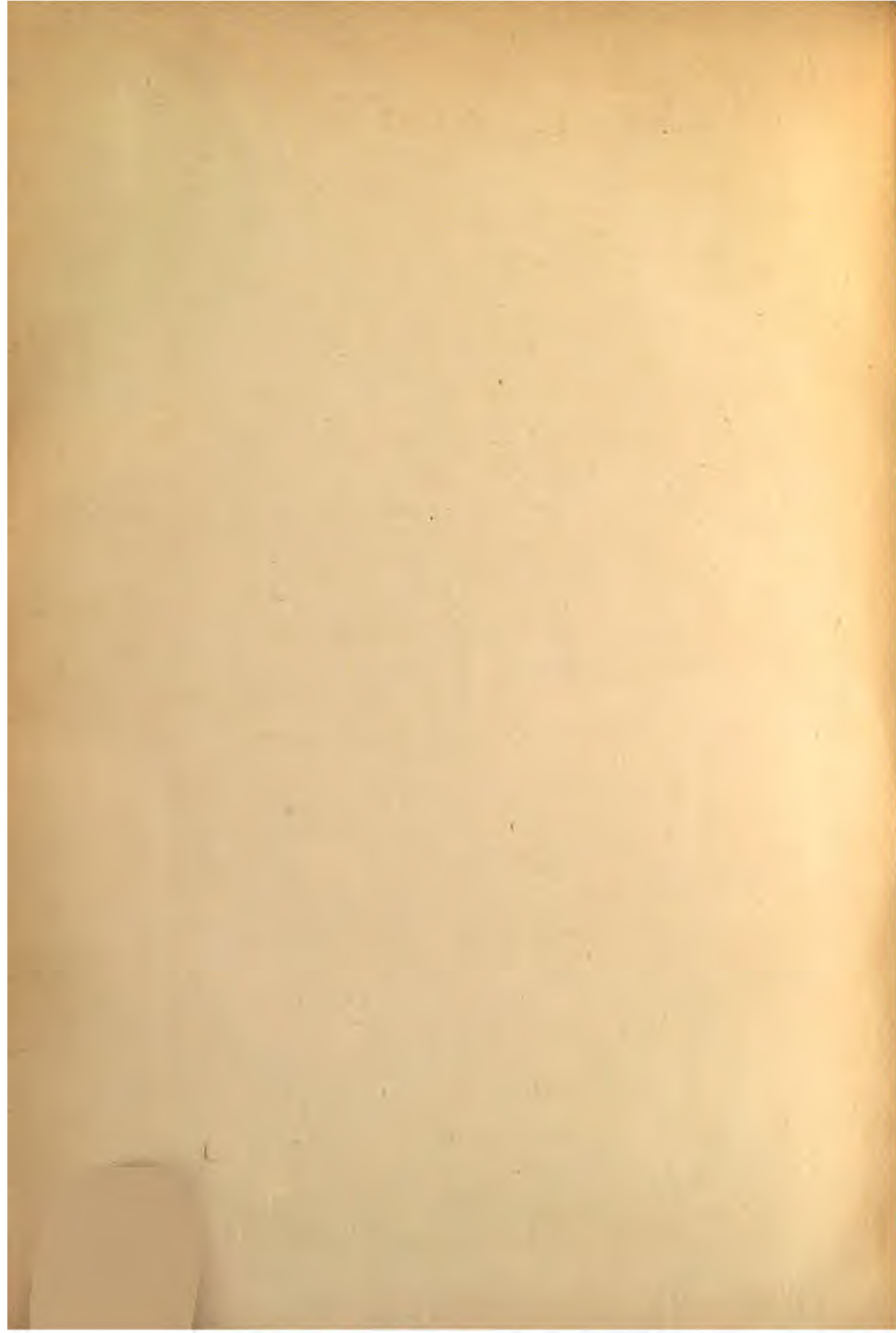
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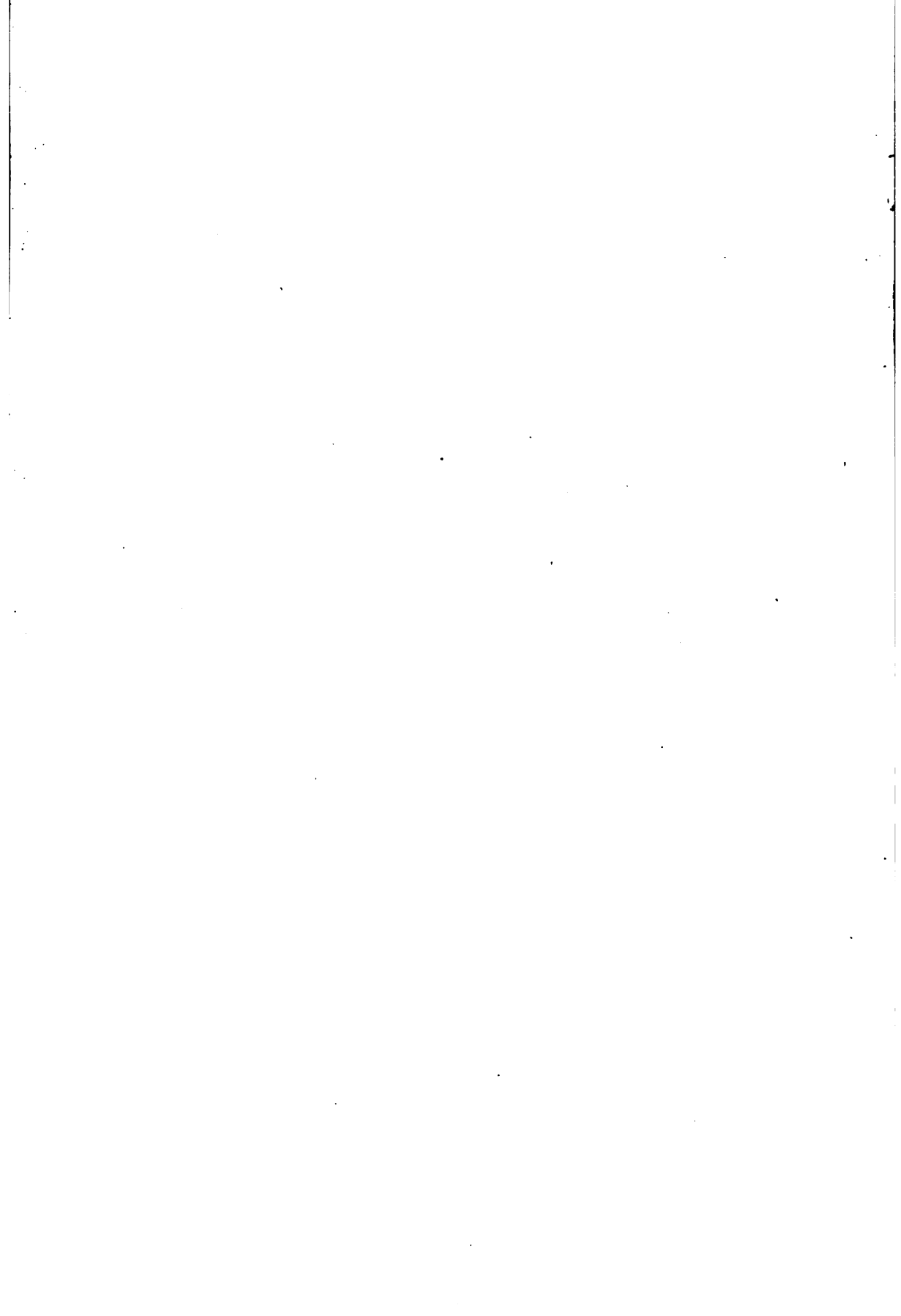


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Vol. 14

No. 1

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

ISSUED APRIL 6, 1918

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DETERMINATION OF THE DEGREE OF UNIFORMITY OF BARS FOR MAGNETIC STANDARDS

By Raymond L. Sanford, Assistant Physicist

STATEMENT OF PROBLEM

Concrete magnetic standards, consisting of straight bars of magnetic material selected and prepared with a view to obtaining the maximum degree of permanence and homogeneity, are of very great value in magnetic testing. By means of readings taken on such standardized bars, permeameters are calibrated and various methods of magnetic testing are compared with a standard method. One of the most important requirements of a magnetic standard bar is that it shall be magnetically uniform along its length. All precision methods for the magnetic measurement of straight bars assume uniformity along the length of the specimen. If this condition is not met, errors may arise, due to the effect of poles which modify the distribution of magnetomotive force. These errors may be of considerable magnitude and can not be calculated or eliminated from the measurements.

Burrows,¹ in his paper on the Determination of the Magnetic Induction in Straight Bars, calls attention to the necessity for uniformity and shows the effect of nonuniformities on the distribution of magnetic induction in a bar magnetized in a double-yoke apparatus. He points out the fact that numbers or identification marks should not be punched or stamped on those portions of a bar which are to be included in the magnetic measurements. Several methods have been suggested² for testing the uniformity

¹ Burrows, this Bulletin, 6, p. 59; 1909.

² Ebeling and Schmidt, Wied. Ann. 58, p. 330, 1896, examine different portions of long bars by measurements with a single-yoke apparatus.

Ebeling, Wied. Ann. 58, p. 342, 1896, examines portions of the bars mentioned above by determining the electrical resistivity of different portions of their length.

Kann Phys. Zs., 7, p. 526, 1906, suggests the detection of flaws by a determination of the mutual inductance between two windings on a core which is arranged to include the material under test.

McCann and Colson, Western Electrician, 48, p. 76, 1908, investigate the continuity of wire ropes, such as mine hoists, etc., by observing changes in self-inductance of a coil surrounding the wire rope under test by means of an alternating current and suitable instruments.

Pierce, Proc. Am. Acad. of Arts and Sciences, 46, p. 197, 1910, determined the suitability of cast-iron pieces for use as permanent magnets by a resistance method.

of bars and the detection of flaws. No one, however, has indicated the magnitude of the effect of nonuniformities on the accuracy of magnetic measurements.

This paper (1) describes a method for determining the degree of magnetic uniformity along the length of a straight bar, (2) indicates the magnitude of the effect of nonuniformities on the accuracy of magnetic measurements, and (3) gives a criterion for the degree of uniformity of magnetic standards.

THEORY OF THE METHOD

Uniformity measurements, as carried out in the present investigation, are made by observing the distribution of magnetic leakage along the length of a specimen when it is magnetized between the poles of a suitable electromagnet. If the bar is uniform, the magnetic induction and leakage vary along its length in a regular manner. Nonuniformities are indicated by irregularities in the distribution of induction and leakage along the bar. In this work the leakage is defined as the change of induction per unit length along the bar. In other words, the leakage at any point on the bar is the first derivative of the induction with respect to the displacement. The curve expressing the rate of change of leakage along the bar, or the leakage slope, has been chosen to represent the degree of magnetic uniformity and is referred to hereafter as the "uniformity curve." This curve gives the second derivative of the induction with respect to the displacement.

The curves given in Fig. 1 show the distribution of induction and leakage for two bars, one of which is uniform while the other has a magnetically hard spot near one end. The induction is greatest at the ends and has a minimum value near the middle of the bar. If the bar is uniform, the induction varies in such a way that the leakage curve is a straight line having a slope which depends upon the magnetic hardness of the material. If the magnetic circuit is symmetrical and the two joint contacts have the same magnetic reluctance, the leakage has a zero value at the middle of the bar. If the two joints are not of the same reluctance, the zero point may be displaced, but the slope remains the same. Consequently, the "uniformity curve" for a uniform bar is a straight line parallel to the axis of the bar. If a bar is nonuniform, the leakage does not vary at a constant rate and the nature and position of the nonuniformity is shown by the deviation of the uniformity curve from a straight line. An increase in the

value of the ordinate at any point along the length of the bar indicates a magnetically hard spot, while a decrease in its value shows a soft spot.

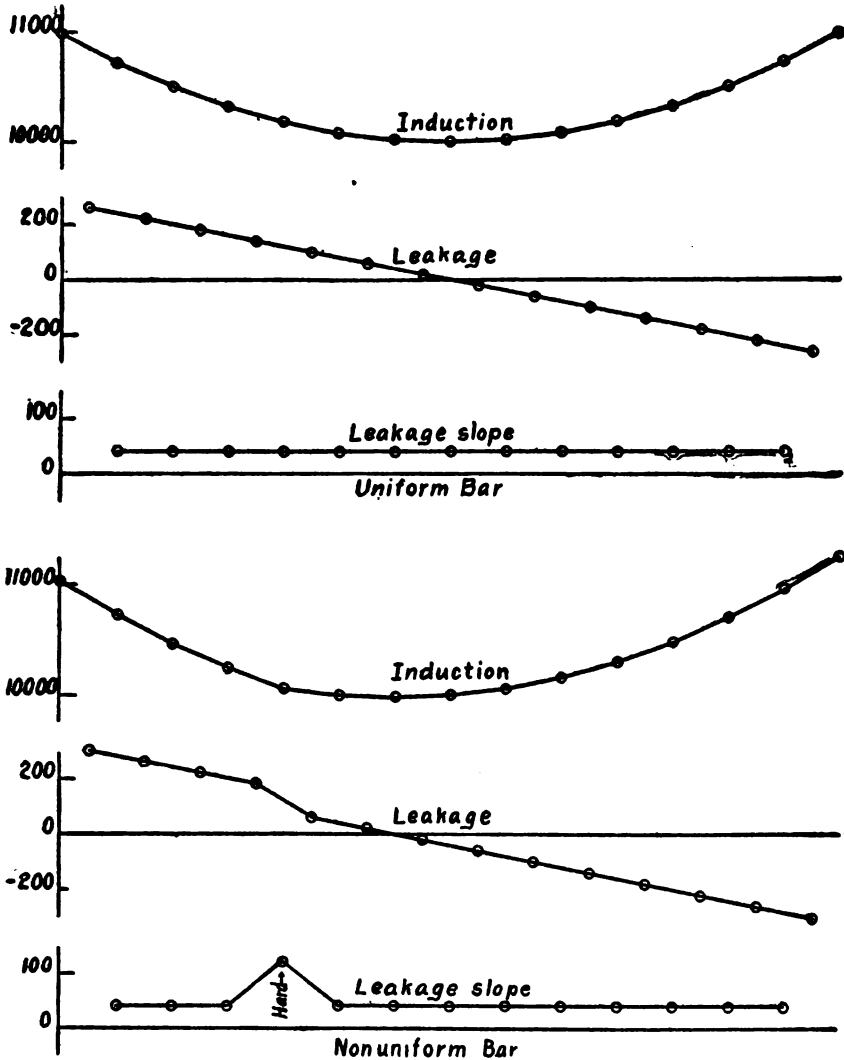


FIG. 1.—Distribution of magnetic induction and leakage for bars magnetized between the poles of an electromagnet

DETAILS OF THE METHOD

The essential features of the apparatus used in this investigation consist of an electromagnet and a pair of specially wound test coils (Fig. 2). The electromagnet has pole pieces adapted for

clamping the specimens. The two test coils, which are slipped over the bar under test, are of 100 turns each and are wound on a common form. Each coil extends over a space of 1 cm and the distance between centers is 2 cm. The displacement of the coils along the bar is read by means of a suitable scale mounted on the apparatus. The ballistic galvanometer is calibrated by means of a mutual inductance, and its sensitivity and damping are adjusted by means of the usual combination of series and parallel resistances. With a single test coil in circuit, the galvanometer deflection on reversal of the magnetizing current is proportional to the induction in that part of the bar under the test coil. With

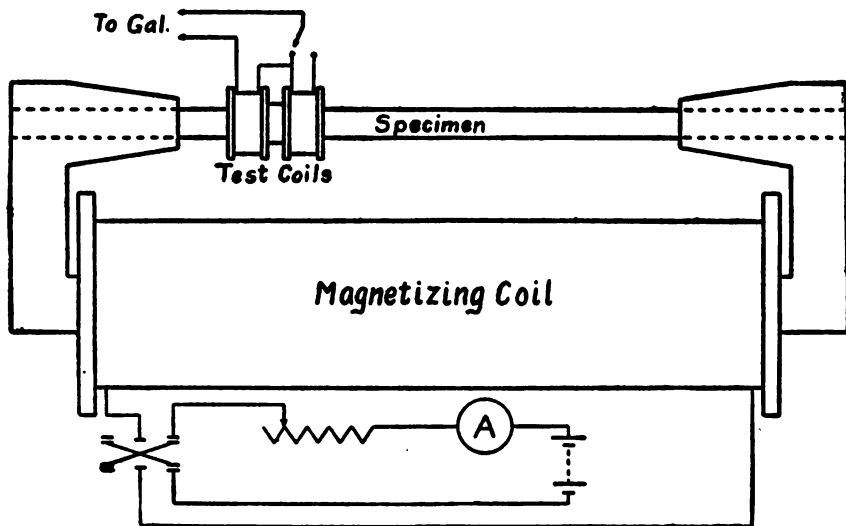


FIG. 2.—Diagram of apparatus for the determination of magnetic uniformity along the length of a straight bar

the two coils connected in opposition, the deflection on reversal is proportional to the difference in induction between the parts of the bar under the two coils. This difference divided by the distance between coils is the average leakage over that region. From leakage data taken at equal intervals along the length of the bar the ordinates of the uniformity curve are calculated, and these points are plotted against displacement along the bar.

A continuous record for the whole length of the bar is possible by a modification of the apparatus. If the opposed test coils are moved at a uniform rate along the bar, the emf induced is proportional to the rate of change of leakage along the bar. This

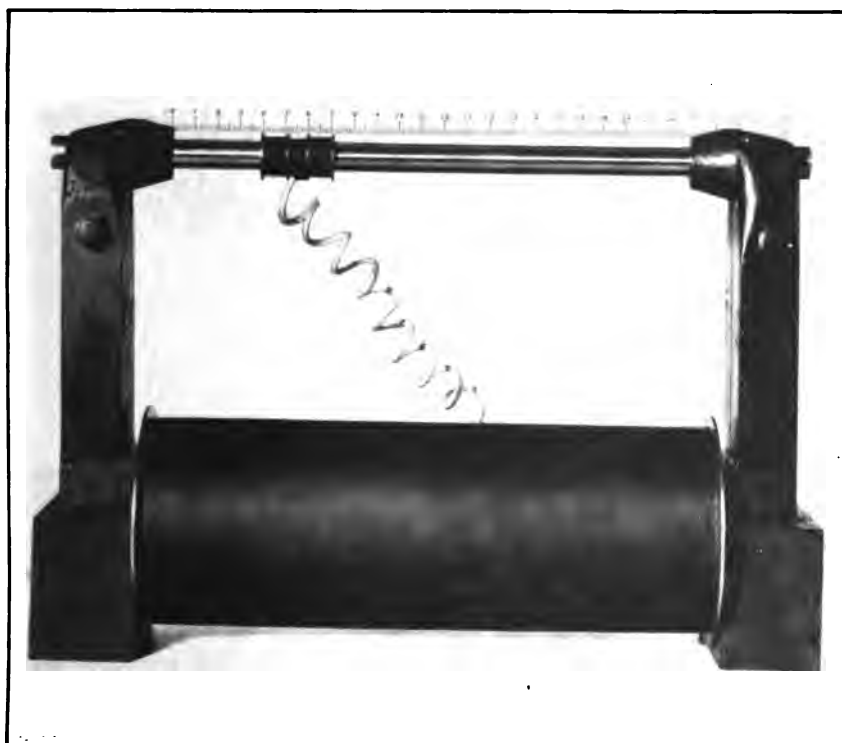


FIG. 3.—*Apparatus for the determination of magnetic uniformity along the length of a straight bar*

emf is constant for a uniform bar and variations indicate non-uniformities. It is possible with suitable apparatus to obtain a photographic record showing the degree of uniformity along the length of a bar.

The choice of the induction at which to make uniformity determinations is of considerable importance. If the induction is carried too high, the effect is to raise the value of the ordinates of the

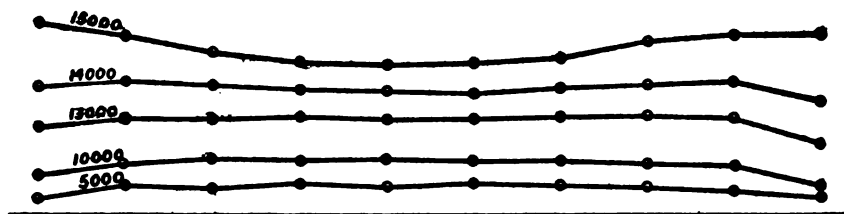


FIG. 4.—Form of the uniformity curve taken at different values of induction

uniformity curve near the ends (Fig. 4). This effect is probably due to the high magnetizing force necessary to produce high inductions. Readings taken at low inductions may be affected by the previous magnetic history of the specimen. The best region in which to work is that in which the induction is changing at the greatest rate, but this involves a preliminary determination of the normal induction, which is not desirable. All of the curves in this paper were taken with an induction at the middle of the bar of approximately 10 000 gauss. This induction for ordinary

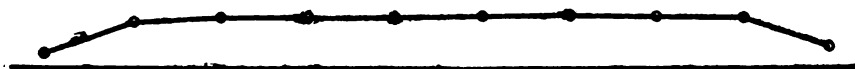


FIG. 5.—Showing depression of the ends of the uniformity curve for a uniform bar due to the effect of the poles of the electromagnet

materials does not require an excessive magnetizing force and gives the required sensitivity.

The form of the uniformity curve is affected by the poles of the electromagnet which modify the distribution of induction near the ends of the bar. The effect is to depress the curve at the ends (Fig. 5). This limits the proportion of the total length of the bar that can be examined by this method. This limitation is not serious, however, as the ends of the bar are usually clamped in yokes or pole pieces and do not enter directly into the measurements. Values can be obtained for points near the ends of the

bar by the use of extensions of the same cross section, which are butted against the ends of the bar and held in place by means of iron sleeves. This arrangement allows readings to be taken nearer the ends of the bar, as the effect of the poles is decreased and the effect of the additional joints is of less importance.

The consistency on repetition is satisfactory, as shown by repeated tests on the same bars. The results can be duplicated within the limits of the precision of reading the galvanometer and setting the position of the test coils. The two curves of Fig. 6 were taken nearly a year apart. The slight differences are immaterial and may be attributed to different positions of the bar in the apparatus and consequent different settings of the test coils.

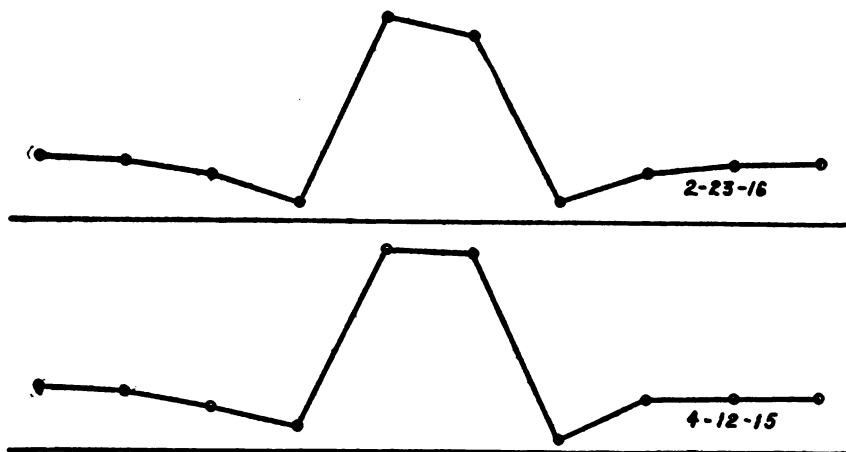


FIG. 6.—Showing the consistency on repetition of the uniformity curve

OBSERVATIONS AND RESULTS

Tests have been made on a large number of bars to determine the degree of magnetic uniformity that may be expected in ordinary test bars as received and also to determine the magnitude of the effect of nonuniformities on the accuracy with which magnetic measurements can be made.

The curves of Fig. 7 give the results of uniformity tests on a number of cast-steel bars that have been submitted to the Bureau of Standards for test. The curves all show large deviations from uniformity and are characteristic of this class of material. Most of these bars have smooth surfaces and give no indication to the eye of the presence of nonuniformities. Another class of material that gives trouble is wrought iron, which in general is

very inhomogeneous. Uniformity curves (Fig. 8) were taken for 12 bars of this material, which had been carefully annealed. These curves also show large deviations and indicate extremely inhomogeneous material.

The effect of nonuniformities on the accuracy of the magnetic measurements may be illustrated by the results of tests on a long bar which has a magnetically soft spot near the middle. Normal induction curves (Fig. 9) taken with the middle test coil of a Burrows permeameter over different portions of this bar show that the observed values of the magnetizing force necessary to produce given inductions are lower when the test coil is over the region which is magnetically soft as indicated by the uniformity curve. In this case there are differences as great as 15 per cent in the observed values of the magnetizing force giving the same induction.

A quantitative statement of the effect of nonuniformities on the accuracy of magnetic measurements is difficult because the magnitude of the error depends upon a number of variables. Among these variables are the induction in the specimen, the nature, number, magnitude, position, and extent of the nonuniformities present and the type of apparatus used for the measurements. It is possible, however, to set a limit for the magnitude of nonuniformity that may exist, when a certain accuracy is desired, from the effect on the normal induction measurements of a single nonuniformity which is located in the position of maximum effect. The effect on the data obtained with the Burrows permeameter is greatest when the nonuniformity is at the middle of the bar. For the purpose of setting this limit, normal induction measurements were made with a Burrows permeameter on a number of bars of various materials which originally were approximately uniform. After nonuniformities had been artificially produced at the middle, the measurements of uniformity and normal induction were repeated. The curves of Fig. 10 which show the effect of drilling a 2.5 mm hole in a bar of low carbon steel 1.27 cm in diameter are typical. The effect of these artificially produced hard spots is to increase the observed value of the magnetizing force necessary to produce a given induction. The data show that for any given induction the percentage error in the observed magnetizing force is, roughly, proportional to the percentage deviation of the ordinate of the uniformity curve from the mean value. Observations made with a number of modifications of the

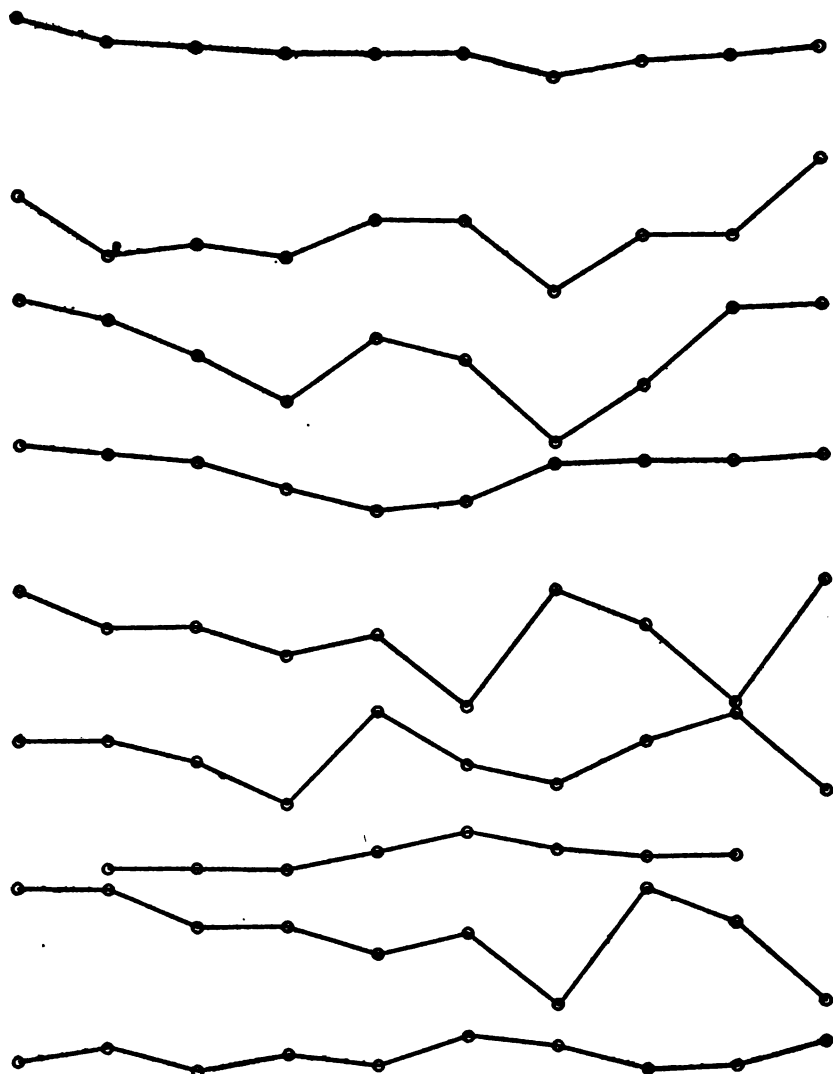


FIG. 7.—Uniformity curves for a number of characteristic cast-iron bars.

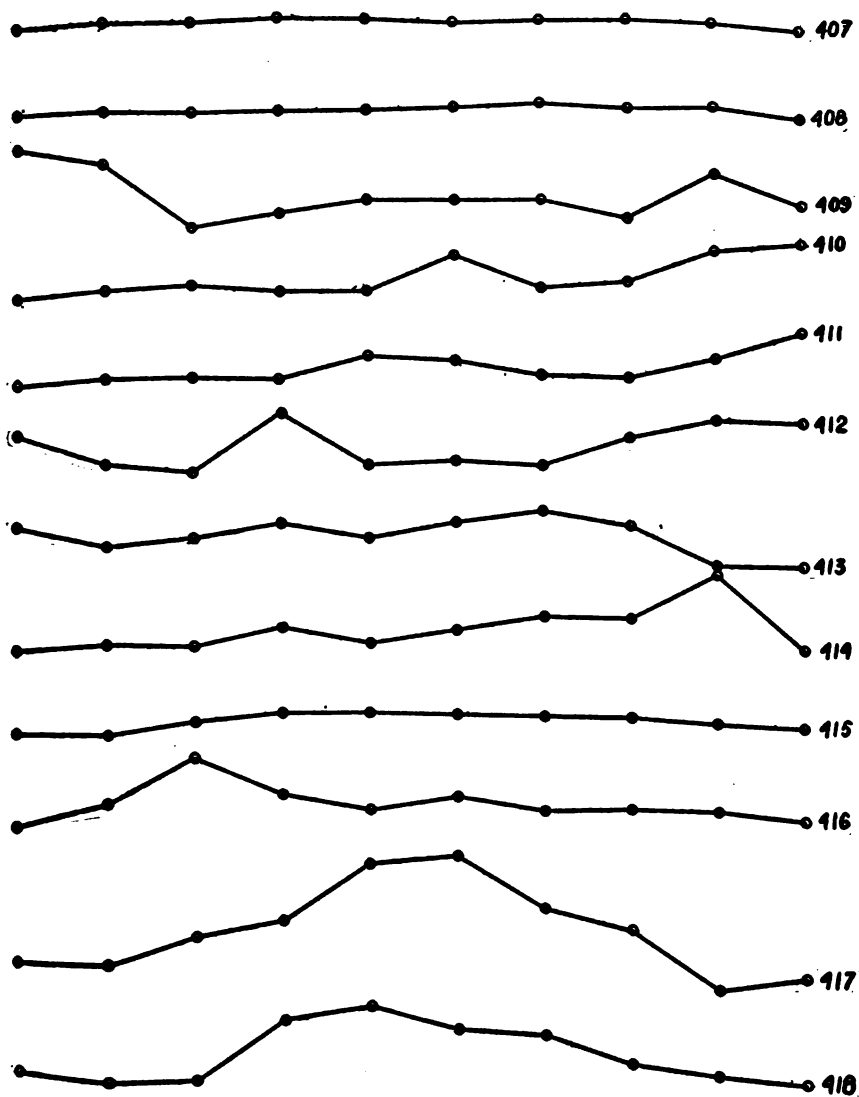


FIG. 8.—Uniformity curves for twelve specially prepared bars of wrought iron

form of the magnetizing system give substantially the same results. The quantitative results may, however, be affected by variations in the proportions of the apparatus. With the apparatus used it was found that for an accuracy of 1 per cent variations in the value of the ordinate of the uniformity curve from the mean must not exceed 10 per cent.

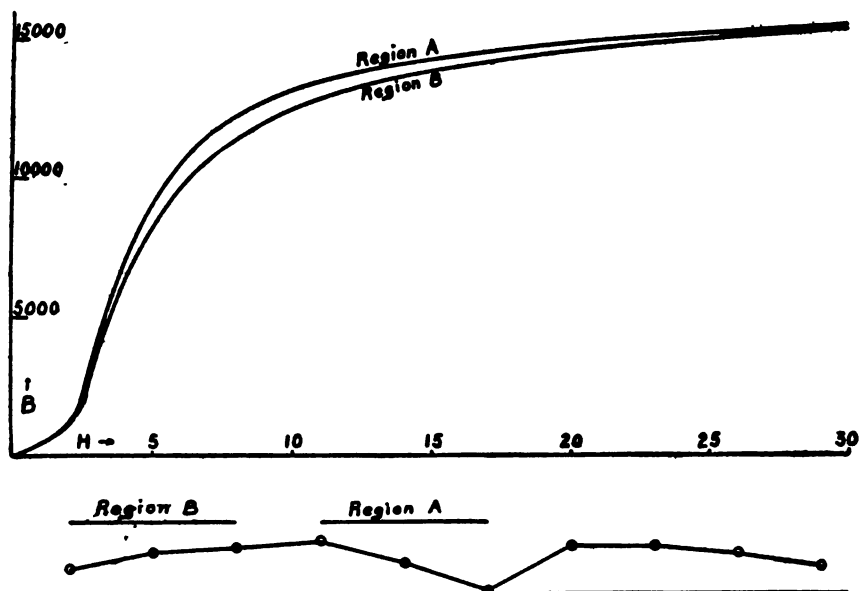


FIG. 9.—Normal induction curves taken with the permeameter test coils over the regions A and B as indicated on the uniformity curve, showing the effect of the magnetically soft spot near region A

Numerical data for the 12 wrought-iron bars of Fig. 7 (Table 1) show that not one of these bars comes within this limit. The best one shows a maximum deviation of 16 per cent and may be used where an accuracy better than 2 per cent is not necessary. The next best bar has a maximum deviation of 38 per cent and is used as an auxiliary. If two bars have equal maximum deviations, the one whose deviations have the smaller arithmetic sum is the better.

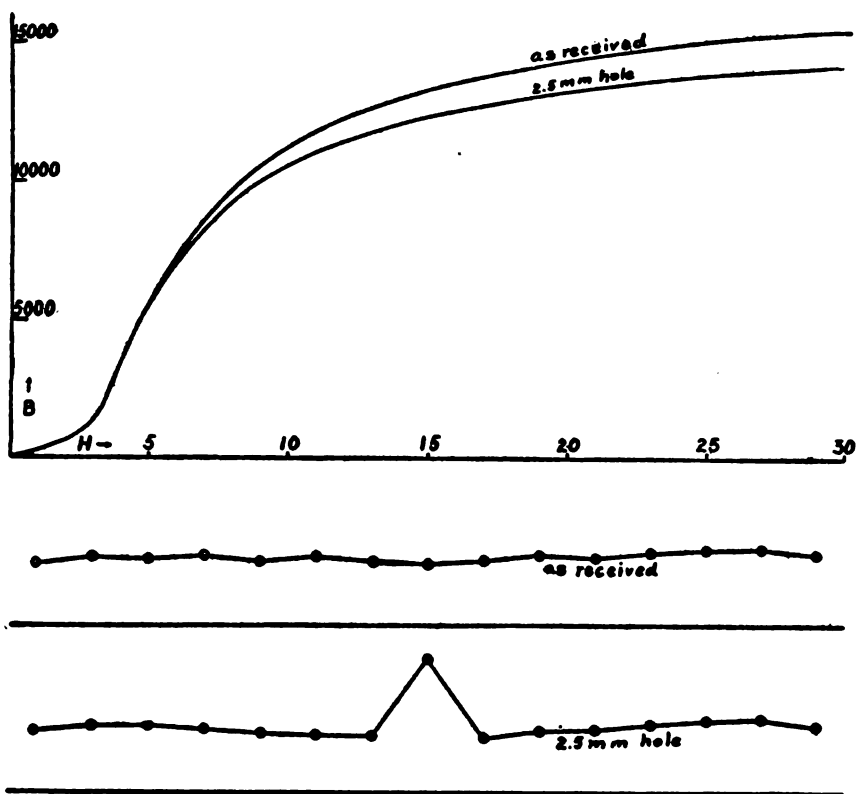


FIG. 10.—Showing the effect on uniformity and normal induction curves of drilling a 2.5 mm hole through the middle of a bar 1.25 cm in diameter

TABLE 1

Per Cent Deviation From the Mean of Ordinates of the Uniformity Curves for 12 Wrought-Iron Bars

Bar No.....	407	408	409	410	411	412	413	414	415	416	417	418
Distance from middle:												
6 cm.....	- 2	-15	+288	- 37	- 48	- 48	-56	- 78	-104	- 22	-100	-103
4 cm.....	- 2	-15	-250	+ 4	- 30	-103	-22	- 96	- 13	+203	- 45	- 98
2 cm.....	+15	+ 4	-100	- 46	- 48	+232	+24	+ 39	+ 54	+ 19	- 17	+ 73
0 cm.....	+ 8	- 8	+ 31	- 50	+122	- 58	-26	- 78	+ 46	- 58	+ 93	+110
2 cm.....	-16	+12	+ 25	+208	+ 83	- 35	+24	- 17	+ 25	- 3	+108	+ 42
4 cm.....	+ 1	+38	+ 25	- 25	- 35	- 68	+58	+113	+ 13	- 72	+ 2	+ 25
6 cm.....	- 2	- 8	-200	+ 12	- 57	+ 84	+ 4	+ 83	- 17	- 69	- 43	- 64

The magnetic laboratory of the Bureau of Standards maintains a set of magnetic standards of different materials. A large number of bars have been examined and only those whose uniformity is satisfactory are included in this set. (The wrought-iron bar mentioned above is included for want of a better bar of this class of material.)

The uniformity curves for these standard bars, a few of which are given in Fig. 11, show that the comparative magnetic hardness is indicated by the average ordinate. The higher the ordinate the harder magnetically is the bar. The numerical data (Table 2) show that, with the single exception already noted, the deviations are all well within the specified limit.

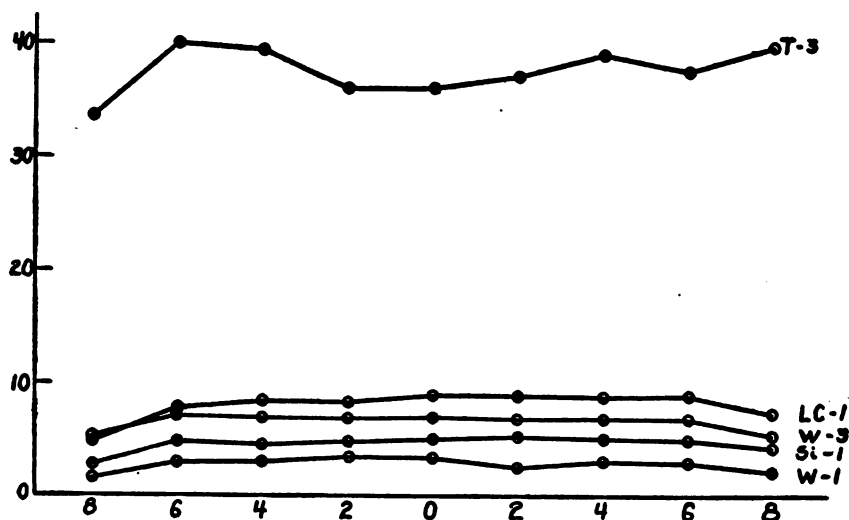


FIG. 11.—Uniformity curves for a number of magnetic standard bars.

TABLE 2

Per Cent Deviation From the Mean of Ordinates of the Uniformity Curves for Five Standard Bars

Bar No.	W-1	W-3	St-1	LC-1	T-3
Distance from middle:					
6 cm.	-2	+3	-3	-3	+6
4 cm.	-2	+2	-7	-1	+3
2 cm.	+15	0	-1	-2	-5
0 cm.	+8	+2	+3	+5	-5
2 cm.	-16	-3	+5	+5	-2
4 cm.	+1	0	+3	0	+3
6 cm.	-2	-2	+1	+2	-1

OTHER APPLICATIONS

The application of this method to the detection of mechanical inhomogeneities and flaws opens up a wide field of possible applications. Space will only be taken here to describe the results of one out of a number of such tests that have already been made.

Fig. 12 gives uniformity curves for a bar which was originally uniform. After the bar had been bent in a vise till it had a permanent bend of about 10° and then straightened the uniformity curve shows a decided kink which indicates a magnetically hard spot. The effect is shown to be greatest at the point where the bar was clamped and gradually decreases toward the free end of the bar. After annealing to remove the effect of these strains, the uniformity curve shows even less deviation than the original.

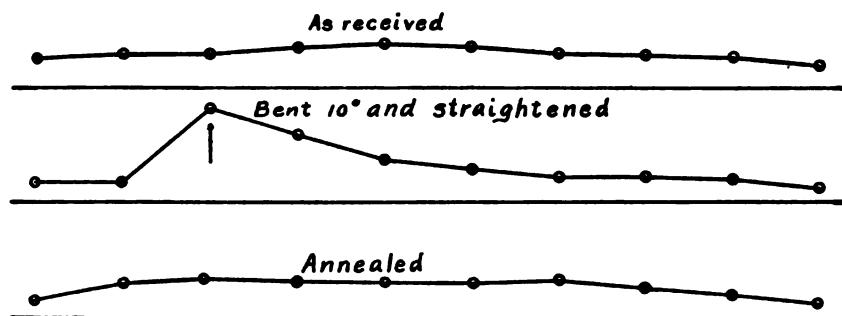


FIG. 12.—Showing the effect of bending and restraightening and of annealing on the magnetic uniformity along the length of a bar.

SUMMARY

Magnetic standard bars are used for the calibration of permeameters and the comparison of methods of magnetic testing with a standard method. One requisite of a magnetic standard bar is that it shall be magnetically uniform along its length. If this condition is not met, errors may arise which can not be calculated or eliminated from the measurements, and which may be of considerable magnitude.

The degree of magnetic uniformity of a bar may be determined from observations of the distribution of magnetic leakage along the length of a specimen when it is magnetized between the poles of a suitable electromagnet. The degree of uniformity is indicated by values of the rate of change of leakage along the length of the bar. Deviations of these values from a constant indicate the presence of nonuniformities. An increase in the value indicates a

magnetically hard spot, while a decrease indicates a soft spot. The degree of uniformity can be indicated by means of a curve plotted between values of the rate of change of leakage and displacement along the bar. This curve is called the uniformity curve and for a uniform bar is a straight line parallel to the axis of displacement.

Errors in magnetic measurements due to nonuniformities depend upon the induction, the nature, location, magnitude, and extent of the nonuniformities and upon the type of apparatus. For a given accuracy no ordinate of the uniformity curve must deviate from the mean by more than a given amount.

This method may be applied to the examination of magnetic materials for mechanical inhomogeneities and the detection of flaws.

WASHINGTON, July 26, 1916.

THERMOELECTRIC MEASUREMENT OF CRITICAL RANGES OF PURE IRON

By George K. Burgess, Physicist, and H. Scott, Aid

INTRODUCTION

There have been a number of thermoelectric observations made over the critical ranges of iron, A_1 and A_2 , using an iron wire as one element of a thermocouple and usually a platinum or copper wire as the other.¹ These observations possess the disadvantage of giving the total electromotive force developed between the cold junction kept at ordinary temperatures and the hot junction immersed in the heated region of a furnace; there may thus be ambiguity or superposition of thermoelectric effects along the iron wire. Messrs. Boudouard and Le Chatelier endeavored to overcome this difficulty, in the case of certain steels,² by using short bars to the ends of which platinum wires were attached, but were unable to get very satisfactory observations due to temperature irregularities, nor did they work with pure iron. Dr. Benedicks has recently devised a method³ which, as described, has the disadvantages above noted of ambiguity, and in addition seems to require for the detection of a transformation point that it exist at different temperatures on heating and on cooling.

It has been demonstrated, however, that the thermal effect for the transformation A_1 occurs at the identical temperature 768°C on heating and cooling,⁴ and the same might be expected to be true for the thermoelectric effect. Benedicks was unable to detect any thermoelectric effect at A_1 .

¹ Dewar and Flemming, *Phil. Mag.*, IV, 40, p. 95, 1895; Harrison, *Phil. Mag.*, V, 8, p. 177, 1902; Broniewski, *C. R.*, 156, pp. 699, 1903, 1913.

² Boudouard, *Rev. de Métallurgie*, 1, p. 80, 1904; Le Chatelier, *ibid.*, *C. R.*, 102, p. 134.

³ Benedicks, *C. R.*, 162, p. 297, 1916. *Jl. Iron and Steel Institute*, May, 1916.

⁴ Burgess and Crowe, *this Bulletin*, 10: p. 315; 1914.

METHOD

With the object of furnishing a more exact experimental basis on which to construct an adequate theory of the iron transformations, it seemed desirable to investigate the thermoelectric phenomena of pure iron more accurately than has hitherto been done. In the following experiments we have used a modification of the method of Le Chatelier giving the thermoelectric power directly, but have succeeded in eliminating the uncertainties of temperature control and distribution, have carried out the experiments in vacuo, and have used iron of exceptional purity. The observations extend over the range 0° to 1000° C and the thermoelectric characteristics of the transformations A_2 and A_1 have been determined with considerable accuracy. It has been found possible to measure the thermoelectric power (dE/dt) of iron against platinum for intervals of 2° up to a temperature of 1000° C, from observations in which the fall of temperature along the iron remained quite constant and was usually less than 10° C.

MATERIALS

The iron used in these experiments was from an electrolytic deposit which had been subsequently melted in vacuo and was of an exceptionally high degree of purity, 99.968 Fe, being from sample F4 of Scientific Paper No. 213. Its analysis, in per cent, gave $C=0.009$, $S=0.009$, $P=0.001$, $Si=0.006$, $Mn=0.001$, $Cu=0.006$. This was drawn through sapphire dies into wire of 0.05 cm diameter. The platinum of the thermocouples, and also against which the thermoelectric power of iron was determined, was from Johnson and Matthey. The thermoelectric force developed by this platinum against the purest samples of normal Heraeus thermocouple platinum was determined at 1100° C.⁵ The difference in thermoelectric power was less than 0.1 microvolt, except for one series with a contaminated platinum couple.

THE APPARATUS

The apparatus is essentially the same as that used for the thermal study of the critical ranges of iron described in detail in Scientific Paper No. 213 of this Bureau. Its principal characteristics are a double-walled electric resistance furnace of 60 cm length in which a vacuum is maintained; an automatically controlled alternating-current supply for heating and cooling the furnace uniformly at any desired rate; a drum chrono-

⁵ Burgess and Sale, this Bulletin, 12, p. 289; 1915.

graph for recording times of observation to 0.1 second; a potentiometer reading to 0.1 microvolt; two galvanometers and a suitable commutator, by means of which temperatures and temperature differences may be measured to 0.01 °C and emf's of the iron-platinum couple to 0.07 microvolt.

The arrangement of the iron sample in the furnace is shown in Fig. 1. A length of iron wire 5 to 7 cm is inserted, either by electric welding or with the oxy-gas flame, between the junctions of two Le Chatelier thermocouples within an outglazed porcelain tube which has a small opening communicating with the vacuum space of the furnace. The iron wire is displaced slightly from the

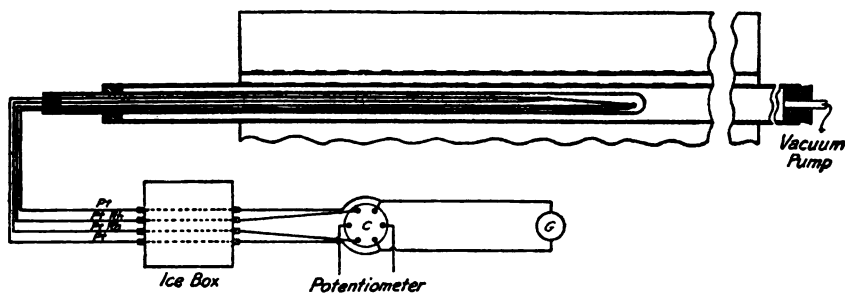


FIG. 1.—Arrangement of iron wire in furnace

hottest region within the furnace so that a small temperature difference will always exist between the two ends of the wire; and care is taken that one end—and not an intermediate region of the wire—is its hottest part.

OBSERVATIONAL METHOD

With two observers, simultaneous observations were taken at 2° intervals of (1) the temperature of one end of the iron wire by means of a thermocouple and potentiometer (see Fig. 1), and (2) the deflection of a galvanometer (*G*) connected across the platinum wires having the iron in the circuit; every 10° the temperature of the other end of the iron wire was taken with the second thermocouple. The instant of each observation was recorded on the chronograph. The true emf's of the iron-platinum couple caused by the temperature drop along the iron wire were then calculated (from 2), correction being made for the quality of platinum (negligible, except for one series) and resistance of circuit and galvanometer. This emf divided by the corresponding difference in temperature—that is, between the two ends of the iron wire (2° to 13°)—was taken as the true thermoelectric power of the iron-platinum couple at the temperature of the hot end of the iron

wire. A few observations were also taken around 0°C and 100°C by the usual method of using an iron-platinum thermocouple having wires passing from the region of variable temperature to a constant temperature, ice or steam.

The thermocouples were carefully intercompared and calibrated several times during the experiments and several calibrations were taken of the galvanometer (G) and scale giving emf's across the iron-platinum couple. Two sets of thermocouples were used, and four lengths of iron wire, all of the latter from the same analyzed sample. The rate of heating or cooling was usually of the same order of magnitude and was about 0.1° per second.

EXPERIMENTAL RESULTS

These are given in Fig. 2. For the dotted curves, the ordinates are proportional to the observed values of galvanometer deflections (G of Fig. 1), which again are approximately proportional to the thermoelectric power of the iron-platinum couple. The abscissæ are temperatures. Each dot represents an observation on the respective heating or cooling curve marked "up" or "down." The dashed curves represent the differences in temperature between the two ends of the iron wire for the adjacent heating or cooling curve.

It is seen that, for the several series, A_2 is indicated by a marked discontinuity in thermoelectric power, as already noted by Benedicks and others for the emf-temperature curve. As would be expected from the thermal⁴ and electrical⁶ observations, A_2 always occurs at a somewhat higher temperature than A_1 .

The A_2 transformation manifests itself, in all cases, by an abrupt change in direction of the curve representing thermoelectric power against temperature. As shown, this is slightly above 768°C , the maximum of the thermal transformation. The actual location of the break in the thermoelectric curve appears to be related to the length of iron wire and the temperature distribution. The thermal transformation, absorption or evolution of heat, is also sharply marked on the curves as a protuberance or dent at exactly 768°C . The fact that this thermal phenomenon is always recorded, even at the junction of wires only 0.05 cm in diameter, is further evidence of the sensitiveness of the method of measurement and of the persistence of A_2 to assert itself when given an opportunity.

The lower portions of the curves below 400°C are less satisfactory than at higher temperatures and for several reasons. There is

⁴ Burgess and Kellberg, this Bulletin, 11, p. 457; 1914.

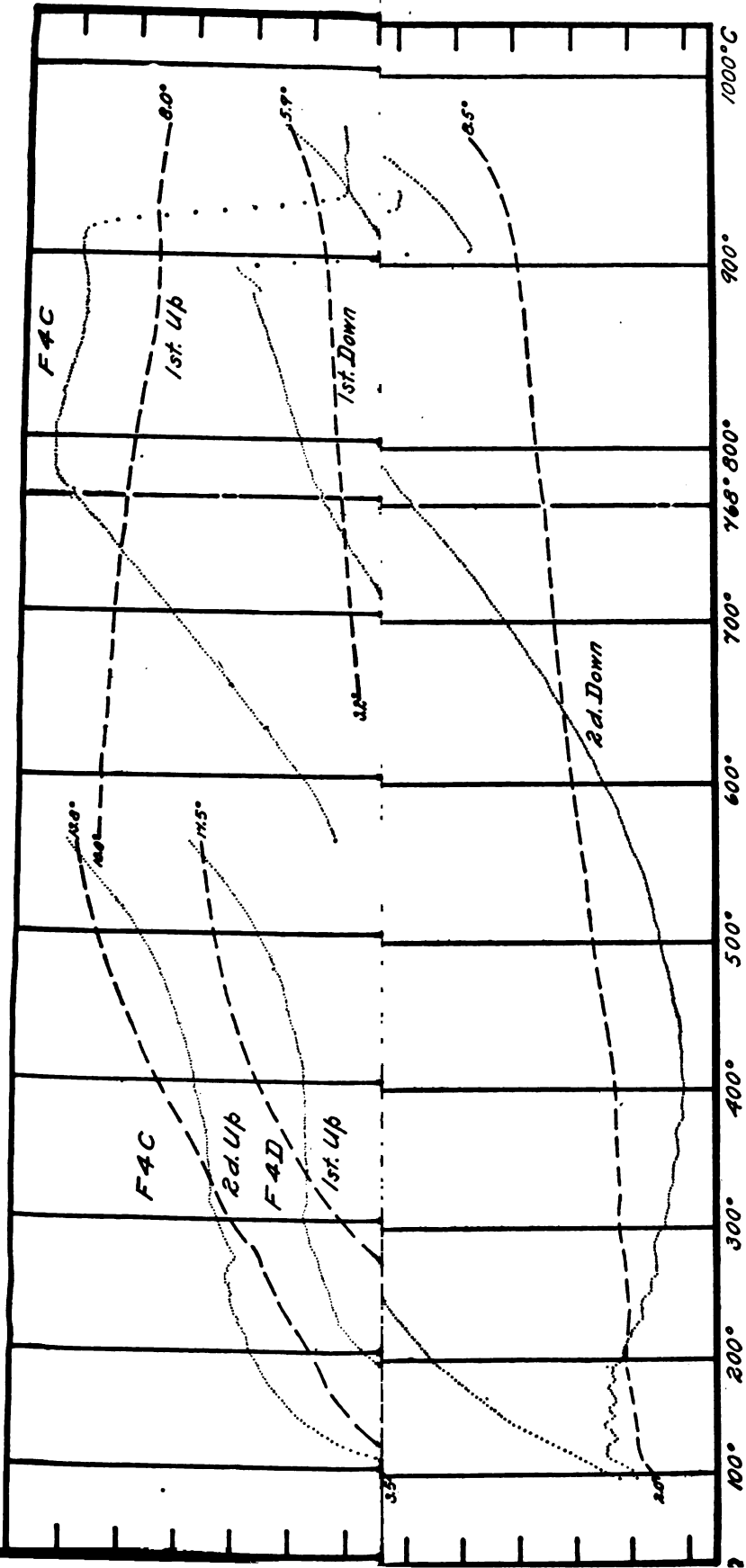


FIG. 2.—Ordinates are (1) galvanometer deflections (approximately thermoelectric power) and (2) difference in temperature between ends of iron wire. Abscissae are temperatures

greater difficulty in obtaining uniformity of temperature distribution, greater sensitiveness to drafts, and at times there were galvanometer and pump troubles; also the heating currents below 400°C , which protect thermally the sample, were almost if not quite zero.

It will be noted that for all the curves, the A_1 transformation at about 700°C , due to the presence of carbon, is absent. For iron containing as little as 0.1 per cent carbon, this A_1 point is, however, very marked, and was obtained accidentally by us when oil from the vacuum pump got into a sample of pure iron. There appears to be no permanent change in direction of the thermoelectric power curve at A_1 , but this transformation produces a jagged swinging back and forth during its progress. We expect to be able to publish shortly an account of the thermoelectric properties of carbon steels.

In Table 1 and Fig. 3 are shown the true thermoelectric power (dE/dt) versus temperature between 0° and 1000°C of the iron-platinum couple, being the most probable values computed from all the series of observations shown in Fig. 2, after all corrections are applied. The Peltier effect, $T dE/dt$ or the product of the absolute temperature and thermoelectric power is also shown. There is also given the values of d^2E/dt^2 , which is proportional to the Thomson effect for iron.

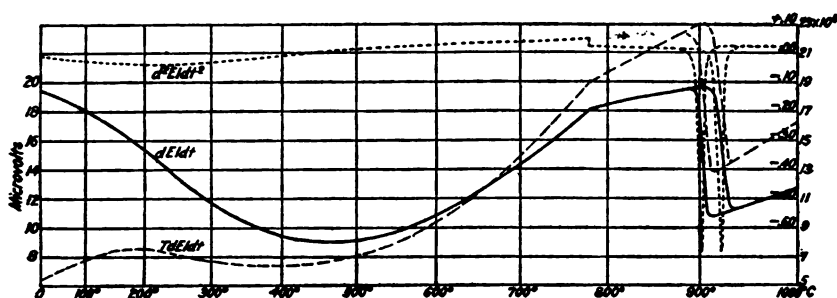


FIG. 3.—Thermoelectric power, Peltier effect and Thomson effect for iron-platinum

TABLE 1
Thermoelectric Power of Iron Against Platinum

Temperature, centigrade, t	Microvolts per degree, dE/dt	Peltier effect, TdE/dt	Thomson effect, d^2E/dt^2			
0.....	19.5	5320	-0.010			
100.....	18.1	6750	- .027			
200.....	15.4	7280	- .085			
300.....	11.7	6700	- .083			
400.....	9.5	6390	- .010			
500.....	9.1	7080	+ .009			
600.....	10.8	9430	+ .026			
700.....	14.3	13 910	+ .036			
780.....	18.1	18 980	+ .045			
800.....	18.4	19 740	+ .014			
880.....	19.4	22 350	+ .010			
	Heating	Cooling	Heating	Cooling	Heating	Cooling
900.....	19.7	17.5	23 100	20 510	0.000	-0.400
910.....	19.4	10.8	22 940	12 770	-.050	- .040
920.....	16.6	10.9	19 800	13 000	-.575	+ .010
930.....	11.4	11.1	13 710	13 350	-.023	+ .017
1000.....	12.6		16 030		+ .017	

SUMMARY AND CONCLUSIONS

There is described a sensitive method for the measurement of the change of true thermoelectric power with temperature, by means of which observations may be taken in vacuo at 2° intervals.

The thermoelectric power of the iron-platinum couple has been determined for the temperature range 0° to 1000° C, using iron of 99.968 Fe purity.

The A_1 transformation in iron is indicated by a marked discontinuity, A_c , being always above A_r . They are located at the same temperatures, as closely as could be desired by the thermal, crystallographic, magnetic, electrical resistance, dilatational, and thermoelectric methods, A_r , just below 900° C and A_c , slightly above 910° C. These physical discontinuities are apparently all a measure of the same transformation which it is generally agreed to call "allotropic" and which appears to delimit two phases in the iron.

At A_2 the thermoelectric power-temperature curve has a break or change in direction; and the other physical properties, except crystallographic change which has not been detected at A_2 , appear to undergo abrupt but slight modification, except for the magnetic change which is relatively enormous here.

As before stated with reference to the electrical resistance change,⁶ it is here again shown—this time for the thermoelectric effect—that the A_2 and A_1 transformations are essentially different in kind, but nevertheless both have a distinct physical existence; or in other words, A_2 and A_1 are critical points that properly may be said to delimit α , β , and γ iron.

There appears to be no break or discontinuity in the thermoelectric properties of iron between 0° and 768°C ; the actual shape of the thermoelectric power curve is, of course, contributed to by both the iron and the platinum.

At about 880°C , in the β range, there appears to be a slight discontinuity in the thermoelectric properties of iron. A break in the β region, first claimed by Weiss for the magnetic properties, is also noticed in some of the dilatation curves of Benedicks, and Burgess and Crowe called attention to it in their thermal study. The existence and meaning of this phenomenon are, however, too uncertain to describe them with positiveness.

WASHINGTON, August 21, 1916.





FIG. 1.—*Galvanometer B*

A STUDY OF ELECTROMAGNET MOVING COIL GALVANOMETERS FOR USE IN ALTERNATING-CURRENT MEASUREMENTS

By Ernest Weibel

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I. INTRODUCTION

The increasing use of alternating-current bridge and potentiometer methods of measurement emphasizes the need of an instrument for the detection and measurement of small alternating voltages. Instruments which have been used for this purpose are: The telephone receiver, the vibration galvanometer, the commutated direct-current galvanometer, the series and separately excited electrodynamicometer, the vibration string galvanometer, and the vibration electrometer. Of these the telephone receiver and the vibration galvanometer are in most common use. The

performances of the different instruments are so very different that some are more suitable for certain measurements than others. None of the instruments yet constructed has as good a performance as the sensitive moving coil galvanometers used in the corresponding direct-current measurements.

The separately excited electro-dynamometer has been used to some extent as an alternating-current galvanometer, but usually the instruments were originally constructed for use in accurate measurements by deflection methods and therefore on account of the stiff suspensions, large coils, weak fields, etc., were not very sensitive. Rowland¹ used the electro-dynamometer as a detector in various alternating-current bridge measurements of resistance, inductance, and capacity, but his instruments were not very sensitive. A more sensitive one has recently been designed by Palm.²

Stroud and Oates³ were the first to use a moving coil galvanometer with an electromagnet excited by alternating current. This was found to be very sensitive and was used in bridge measurements of inductance and capacity. They observed effects due to the induced currents in the moving coil and in the frame on which the coil was wound and found that it was necessary to work with a false zero. The complete explanation of the effects observed was not given.

Sumpner and Phillips⁴ designed iron-cored instruments and devised a number of methods of measurements, using them separately excited as detectors. Sumpner⁵ gave special attention to the design of alternating-current electromagnets in which the magnetic flux is very nearly in quadrature with the impressed voltage. Moving coil instruments with such electromagnets were found to be useful in measurements by deflection methods.

H. Abraham⁶ in 1906 described briefly an electromagnet moving coil galvanometer and showed that its performance depended upon the constants of the moving coil circuit. The effects of electromagnetic damping were observed and it was found possible by suitable adjustment of the constants of the moving coil circuit to make the motion aperiodic. He also observed that, on account of the induced current, the period and hence the sensitivity

¹ Physical Papers, pp. 294, 314; 1897.

² Zeit. Instr. **23**, p. 368, 1913; **24**, p. 281, 1914.

³ Phil. Mag., **6**, p. 707; 1903.

⁴ Proc. Phys. Soc., **22**, p. 395; 1908.

⁵ Proc. Roy. Soc., **80**, p. 310, 1908; Jour. Inst. E. E., **26**, p. 421, 1905.

⁶ Comptes Rendus, **142**, p. 993; 1906.

depended upon the reactance of the moving coil circuit. He found that, by compensating for the inductance of the moving coil by means of a suitable condenser in parallel with a resistance, the effect of the induced current could be made zero. This method of compensation for inductance was pointed out before by Sumpner⁷ and applied to telephone circuits. The same method was used by Trowbridge at the University of Wisconsin in 1905 to compensate for the inductance of the moving coil of an electromagnet galvanometer. Rosa⁸ used this method in 1906 to compensate for the inductance of the moving coil of an electro-dynamometer.

The step from the separately excited electro-dynamometer to the electromagnet moving coil galvanometer is a natural one in the development of a sensitive alternating-current galvanometer. The use of iron in the fixed coil of the instrument results in a marked increase in the strength of the magnetic field, so that the induced current is sufficient to affect the performance. It was therefore considered necessary for the proper design, construction, and use of such instruments to make a theoretical study taking these effects into account. The investigation has led to the construction of instruments having sensitivities much greater than those previously obtained and equal to those of the best permanent magnet moving coil galvanometers for direct-current measurements. The high sensitivity and good values for the other working constants result from the use of a strong magnetic field and a moving system properly designed for the conditions of use. Methods of testing and using have been worked out. The use of such instruments is not as simple as that of the telephone receiver or the vibration galvanometer, but it is believed that in many cases the gain in sensitivity and usefulness more than compensates for the trouble involved.

II. THEORY

1. ASSUMPTIONS

In order to determine the operation of electromagnet moving coil galvanometers of various kinds and under various conditions it was necessary to derive the general equation of motion of a coil carrying an alternating current in an alternating magnetic field. To do this it is necessary to consider all of the torques acting on the moving coil.

⁷ Journ. Soc. Tel. Eng., 16, p. 344; 1887.

⁸ This Bulletin, 8, p. 43; 1907.

In most cases the moving coil is made of fine wire, so that the assumption of a linear moving coil circuit introduces no appreciable errors arising from the variation of the current distribution in the cross section of the wire. Therefore, we consider the electro-magnet moving coil galvanometer to consist of a rigid linear circuit suspended so as to rotate through a small angle in an alternating magnetic field. The magnetic field will be assumed to be approximately uniform and to have no rotating components⁹ around the axis, so that it is possible to turn the coil to a position in which the resultant total magnetic flux is always zero. This is the normal position of the coil and only small deflections from a reference position near this will be considered.

2. NOTATION

To show the relations between the intrinsic constants, which depend upon the construction, and the performance under certain conditions it will be convenient to use the following symbols. Unless otherwise stated the quantities are to be expressed in electromagnetic cgs units.

Symbol	Quantity
θ	= Angular deflection from reference position.
t	= Time.
K	= Moment of inertia of moving system.
D	= Moment of damping of moving system.
U	= Moment of restoration acting on moving system.
$\omega = 2\pi$	times the fundamental frequency.
n	= Degree of the harmonic ($n=1$ for fundamental).
ϕ	= Total instantaneous magnetic flux through the coil produced by the separate excitation.
G_n	= Rate of change with deflection of the effective value of the n^{th} harmonic of the total flux.
α_n	= Phase angle of lead of the n^{th} harmonic flux ahead of the fundamental flux.
θ_0	= Deflection for no flux through the coil.
i	= Total instantaneous current through the moving coil.
e_1	= Instantaneous value of the impressed electromotive force in the moving coil circuit.
E_n	= Effective value of the n^{th} harmonic of the impressed electromotive force.
β_n	= Phase angle of lead of the n^{th} harmonic of the impressed electromotive force ahead of the fundamental flux.
e_2	= Instantaneous induced electromotive force in coil.
Z_n	= Impedance of the complete moving coil circuit to current of n^{th} harmonic frequency.
γ_n	= Phase angle of lead of the n^{th} harmonic of the current ahead of electromotive force producing it.

⁹ In general, the alternating field at any point in space may be considered as made up of two unidirectional alternating fields at right angles and a constant rotating field always perpendicular to one of these.

T = Complete undamped (free) period of the moving system.

T_1 = Complete period on open circuit.

λ = Logarithmic decrement.¹⁰

λ_1 = Logarithmic decrement on open circuit.

S_n = Sensitivity to n^{th} harmonic electromotive force.

S = Sensitivity when critically damped.

r = Resistance of moving system.

R' = Total resistance of moving coil circuit.

R = External critical resistance.

3. EQUATION OF MOTION

Since, in general, there will be harmonics in the magnetic field of the electromagnet depending upon the manner of excitation, the flux through the moving coil due to the excitation is expressed as

$$\phi = \sum_{n=1}^{n=\infty} \sqrt{2} G_n (\theta - \theta_0) \sin (n\omega t + \alpha_n) \quad (1)$$

or this can be written

$$\phi = \theta \sum_{n=1}^{n=\infty} \sqrt{2} G_n \sin (n\omega t + \alpha_n) - \theta_0 \sum_{n=1}^{n=\infty} \sqrt{2} G_n \sin (n\omega t + \alpha_n) \quad (2)$$

Now the torque acting on the moving coil when the current i flows is given by ¹¹

$$i \frac{d\phi}{d\theta} = i \sum_{n=1}^{n=\infty} \sqrt{2} G_n \sin (n\omega t + \alpha_n) \quad (3)$$

On equating this to the sum of the moments due to angular acceleration, damping, and restoration we obtain the differential equation of motion

$$K \frac{d^2\theta}{dt^2} + D \frac{d\theta}{dt} + U\theta = i \sum_{n=1}^{n=\infty} \sqrt{2} G_n \sin (n\omega t + \alpha_n) \quad (4)$$

The current in the moving coil can be considered to be the result of two electromotive forces, the impressed electromotive force and the induced electromotive force. If the impressed electromotive force contains harmonics and if it is in synchronism with the magnetic field we can represent it by

$$e_1 = \sum_{n=1}^{n=\infty} \sqrt{2} G_n \sin (n\omega t + \beta_n) \quad (5)$$

¹⁰ The logarithmic decrement is here taken to be the natural logarithm of the ratio of one deflection to the following deflection in the opposite direction.

¹¹ The reason for neglecting the effect of the flux due to i is given on p. 40.

The induced electromotive force is

$$e_2 = -\frac{d\phi}{dt} = -\theta \sum_{n=1}^{n=\infty} \sqrt{2n\omega G_n} \cos(n\omega t + \alpha_n) - \frac{d\theta}{dt} \sum_{n=1}^{n=\infty} \sqrt{2} G_n \sin(n\omega t + \alpha_n) + \theta_0 \sum_{n=1}^{n=\infty} \sqrt{2n\omega G_n} \cos(n\omega t + \alpha_n) \quad (6)$$

giving for the total current

$$i = \sum_{n=1}^{n=\infty} \frac{\sqrt{2} E_n}{Z_n} \sin(n\omega t + \beta_n + \gamma_n) - \theta \sum_{n=1}^{n=\infty} \frac{\sqrt{2n\omega G_n}}{Z_n} \cos(n\omega t + \alpha_n + \gamma_n) - \frac{d\theta}{dt} \sum_{n=1}^{n=\infty} \frac{\sqrt{2} G_n}{Z_n} \sin(n\omega t + \alpha_n + \gamma_n) + \theta_0 \sum_{n=1}^{n=\infty} \frac{\sqrt{2n\omega G_n}}{Z_n} \cos(n\omega t + \alpha_n + \gamma_n) \quad (7)$$

Substituting in (3) we get for the moment of displacement

$$\sum_{n=1}^{n=\infty} \sqrt{2} G_n \sin(n\omega t + \alpha_n) \left[\begin{aligned} & \sum_{n=1}^{n=\infty} \frac{\sqrt{2} E_n}{Z_n} \sin(n\omega t + \beta_n + \gamma_n) \\ & - \theta \sum_{n=1}^{n=\infty} \frac{\sqrt{2n\omega G_n}}{Z_n} \cos(n\omega t + \alpha_n + \gamma_n) \\ & - \frac{d\theta}{dt} \sum_{n=1}^{n=\infty} \frac{\sqrt{2} G_n}{Z_n} \sin(n\omega t + \alpha_n + \gamma_n) \\ & + \theta_0 \sum_{n=1}^{n=\infty} \frac{\sqrt{2n\omega G_n}}{Z_n} \cos(n\omega t + \alpha_n + \gamma_n) \end{aligned} \right] \quad (8)$$

From this it follows that the resultant torque due to the alternating current and magnetic field of frequency $\frac{\omega}{2\pi}$ is the sum of a constant torque (the average of the above) and a series of sinusoidal torques of frequency $\frac{\omega}{2\pi}$ and higher. If, as is usually the case, no even harmonics are present, the sinusoidal torques are of frequency $\frac{\omega}{\pi}$ and higher; that is, have frequencies equal to and greater than twice the fundamental frequency. Now, this is large

compared with $1/T$, so that the motion will be very nearly the same as if a torque equal to the average of the above were acting.¹² The average torque as calculated from the instantaneous torque given above is

$$\begin{aligned} & \sum_{n=1}^{n=\infty} \frac{G_n E_n}{Z_n} \cos (a_n - \beta_n - \gamma_n) + \theta \sum_{n=1}^{n=\infty} \frac{n\omega G_n^2}{Z_n} \sin \gamma_n \\ & - \frac{d\theta}{dt} \sum_{n=1}^{n=\infty} \frac{G_n^2}{Z_n} \cos \gamma_n - \theta_0 \sum_{n=1}^{n=\infty} \frac{n\omega G_n^2}{Z_n} \sin \gamma_n \end{aligned} \quad (9)$$

We can now write the equation of motion as follows:

$$\begin{aligned} K \frac{d^2\theta}{dt^2} + \left(D + \sum_{n=1}^{n=\infty} \frac{G_n^2}{Z_n} \cos \gamma_n \right) \frac{d\theta}{dt} + \left(U - \sum_{n=1}^{n=\infty} \frac{n\omega G_n^2}{Z_n} \sin \gamma_n \right) \theta \\ = \sum_{n=1}^{n=\infty} \frac{G_n E_n}{Z_n} \cos (a_n - \beta_n - \gamma_n) - \theta_0 \sum_{n=1}^{n=\infty} \frac{n\omega G_n^2}{Z_n} \sin \gamma_n \end{aligned} \quad (10)$$

which is of the form

$$K \frac{d^2\theta}{dt^2} + D' \frac{d\theta}{dt} + U' \theta = A + B \quad (11)$$

where D' , U' and B depend upon the constants of the galvanometer and the moving coil circuit and A depends upon the impressed electromotive force in the moving coil circuit. This is the familiar

¹² Actually the motion is the sum of the slow motion resulting from the average torque and a vibration containing all of the harmonics to some extent. It can be expressed as a Fourier's series, but the coefficients are very complicated functions of the intrinsic constants. The magnitude of the error in determining the deflection due to the blurring of the image on account of the vibration of the coil can be estimated by inspection of the equation of motion for the simple case of sine wave flux and impressed electromotive force which is

$$\begin{aligned} K \frac{d^2\theta}{dt^2} + \left(D + \frac{G^2}{Z} \cos \gamma \right) \frac{d\theta}{dt} + \left(U - \frac{\omega G^2}{Z} \sin \gamma \right) \theta = \frac{GE}{Z} \cos (\beta + \gamma) - \theta_0 \frac{\omega G^2}{Z} \sin \gamma - \frac{GE}{Z} \cos (2\omega t + \beta + \gamma) \\ - \frac{d\theta}{dt} \frac{G^2}{Z} \cos (2\omega t + \gamma) + (\theta - \theta_0) \frac{\omega G^2}{Z} \sin (2\omega t + \gamma) \end{aligned}$$

The maximum values of the alternating torques are

$$\frac{GE}{Z}, \frac{d\theta}{dt} \frac{G^2}{Z} \sin \gamma, \text{ and } (\theta - \theta_0) \frac{\omega G^2}{Z}$$

and the vibration produced by these can be calculated approximately by the equation

$$\psi' = \left(\frac{\pi/\omega}{T} \right)^2 \psi$$

where ψ' is the amplitude of the vibration and ψ is the deflection which a constant torque equal to the maximum value of the alternating torque would produce. This formula is not correct when T approaches the value $\frac{\pi}{\omega}$; that is, near mechanical resonance. It is interesting to note that with a tight suspension and a small coil—that is, when T is small—the instrument becomes a "vibration galvanometer" which may be made to vibrate with twice the frequency of the magnetic field and hence be tuned to the frequency of the alternating torques.

equation of an oscillating system acted upon by the torques A and B and the particular solution shows that the motion may be either underdamped, aperiodic (critically damped or dead beat) or overdamped according as $(D')^2$ is less than, equal to, or greater than $4KU'$. The free period is

$$T = 2\pi\sqrt{\frac{K}{U'}} \quad (12)$$

This is the undamped period and equals the time required for the coil to deflect 99 per cent of the total deflection when critically damped. It is therefore a measure of the speed of the instrument. The logarithmic decrement is

$$\lambda = \frac{\pi D'}{\sqrt{4KU' - (D')^2}} \quad (13)$$

The torques A and B produce deflections from the equilibrium position equal to A/U' and B/U' .

The period not only depends upon the moment of inertia of the moving system and moment of restoration of the suspension, but also upon the constants of the moving coil circuit and the excitation and may be larger or smaller than the period without excitation. The damping is always larger with excitation. The torque B produces a deflection of the coil from its equilibrium position even when there is no impressed electromotive force. This deflection is proportional to the angle θ_0 and is due to the induced current in the circuit. The torque A which results from the impressed electromotive force produces a deflection depending upon the magnitudes of the harmonics in the excitation and in the electromotive force, upon the phase relations and upon the constants of the moving coil circuit.

4. SIMPLE CASE

For the simple ideal case of a sine-wave magnetic field and a sine wave of impressed electromotive force of the same frequency the equation (10) reduces to

$$\begin{aligned} K\frac{d^2\theta}{dt^2} + \left(D + \frac{G^2}{Z} \cos \gamma\right) \frac{d\theta}{dt} + \left(U - \frac{\omega G^2}{Z} \sin \gamma\right) \theta \\ = \frac{GE}{Z} \cos(\beta + \gamma) - \theta_0 \frac{\omega G^2}{Z} \sin \gamma \end{aligned} \quad (14)$$

In some cases γ , if not already zero, is made so by the use of suitable impedance in the moving coil circuit. Under these conditions (14) becomes

$$K \frac{d^2\theta}{dt^2} + \left(D + \frac{G^2}{R'}\right) \frac{d\theta}{dt} + U\theta = \frac{GE}{R'} \cos \beta \quad (15)$$

This shows that when an electromotive force E is inserted in the moving coil circuit the coil deflects through an angle

$$\theta = \frac{GE}{R'U} \cos \beta \quad (16)$$

so that the deflection is proportional to the component of the electromotive force in phase with the magnetic field. The motion of the coil to this new position depends upon the resistance R' and G and if

$$\left(D + \frac{G^2}{R'}\right)^2 = 4KU \quad (17)$$

or

$$R' = \frac{G^2}{\sqrt{4KU} - D} = R + r \quad (18)$$

the motion is aperiodic.

The operating constants of importance when the electromagnet moving coil galvanometer is used as a detector in bridge or potentiometer measurements are the sensitivity S , the free period T , and the external critical resistance R . This sensitivity is defined as the deflection per unit electromotive force in the moving coil circuit when the motion is aperiodic and is expressed in millimeters deflection at 1 meter of the reflected ray per microvolt. The free period is in seconds and the resistance in ohms. The relations between these constants and the intrinsic constants follow from equations (12), (16), and (18), and are

$$S = 200\,000 \frac{\sqrt{4KU} - D}{GU} \quad (19)$$

$$T = 2\pi \sqrt{\frac{K}{U}} \quad (20)$$

$$R = 10^{-9} \frac{G^2}{\sqrt{4KU} - D} - r \quad (21)$$

For these conditions the operating constants are independent of frequency and are the same as for the direct-current galvanometer.

In order to design an instrument for use as a detector in a certain kind of measurement it is necessary to be able to calculate the intrinsic constants from the required operating constants. The necessary relations are readily obtained from equations (19), (20), and (21), and are

$$K = 0.161 \frac{T^2}{S^2(r+R)} m \quad (22)$$

$$U = 6.4 \frac{T}{S^2(r+R)} m \quad (23)$$

$$G = 31\,800 \frac{T}{S} m \quad (24)$$

where

$$m = 1 \pm \sqrt{1 - 0.99 \frac{S^2(r+R)}{T^2}} D$$

We see from this that the damping D can not be greater than $\frac{T^2}{0.99S^2(r+R)}$. Furthermore, the $\pm$ sign indicates that two designs are possible, one with all of the constants K , U , and G larger than the corresponding constants in the other design. The larger constants are the more easily obtained.¹³

5. EFFECTS OF REACTANCE

If the moving coil circuit is inductive we see from equation (14) that one effect is a deflection of the coil on account of the additional torque

$$-\theta_0 \frac{\omega G^2}{Z} \sin \gamma$$

If the circuit has inductance (γ negative) the coil turns toward the position of zero flux and if the circuit has capacity it turns in the opposite direction. The effect is zero, however, if θ_0 is zero; that is, if the equilibrium position of the coil with the circuit open is such that no flux links with it. A second effect of γ being different from zero is that the period is changed on account of the torque given by the term

$$-\theta \frac{\omega G^2}{Z} \sin \gamma$$

¹³ For a discussion of the design of critically damped galvanometers see Wenner, this Bulletin, 18, 1916. Scientific Paper No. 273.

Thus if γ is negative (lagging current) the period is shortened on account of the increased restoring torque, and if γ is positive the period is lengthened. If the factor

$$\frac{\omega G^2}{Z} \sin \gamma \text{ becomes greater than } U$$

the coil is in unstable equilibrium and will, if displaced slightly from its zero position, turn suddenly toward a position at right angles to the field. Stroud and Oates¹⁴ observed the shortening of the period, and the fact that the equilibrium position depends, in general, upon the impedance of the moving coil circuit. A third effect is the increase in damping, which for small angles of lead or lag is negligible.

The change in the restoring torque resulting from the reactance in the moving coil circuit causes a change in the period, damping, and sensitivity according to the following relations:

$$T = 2\pi \sqrt{\frac{K}{U - \frac{\omega G^2}{Z} \sin \gamma}} \quad (25)$$

$$\lambda = \frac{\pi (D + \frac{G^2}{Z} \cos \gamma)}{\sqrt{4K(U - \frac{\omega G^2}{Z} \sin \gamma) - (D + \frac{G^2}{Z} \cos \gamma)^2}} \quad (26)$$

$$S_1 = 200\,000 \frac{G}{Z(U - \frac{\omega G^2}{Z} \sin \gamma)} \quad (27)$$

These follow immediately from equations (12), (13), and (14), and give the performance of the electromagnet moving coil galvanometer when the motion is underdamped.

When the damping D and the resistance r are small, the equations (19), (20), and (21) can be put in the more convenient form

$$T = 2\pi \sqrt{\frac{K}{U}} \quad (28)$$

$$S = \frac{200\,000}{\pi} \frac{T}{G} \quad (29)$$

$$R = \frac{10^{-9}}{4\pi} \frac{G^2 T}{K} \quad (30)$$

¹⁴ Phil. Mag., 6, p. 707; 1903.

and if the U becomes $U - \frac{\omega G^2}{Z} \sin \gamma$ these become

$$T' = 2\pi \sqrt{\frac{K}{U - \frac{\omega G^2}{Z} \sin \gamma}} \quad (31)$$

$$S' = \frac{200\,000}{\pi} \frac{T'}{G} = \frac{T'}{T} S \quad (32)$$

$$R' = \frac{10^{-8}}{4\pi} \frac{G^2 T'}{K} = \frac{T'}{T} R \quad (33)$$

showing that the period, sensitivity, and the external critical resistance change in the same proportion when the constants of the moving coil circuit are changed. We can therefore in this manner change the operating constants of the instrument without an actual change in the construction.

The intrinsic constants K , D , and U can readily be determined for any particular galvanometer by methods described later. G can also be determined by these methods for the value of the excitation current used and being approximately proportional to this current can be calculated for other values. If the electromagnet moving coil galvanometer is to be used at a certain frequency with apparatus having a known resistance and reactance between its "galvanometer terminals," it is possible by the use of equations (25), (26), and (27) to adjust Z , γ , and the excitation to give the most satisfactory operation under the conditions of use.

In the adjustment of Z and γ to the best values it is sometimes necessary to use series or parallel resistance, inductance, or capacity. Sometimes the circuit containing the small electromotive force to be measured has a reactance which is large compared with its resistance, for example, a bridge for measuring large inductances or small capacities. The general aim in these cases is to produce an electrical resonating system¹⁵ so that the total reactance in the moving coil circuit is zero or such a value as will give the best operating constants. Obviously if such a system is used it is necessary to maintain the frequency constant.

¹⁵ Wenner, Weibel, and Silsbee, this Bulletin, 12, p. 18; 1915.

6. EFFECTS OF HARMONICS

To return now to the more general case in which the motion is that given by equation (10) we see that the effects of harmonics in the magnetic field and the impressed electromotive force can be considered as additions to the various torques acting on the coil. The result of the presence of the harmonics is to change the equilibrium position for no impressed electromotive force, to change the period, to change the damping, and to give a deflection proportional to the sum of the products of the harmonic flux and harmonic electromotive force and the cosine of the angle between them. If a given n^{th} harmonic electromotive force be impressed, the maximum deflection produced as its phase is changed gives a measure of G_n/Z_n and therefore of G_n since Z_n can be measured. This furnishes a means of measuring the selectivity of the galvanometer. Usually the magnitudes of the harmonics are small compared with the fundamental. The selectivity can be still further increased by using an electrical resonating system in the moving coil circuit. The effect of this is to make the impedance to the harmonics very large compared with the impedance to the fundamental, so that G_n/Z_n and hence the sensitivity to the harmonics is very small. The use of such a device, however, decreases the deflection with no impressed electromotive force. These changes are very small since they diminish as G_n/Z_n diminishes.

7. DETERMINATION OF CONSTANTS

The constants Z_1, Z_2, Z_3 , etc., and $\gamma_1, \gamma_2, \gamma_3$, etc., depend upon the resistance, inductance, and capacity of the moving coil circuit and can usually be easily measured. E_1, E_2, E_3 , etc., and $\beta_1, \beta_2, \beta_3$, etc., depend upon the conditions of use. The intrinsic constants $K, D, U, \theta_0, G_1, G_2, G_3$, etc., and a_1, a_2, a_3 , etc., depend upon the construction and excitation, and can all be determined experimentally as follows:

(a) The period T_1 and logarithmic decrement λ_1 are measured on open circuit.

(b) The period T_2 and logarithmic decrement λ_2 are measured with the moving coil connected to a fairly noninductive resistance R_2 of such a value that the decrement is about 10 times that on open circuit.

(c) With the moving coil connected to a very high noninductive resistance R_1 the sensitivity S_n of the instrument to electromotive

forces of the frequencies of the harmonics is determined for various phase relations of the harmonics with respect to the fundamental.

(d) The deflection of the coil when it is connected to an inductance gives θ_0 and by adjustment of the reference position of the coil—that is, by twisting the suspension slightly— θ_0 can be made zero.¹⁶

These data are sufficient for the calculation of all of the intrinsic constants under the conditions of excitation, etc., that exist during the measurements. The most important of the intrinsic constants can be calculated from the observed data by the use of the following formulas in the order given.

$$K = \frac{(r + R_2) T_1^4}{4\pi^4 (r + R_1)^2 \sum_{n=1}^{n=\infty} S_n^2} \left(\frac{\lambda_2}{T_2} - \frac{\lambda_1}{T_1} \right) \left(1 - \frac{\lambda_1^2}{\pi^2} \right)^2 \quad (34)$$

$$U = \frac{4\pi^2}{T_1^2} \left(1 - \frac{\lambda_1^2}{\pi^2} \right)^{-1} K \quad (35)$$

$$D = \frac{4\lambda_1}{T_1} K \quad (36)$$

$$G_1 = S_1 (r + R_1) U, \quad G_2 = S_2 (r + R_1) U, \text{ etc.} \quad (37)$$

The deduction of these relations is based upon the general equation of motion as given above (equation 10) and is as follows:

(a)

In this case the equation of motion becomes

$$K \frac{d^2\theta}{dt^2} + D \frac{d\theta}{dt} + U\theta = 0 \quad (38)$$

which gives for the period

$$T_1 = 2\pi \sqrt{\frac{K}{U} \left(1 - \frac{D^2}{4KU} \right)^{-1/2}} \quad (39)$$

and for the decrement

$$\lambda_1 = \frac{\pi D}{\sqrt{4KU} \left(1 - \frac{D^2}{4KU} \right)^{-1/2}} \quad (40)$$

¹⁶ If a small variable mutual inductance be used with one coil in the moving coil circuit and the other in the fixed coil circuit, this adjustment need only be made approximately. Then adjustment of the mutual inductance will make the total flux through the moving coil circuit zero for the position of the moving coil on open circuit. With this adjustment made the vibration is also least, and the equilibrium position of the coil with no impressed electromotive force is independent of the impedance of the circuit. If the magnetic flux is proportional to the current, the adjustment is made when

$$M I = G\theta_0.$$

(b)

The equation of motion is now

$$K \frac{d^2\theta}{dt^2} + \left(D + \sum_{n=1}^{\infty} \frac{G_n^2}{r + R_2} \right) \frac{d\theta}{dt} + U_2 \theta = 0 \quad (41)$$

which we see by comparison with (10) is true if the resistance is sufficiently noninductive, so that the cosine differs but little from one and Z differs but little from R_2 at all frequencies for which G_n is at all appreciable. The moment of restoration may be different, as is likely even if the resistance is sufficiently noninductive to make the above assumption valid.

As in (a) above this gives the period and decrement as follows:

$$T_2 = 2\pi \sqrt{\frac{K}{U_2}} \left[1 - \frac{\left(D + \sum_{n=1}^{\infty} \frac{G_n^2}{r + R_2} \right)^2}{4KU_2} \right]^{-1/2} \quad (42)$$

$$\lambda_2 = \frac{\pi \left(D + \sum_{n=1}^{\infty} \frac{G_n^2}{r + R_2} \right)}{\sqrt{4KU_2}} \left[1 - \frac{\left(D + \sum_{n=1}^{\infty} \frac{G_n^2}{r + R_2} \right)^2}{4KU_2} \right]^{-1/2} \quad (43)$$

(c)

With a very high noninductive resistance in the moving coil circuit the equation of motion is assumed to be

$$K \frac{d^2\theta}{dt^2} + D \frac{d\theta}{dt} + U\theta = \sum_{n=1}^{\infty} \frac{G_n E_n}{r + R_1} \cos(\alpha_n - \beta_n - \gamma_n) \quad (44)$$

Referring to (10) we see that the errors due to the omission of the damping term, the restoring term, and the deflecting term can be made as small as desired by making Z —that is, R_1 —large. Further, the period, decrement, and zero can be observed and should check with the observations in (a). An observation under these conditions gives

$$S_n = \frac{\Delta\theta}{\Delta E_n} = \frac{G_n}{(r + R_1)U} \quad (45)$$

so we get

$$S_1 = \frac{G_1}{(r + R_1)U}, S_2 = \frac{G_2}{(r + R_1)U}, \text{ etc.} \quad (46)$$

(d)

From (10) we see that the rest point with an inductance in the moving coil circuit and no impressed electromotive force is

$$\theta = \frac{\sum_{n=1}^{n=\infty} \frac{nG_n^2}{L}}{U + \sum_{n=1}^{n=\infty} \frac{nG_n^2}{L}} \theta_0 \quad (47)$$

and if $\sum_{n=1}^{n=\infty} \frac{nG_n^2}{L}$ is large compared with U , this becomes θ_0 . When this is true the period is very short, since U' is now many times U .

In (a), (b), and (c) the observed quantities are $T_1, \lambda_1, T_2, \lambda_2, R_2, R_1, S_1, S_2, S_3$, etc., and from these we are to determine K, D, U, G_1, G_2, G_3 , etc., by using equations (39), (40), (42), (43), and (46). From (39) and (40) we get by division and rearrangement

$$D = \frac{4K}{T_1} \lambda_1 \quad (48)$$

In a similar manner we get from (42) and (43)

$$D + \sum_{n=1}^{n=\infty} \frac{G_n^2}{r + R_2} = \frac{4K}{T_2} \lambda_2 \quad (49)$$

Substituting (48) in (39) gives

$$T_1 = 2\pi \sqrt{\frac{K}{U} \left(1 - \frac{\lambda_1^2}{\pi^2}\right)^{-\frac{1}{2}}} \quad (50)$$

so that

$$U = \frac{4\pi^2}{T_1^2} \left(1 - \frac{\lambda_1^2}{\pi^2}\right)^{-1} K \quad (51)$$

Using this in (45) we get

$$G_n = \frac{4\pi^2 S_n (r + R_1)}{T_1^2} \left(1 - \frac{\lambda_1^2}{\pi^2}\right)^{-1} K \quad (52)$$

Now, on substituting this and (48) in (49) we find that

$$K = \frac{(r + R_2) T_1^4}{4\pi^4 (r + R_1) \sum_{n=1}^{n=\infty} S_n^2} \left(\frac{\lambda_2}{T_2} - \frac{\lambda_1}{T_1} \right) \left(1 - \frac{\lambda_1^2}{\pi^2}\right)^3 \quad (53)$$

Which, together with (52), (51), and (48) give the relations desired.

The phase angles $\alpha_1, \alpha_2, \alpha_3$, etc., can be determined from the observed phase changes necessary to give maximum (or zero, which is more accurate) deflection for the various harmonics.

From a few observations it is therefore possible to obtain all of the intrinsic constants and hence determine the performance of the electromagnet moving coil galvanometer under various conditions of use.

8. DISTURBING TORQUES

It has been assumed in the preceding development of the theory of the electromagnet moving coil galvanometer that the only torques acting were those contained in equation (10), which are: (1) The torque of reaction due to the acceleration; (2) the torque due to the constant damping D ; (3) the torque due to the suspension; (4) the torque due to the current resulting from the impressed electromotive force; and (5) the torque due to the current resulting from the induced electromotive force.

Besides these torques there are, in general, others which we shall call disturbing torques. An analysis of these torques, using the results of the preceding considerations, shows the causes and indicates the means of making the disturbances small. These disturbing effects will now be considered in turn. Among them there are several which are similar in that their magnitude is proportional to the deflection from some position. The effect of such a torque is, in general, to change the period and cause a permanent deflection. For if we let the disturbing torque be $-u(\theta - \theta')$ and the torque of the suspension is $-U\theta$, we get for the total torque

$$-U\theta - u(\theta - \theta') = -(U + u)\theta + u\theta' \quad (54)$$

and if for simplicity we equate this to $K\frac{d^2\theta}{dt^2}$ we see that the resulting period is

$$T' = 2\pi\sqrt{\frac{K}{U+u}} = 2\pi\sqrt{\frac{K}{U}}\left(1 + \frac{u}{U}\right)^{-\frac{1}{2}} = T_0\left(1 + \frac{u}{U}\right)^{-\frac{1}{2}} \quad (55)$$

and if u/U is small, this becomes

$$T_0\left(1 - \frac{1}{2}\frac{u}{U}\right) \quad (56)$$

Also we see that the new rest point is

$$\theta'' = \frac{u}{U+u}\theta' \quad (57)$$

This is a position of stable equilibrium if u is between $-U$ and ∞ . As u decreases approaching $-U$ the period becomes longer and for smaller values the coil is not in stable equilibrium.

One disturbing torque of this kind is that due to the electrostatic attraction between the coil and the case, pole faces, etc., due to a difference of potential. This torque is proportional to the square of the potential difference and to the rate of change of the capacity between the coil and the case with deflection. The coil will therefore tend to a definite position and the disturbing torque will be approximately proportional to the deflection from this position. Obviously this disturbance requires only one lead to the coil for its existence. This torque has been definitely identified in several instruments and it was found that a potential difference of 20 or 30 volts was necessary in order to cause serious disturbances. The coil and the case should therefore be maintained at approximately the same potential.

A similar torque is that due to the change of the self-inductance of the coil as it turns. In the equation (3) we have used only the flux due to the excitation. The total torque is really

$$i \frac{d\phi}{d\theta} + \frac{i^2}{2} \frac{dL}{d\theta} \quad (58)$$

Now the torque $\frac{i^2 dL}{2d\theta}$ has been neglected. For the normal position of the coil the inductance will be a minimum, so that $\frac{d^2L}{d\theta^2}$ will be positive and for small angles $\frac{dL}{d\theta}$ is proportional to the deflection. Hence this torque tends to turn the coil out of the normal position and corresponds to u having a negative value. In sensitive instruments i is never large and $\frac{dL}{d\theta}$ is small, so that $\frac{i^2}{2} \frac{dL}{d\theta}$ is negligible compared with $i \frac{d\phi}{d\theta}$.

A very serious disturbing torque of this kind is that resulting from the fact that the materials of which the moving system is made have a finite susceptibility.¹⁷ If magnetic impurities are present in the form of small particles, these will be acted upon by forces proportional to the gradient of H^2 and hence to H^2 since the field remains similar to itself as the intensity is varied. The same is true to a less extent for the materials of the coil with a

¹⁷ Ayrton and Mather, *Phil. Mag.*, 42, p. 442, 1896; Anthony Zeleny, *Phys. Rev.*, 22, p. 401, 1906.

susceptibility a little different from that of the surrounding air. Now, the resultant torque will obviously be zero for some position near the normal one, for usually the field is very nearly uniform and the lack of uniformity is symmetrical with respect to this position. This disturbing torque is very evident in sensitive instruments and, if care is not taken, may be much greater than that due to the suspension. The magnetic dirt which is usually on the surface can be removed by suitable treatment (p. 45). The magnetic field should be of suitable distribution. Thus, for a moving system whose parts are cylindrical, with axes parallel to the axis of rotation, a uniform field gives no torque. For other shaped systems the field should be radial, with its center on the axis of rotation of the coil.

A disturbing torque which is of this same kind is that resulting from the capacity current between turns of the moving coil. Let us replace this distributed capacity by a capacity localized at the terminals. Then it follows from equation (14) that in the simple case this would give a torque

$$\frac{\omega G^2}{Z} \sin \gamma (\theta - \theta_0) = \omega^2 G^2 C (\theta - \theta_0) \quad (59)$$

which is directly proportional to the capacity, to the square of the frequency, and to the square of the exciting current. The only remedy for this is to diminish the capacity between turns by the use of thicker insulation or to diminish G by using fewer turns. The effect of this torque is to diminish the total restoring torque; that is, to lengthen the period so that it is only necessary to change the external circuit constants to bring the control back to its original value. The change of rest point is zero if θ_0 is zero, which can be made so by adjustment of either the coil or the mutual inductance.

If there is an alternating potential difference between the case and one terminal of the coil, there will be capacity current flowing through the coil, at least in part, and if this current has a component in phase with the magnetism, a torque acts tending to deflect the coil. The torque is proportional to the potential difference and has been observed to be very large in a sensitive instrument when the potential difference was only 5 volts at a frequency of 2100 cycles per second.

Other disturbing torques result from leakage and capacity currents between the parts of the moving coil circuit and external

bodies which, on account of the inevitable dissymmetry in the two leads to the coil, flow through the coil. These disturbances are very serious at high frequencies when the capacity currents are large. The remedy for these and many of the other disturbing torques is the shielding scheme¹⁸ shown in Fig. 2. The metallic shield and the one coil of the mutual inductance are maintained at a potential very nearly equal to the potential of the moving coil circuit.

The problem of the elimination of the mechanical disturbances is the same as for all instruments having delicate suspended systems, the difficulties increasing as the sensitivity increases. In extreme cases special attention is necessary in mounting the instrument.

In the electromagnet moving coil galvanometer it is sometimes necessary to use intense separate excitation, so that considerable

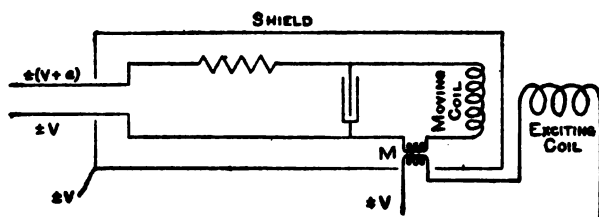


FIG. 2.—Shielding arrangement

heating is liable to occur. At low frequencies this occurs in the copper wire of the exciting coil or coils and at high frequencies the iron losses may be large. The exciting coils should always be separated from the moving system by a shield, so that disturbances caused by moving air will be minimized.

III. CONSTRUCTION AND PERFORMANCE

1. GENERAL DESIGN

The construction of an electromagnet moving coil galvanometer for use in alternating-current measurements differs but little from that of the sensitive moving coil galvanometer for direct currents. In place of the permanent magnet a laminated electromagnet is used. The moving system can be the same as for the direct-current galvanometer, with modifications suggested by the analysis of the disturbing torques. By using equations (22), (23), and (24), it is easy to calculate the values of the moment of

¹⁸ Price, *Phil. Mag.*, 42, p. 150, 1896; White, *Phys. Rev.*, 25, p. 334, 1907.

inertia K , the moment of restoration U , and the moment of displacement G necessary for an instrument which is to be used with apparatus having a resistance between "galvanometer" terminals about equal to R and which is to have a sensitivity S and a period T . The resistance r and the damping are assumed to be zero in this preliminary calculation. A consideration of the construction of such an instrument from the material available then allows estimates of the probable values of r and D to be made, so that more accurate values of the intrinsic constants can be computed.

In the construction of the moving system it is desirable to have the resistance of the coil as low as possible and the material as free from magnetic impurities as possible. The use of thin insulating material (enameled wire) is desirable, for this keeps the moment of inertia low. Furthermore, but little insulation is needed. However, in case the instrument is to be used at high frequencies the use of thin insulation increases the capacity between turns and hence the corresponding disturbing torque. A large mirror adds to the moment of inertia, but some sacrifice is necessary if accurate readings are to be made easily. Much can be gained by the use of well-made mirror of good glass. The resistance of the suspensions should be kept reasonably low. Short, flat suspensions made by rolling small (0.02 mm) copper wire have been found satisfactory.

The construction of the electromagnet should be such as to give a field of the desired strength and shape with as little power loss and distortion of the wave shape as possible. The strength of the field is limited by the magnetic properties of the materials used in the coil and also by the source of excitation available. At high frequencies special attention is necessary to keep the losses due to hysteresis and eddy currents low. It is also necessary to design the exciting winding so that the higher voltages required at the higher frequencies cause no trouble. It is desirable to have the winding made up of several coils which can be connected in series or parallel as the case demands.

Usually the reluctance of the air gap is large compared with that of the iron and also low inductions (500–2000 gauss) are used so that the production of harmonics in the magnetic flux depends upon the source of excitation. Since the inductance of the fixed coil is large compared with the resistance, the exciting current and therefore the magnetic flux will contain fewer harmonics than the impressed electromotive force. The magnitude of the harmonics can be still further reduced by the use of a resonating

system. A large inductance in series and a suitable condenser in parallel with the coils improves the wave shape of the magnetic flux and hence increases the selectivity of the instrument.¹⁹

In order to shield the moving system of the galvanometer from leakage and electrostatic effects, it is convenient to surround it with a metallic case which is connected to some point in the testing circuit such that the difference of potential between it and the coil is always very small. This shield can also be the support for the adjustment and arrestment devices, terminals, etc. The iron core of the fixed coils should be connected electrically to the shield.

Several sensitive galvanometers have been designed and built following the ideas given above. The description of the construction and the performances are given on the following pages.

2. GALVANOMETER A

The moving system of this instrument consists of a coil 8 mm wide and 12 mm high which has $27\frac{1}{2}$ turns of 0.08 mm (No. 40 American wire gage) single silk-covered copper wire and a mirror 1 cm in diameter and 0.6 mm in thickness. The upper and lower suspensions are each about 5 cm long and were made from 0.015 mm copper wire rolled flat.

The coil was suspended in an air gap 10 mm long between pole-faces 8 mm square. The core of the fixed coil was made of laminations of good silicon steel. The exciting coil consisted of 1560 turns of 0.51 mm (No. 24) enameled copper wire. With 120 volts at 60 cycles per second on this winding the current was 0.93 ampere and the performance was as shown in Table 1. Here C is the capacity of the condenser connected to the terminals of the moving coil, T is the period of the instrument, R is the external critical resistance, and S_{80} , S_{180} , and S_{300} are the sensitivities to the fundamental, the third and the fifth harmonics, respectively. The prediction of aperiodic motion is borne out, but it is noted that the external resistance necessary to produce it is low.

TABLE 1
Performance of Galvanometer A

C	T	R	S_{80}	S_{180}	S_{300}
μf	seconds	ohms	mm/ μv	mm/ μv	mm/ μv
0	5.6	0.1	13.5	0.13	0.015
2	8.0	5.1	16.0		

¹⁹ Ryan, Trans. Am. Inst. E. E., 20, p. 1419; 1903.

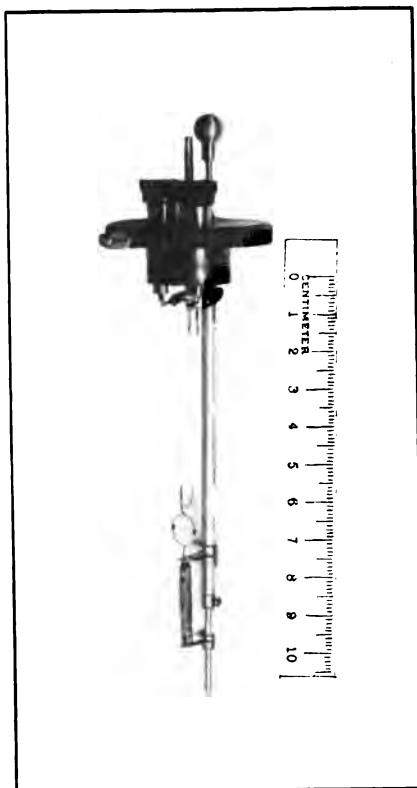


FIG. 3.—*Moving system of galvanometer B*

Approximate values of the intrinsic constants of this instrument are as follows:

$$K = 0.022, D = 0.001, U = 0.028, G_1 = 28\,000, G_2 = 280, G_3 = 30, \text{ and } r = 14 \times 10^9.$$

3. GALVANOMETER B

This instrument was intended for use with apparatus in which the resistance between the terminals to which the galvanometer is to be connected is of the order of 100 ohms. A period of only a few seconds, a sensitivity of several millimeters per microvolt, and aperiodic motion were required so that it was necessary to use a light coil. The coil consists of 100 turns of 0.08 mm (No. 40) enamelled copper wire wound on a wood frame to make a coil 2 cm long, 0.3 cm wide, and 0.1 cm thick. The wire is insulated with cellulose acetate which is very thin. The coil was treated with potassium copper chloride to remove the magnetic impurities and it was then washed, dried, and dipped into collodion, which bound the parts firmly together. The two end wires of the coil are brought out at the top instead of in the usual way in order not to have a large portion of the total magnetic flux through the core linking the moving coil circuit. One of these wires is soldered to a large copper wire, fixed to the wood frame of the coil, and to which a mirror 0.8 cm in diameter is attached. To the end of this large copper wire is soldered the supporting suspension, a flat copper strip similar to those used before. To the other end wire is soldered another strip, which is left loose and runs approximately parallel to the supporting suspension 0.3 cm away. The complete moving system is shown in Fig. 3. The fibre plug at the top fits into the case of the instrument as shown in Fig. 1. The moving system is easily removed, to be replaced by another of different constants.

The coil swings in a gap 0.6 cm long in the laminated steel core 2 by 0.8 cm in section. There are two exciting coils each wound with 650 turns of 0.72 mm (No. 21) enamelled copper wire, and the coils are arranged for either series or parallel connection.

TABLE 2
PERFORMANCE OF GALVANOMETER B

f	Excitation		C	T	R	S
	volts	ampere	μf	seconds	ohms	mm/ μv
~ / second						
0	2.6	0.4	0.00	1.7	25	2.3
0	3.9	.6	.00	1.4	60	1.2
60	29.0	.4	.00	1.9	30	2.6
60	29.0	.4	.22	4.2	100	5.7

The performance of this instrument under certain conditions of use can be seen from Table 2. The intrinsic constants were found to be as follows:

$$K = 0.006, D = 0.0009, U = 0.05 + 0.2 I^2, G = 115\,000 I, \\ r = 25 \times 10^9 \text{ and } c = 0.07.$$

Here I is the exciting current in amperes (coils in series). The restoring force is seen to depend upon the excitation showing that there are magnetic impurities in the moving system. On account of the thin insulation used the capacity between turns is large and the quantity c is the capacity in microfarads which, if connected to the terminals, would affect the period by an amount equal to the effect of the distributed capacity. The resistance of the two fixed coils in series is 6.4 ohms and the inductance is 0.18 henry. The insulation is such that 120 volts can be used for excitation.

4. GALVANOMETER C

The fixed system of this galvanometer is the same as that of Galvanometer B. The moving system consists of a coil of 75 turns of 0.08 mm (No. 40) single silk-covered copper wire wound to form a coil 2 cm long, 0.5 cm wide, and about 0.15 cm thick, and a mirror 0.8 cm in diameter. This coil was made without a form by winding the wire over two copper wires held parallel and about 2 cm apart. It is necessary to remove the twist in the wire before winding, otherwise the coil will warp when taken off of the winding device. The turns of wire were bound together by means of silk fibers. Small copper end pieces are attached to hold the mirror and to serve as connections between the suspensions. After these were attached the coil was treated as before to remove the magnetic dirt and was then washed, dried, and dipped in collodion. The coil is rigid enough for the purpose.

The performance of this instrument is very satisfactory, the capacity effect being small and the coil being very free from magnetic impurities.²⁰

In order to check the theory approximately the quantities K , D , U , and G were measured by the method already described. This was done for various values of the exciting current and fre-

²⁰ After first assembling this instrument its period with the field on was found to be much shorter than without the field, showing the coil to be magnetic. It was assumed that this was due to magnetic dirt picked up by the coil in the assembling. The coil was then treated again and replaced and the above results obtained. The total change in the restoring force when this coil is placed in a field of 4000 gauss is 0.02 dyne cm per radian, which is strikingly small. For Galvanometer B the change is 0.2 dyne cm per radian for the same field strength.

quency. The results are given in Table 3. It appears that the quantities K and U are real constants to the accuracy consistent with that of the determination of T_1 , T_2 , and λ_2 . Further, the damping D and the moment of displacement G are constant for a given excitation. G is roughly proportional to the excitation current as it should be. D increases with excitation in such a manner as to indicate the presence of a damping due to eddy currents.

TABLE 3

Results of the Determination of the Intrinsic Constants of Galvanometer C at Various Excitations and Frequencies

I ampere.....	0.2	0.5	1.0	0.2	0.5	1.0	0.2
$t \sim$ / seconds.....	0	0	0	60	60	60	300

OBSERVED QUANTITIES

T_1 seconds.....	1.65	1.55	1.51	1.69	1.68	1.65	1.68
λ_1	.05	.07	.12	.05	.06	.10	.05
T_2 seconds.....	1.70	1.65	1.65	1.73	1.67	1.65	2.30
λ_2	.28	.53	.74	.38	.68	.77	.44
$(r+R_2)$ ohms.....	115.	305.	715.	67.	214.	650.	88.
$(r+R_2)$ S mm / μ ampere.....	35.	74.	125.	33.	75.	135.	33.

COMPUTED QUANTITIES

K	0.01	0.009	0.009	0.01	0.012	0.011	0.011
D	.0011	.0016	.0029	.0011	.0016	.0027	.0012
U	.14	.15	.16	.14	.17	.16	.15
G	25 000	56 000	99 000	23 000	63 000	105 000	25 000

$r=17$ ohms.

5. GALVANOMETER D

This instrument is the best of several designed for use as detectors of alternating electromotive forces of frequencies up to 2000 cycles per second. It was thought that if the same good performance could be obtained at these frequencies, the range of usefulness of the electromagnet moving coil galvanometer would be considerably extended. A great many difficulties have been encountered and, although a reasonably satisfactory instrument has been obtained, the conditions for satisfactory performance are probably too special to allow of any great use.

One of the first difficulties met with when the electromagnets were excited with alternating currents of these frequencies was a continual vibration of the coil when it was closed on a circuit of fairly low impedance. This vibration sometimes was a con-

tinuous one of constant amplitude and sometimes of varying amplitude indicating a phenomenon similar to beats. It was soon noticed that this vibration about the axis was accompanied by a back and forth motion of the coil in the gap. The disturbance increased with frequency, excitation, and with diminution of the impedance in the moving coil circuit. It was decided that this disturbance was the result of the coil being in a nonuniform field so that the back and forth vibration caused a periodic change in the value of G , and therefore in the apparent restoring torque, such that if the frequency of this vibration was twice that of the rotational vibration the motion would be maintained. If the frequency

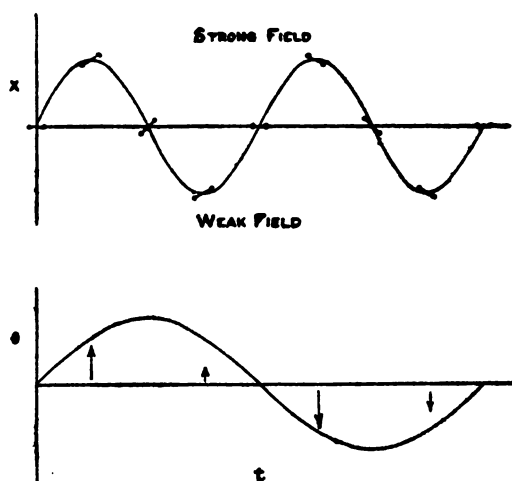


FIG. 4.—Diagram showing the vibration of a coil in a nonuniform alternating magnetic field

was different from this, beats occurred. A rough explanation of the phenomenon, which in detail is very complicated, can be seen by referring to Fig. 4, which represents the conditions during one complete rotational vibration of the coil. For simplicity the induced current is assumed to be that due to a condenser of capacity C . The resulting torque $G^2 \omega^2 C \theta$. (see p. 41) acts on the coil and as the coil vibrates in the nonuniform field G changes in value. In the figure x is the displacement of the coil from its equilibrium position. The arrows indicate the magnitude and the direction of the torque and it is seen to aid the vibration; that is, it causes energy to be given to the system so that the vibration may be maintained. If the circuit is inductive, the rotational vibration is maintained when the phase of x is shifted 180° so that as the coil deflects away from the reference position it is moving into a weaker instead of a stronger field.

In order to eliminate this disturbance the period of vibration of the coil back and forth in the gap was made small by the use of short suspensions. More care was also taken to make the magnetic field uniform. These precautions were found to be sufficient so that there is no serious vibration of the coil of Galvanometer D

at 2100 cycles per second even when the moving coil circuit is very inductive.

Referring to equation (14) we see that as the frequency is increased the term

$$\frac{\omega G^2}{Z} \sin \gamma$$

increases if γ is not zero so that there is a change in the total restoring torque and in the rest point. In order to obtain satisfactory performance this effect must be kept small so that it is necessary to keep γ small. At low frequencies the inductance can be compensated for by the use of a series resistance in parallel with a condenser as explained above. This compensation can be made to hold for a fairly large range of frequencies but at the higher frequencies such is not the case if the resistance is to be kept reasonably low. The frequency must therefore be maintained constant for satisfactory operation.

In the design of an alternating-current electromagnet for use as a frequency of 2000 cycles per second the losses due to hysteresis and eddy currents have to be considered. Since an air gap is necessary in which to swing the coil, there is considerable leakage so that the magnetic induction in the core may be several times the field strength required in the gap. It was found by the measurements of the apparent resistance and inductance²¹ of iron-cored coils that the losses could be computed with sufficient certainty from data obtained at low frequencies.

A rectangular core 5.9 by 8.0 cm outside with a cross section of 1.5 by 1.7 cm was used. This was built up of laminations of silicon steel 0.18 mm thick, the laminations being insulated from each other by thin paper. An exciting coil of 500 turns of 0.64 mm (No. 22) copper wire is wound on one of the long sides of the rectangle. In the middle of the other side are cut two gaps 0.18 cm long and 0.75 cm apart in which the coil swings.

The moving coil of this galvanometer is rectangular in shape of mean area 0.8 by 2.5 cm. It is wound with $71\frac{1}{2}$ turns of 0.08 mm (No. 40) single silk-covered copper wire. The wire is wound on a copper and mica frame made up as shown in Fig. 5. The copper pieces are covered with silk and after the coil is wound the

²¹ These were measured by means of a Wheatstone bridge, three arms containing simply resistance and the fourth containing the test coil in series with a condenser. A telephone receiver was used as a detector and by adjustment of the resistances and the capacity very good balances could be obtained for various currents and frequencies. From the resistances, capacity, and frequency the apparent resistance and inductance could be computed. Then knowing the dimensions, number of turns, etc., the loss for various inductions could be computed.

small copper extensions are bent down and hold the wire in place. The mirror is attached as shown. The suspensions of this instrument are about 1 cm long and are of the flat copper strip used before. The lower suspension is soldered to a small flat copper spring so that the suspensions can be held tight.

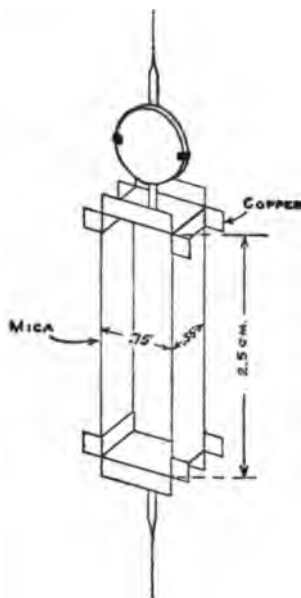


FIG. 5.—Diagram showing construction of moving system of galvanometer D

Table 4 shows the performance of this instrument under various conditions of use. W is the power required for excitation, C is the capacity of the parallel condenser used in controlling the period, and S_1 is the current sensitivity. With a current of 0.5 ampere the motion is aperiodic when the external resistance is 16 ohms. At the low frequencies such excitation is available but at the high frequencies this represents considerable power and was not obtainable from the generators available. The performance with the excitation obtained was satisfactory however. The sensitivity is high and the motion is not far from being aperiodic. The air damping is larger in this instrument than in the others on account of the larger coil

and closer proximity of the pole faces. This accounts for the reasonably satisfactory damping at the high frequencies where the electromagnetic damping is not large.

TABLE 4

Performance of Galvanometer D

f	I	W	C	T	R'	S	S_1
~/second	ampere	watts	μf	seconds	ohms	mm/ μv	mm/ μ ampere
0	0.490	1.0	0.00	2.4	16	1.10	38.
430	.260	3.4	.00	3.3	23	.90	38.
430	.260	3.4	.10	2.5	76	.23	22.
2100	.068	1.2	1.00	1.7	11	.09	2.7
2100	.066	1.1	1.00	2.4	22.5	.12	5.
2100	.063	.9	.25	2.4	42.5	.06	4.9
2100	.063	.9	.05	3.1	100	.07	8.3
2100	.151	6.8	1.00	2.2	23.5	.23	9.8
2100	.198	12.6	1.00	2.4	23.5	.37	16.
2100	.210	14.2	.05	1.5	94	.058	6.6
2100	.210	14.2	.05	4.2	100	.43	51.

The intrinsic constants of this instrument are

$$K = 0.075, D = 0.03, U = 0.6, G = 200\,000 I \text{ and } r = 19 \times 10^9.$$

The direct-current resistance of the exciting coil is 4 ohms. The apparent resistance to alternating current increases with current and frequency approaching the value 320 ohms at 2100 cycles per second and 50 ohms at 430 cycles per second for currents above 0.1 ampere. For currents less than this the resistance seems to approach the direct-current value as the current approaches zero. The apparent inductance increases from the value 0.06 henry for small currents to the value of 0.1 henry for currents above 0.1 ampere. The apparent inductance is independent of frequency indicating that the eddy currents are not large.

IV. USES

1. DETECTOR

In general, the purpose of a detector in null methods of electrical measurement is to show when the current or the electromotive force in a certain circuit is zero or does not change when certain changes are made in the testing current or voltage. Such conditions are brought about by an adjustment of some part or parts of the measuring apparatus. For example, in the measurement with a Wheatstone bridge using alternating current the inductance and resistance in one arm of the bridge might be adjusted to a balance; that is, until the detector shows that the current in the galvanometer circuit is independent of the testing current. The purpose of a detector can therefore be considered to be measure roughly the difference of potential between two terminals of apparatus having a certain impedance. Most of the measurements are of this kind, but there are some in which the current flowing in a circuit of high impedance (leakage) is to be measured or detected. In any case besides the sensitivity the user is interested usually in the time required for an observation. A reasonably short period and reasonably deadbeat motion is required.

It is seen from the preceding discussion that the performance of the electromagnet moving coil galvanometer as a detector is similar to that of the direct-current moving coil galvanometer. If one is dealing with sinusoidal current or electromotive force to be detected and noninductive circuits, the only difference is due to the fact that the small electromotive force or current may be

out of phase with the magnetic field. Since the deflection is proportional to the component of the electromotive force in phase with the magnetic field, zero deflection means only that the electromotive force, if it is detectable, is in quadrature with the magnetic field. In order, therefore, to be sure that the electromotive force is zero, the phase of the excitation is changed. With noninductive circuits the phase of the excitation can be adjusted once for all so that the maximum sensitivity is obtained. Under these conditions the magnetic field is in phase with the unbalanced electromotive forces which are always in the same phase. However, in this case, it is only necessary that the magnetic field be approximately in phase with the testing current, for since the phase of the unbalanced electromotive forces is always the same as that of the testing current, only its magnitude can be changed and hence zero deflection means zero electromotive force..

Usually in alternating-current measurements we have to do with apparatus having both resistance and reactance so that the unbalanced electromotive force may be of any phase whatsoever. There are usually two adjustments, the "resistance" and the "inductance" adjustments. A small change in one results in a change in the unbalanced electromotive force with a phase approximately in quadrature with the change produced by the other adjustment. Now if the phase of the excitation be such that a large change in one adjustment produces no change in deflection, it is possible by shifting the phase of the excitation 90 electrical time degrees to make the one adjustment independent of the other. When such an adjustment of the phase of the magnetic field is not made the balance must be obtained by successive approximations. This is illustrated in Fig. 6. Here ϕ_1 and ϕ_2 are the two phase positions of the magnetic field, R and L are the phases of the resistance and inductance adjustments, respectively, and E is the small electromotive force to be reduced to zero. Suppose to start with, the excitation is at ϕ_1 we will now make some adjustment which reduces the deflection to zero; that is, makes E in quadrature with ϕ_1 . To do this we may make either a resistance or an inductance adjustment. Suppose, first, we make an inductance adjustment which carries the end of E to a . Now shift the excitation to ϕ_2 and make a resistance adjustment which brings us to b , and so on. After a number of adjustments the deflection will be reduced to a negligible amount for both

phase positions of the excitation. But suppose a resistance adjustment to be made first. On following this out it is seen that the electromotive force to be made zero increases. The advantage of having the adjustments independent is shown by comparing Fig. 6 with Fig. 7.

It is obvious that either the phase of the magnetic field or of the testing current can be changed. Various phase-shifting devices can be used. A phase shifter of the rotating-field type is very convenient since any phase relation is obtainable which allows the adjustments to be made absolutely independent irrespective of the phase of the testing current. Any polyphase supply furnishes a definite shift of phase so that adjustment to zero is possible. Sometimes when the testing current is small it can be shifted by means of suitable inductance coils or condensers.

The precautions necessary in the use of an electromagnet moving coil galvanometer as a detector have been discussed with reference to the disturbing torques. The most important precaution is the maintaining of the coil and the metallic shield at the same potential. At the higher frequencies this is particularly true. Fig. 8 shows a good scheme of connections for use when the galvanometer is used as a detector for a Wheatstone bridge measurement. The metallic shield around the moving coil is electrically

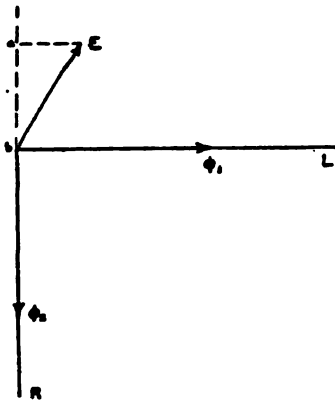


FIG. 7.—Vector diagram showing adjustments with quadrature excitation. Adjustments independent

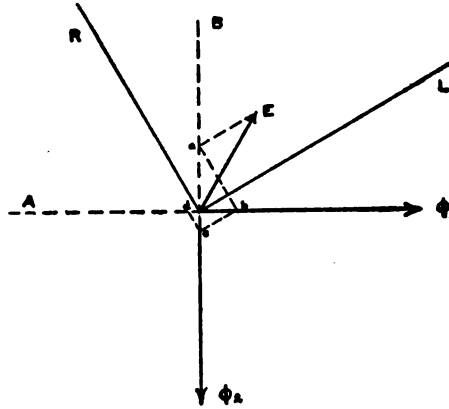


FIG. 6.—Vector diagram showing adjustments with quadrature excitation. Adjustments not independent

connected to a metal tube through which the leads are carried to the control apparatus. This apparatus, consisting of a parallel and a series resistance, a condenser, the mutual inductance, and

a reversing switch, is surrounded by a metallic shield (wire netting) which is connected to the tube from the galvanometer and to the divided resistance as shown. By means of the divided resistance the shield is maintained at a potential near that of the galvanometer branch points of the bridge. The reversing switch in the galvanometer circuit allows the main test current to be left

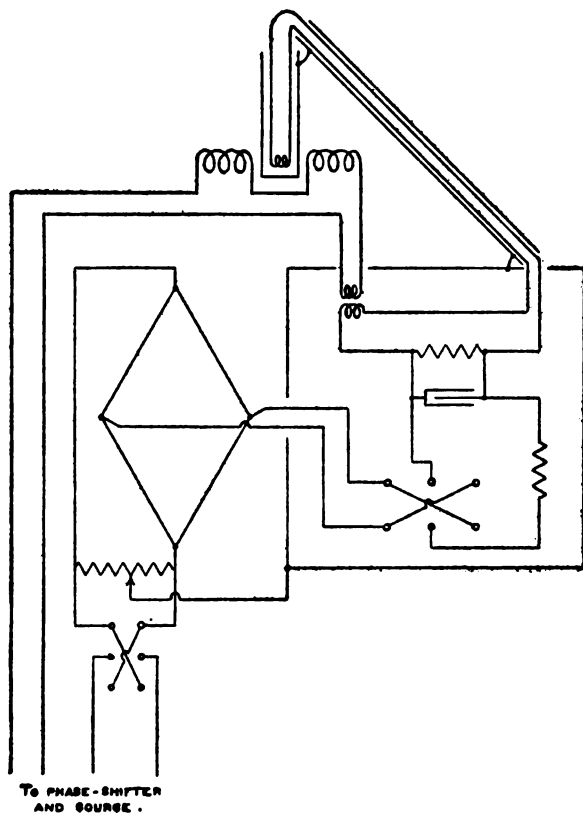


FIG. 8.—Diagram of connections with galvanometer used as a detector in alternating current bridge measurements

constant during the balance. If on reversal the rest point of the galvanometer does not change, the difference of potential between the leads from the bridge is zero. Usually the opening of the moving coil circuit causes a deflection on account of the change in the induced current, as pointed out above, so that this scheme is feasible only if the mutual inductance is adjusted to make $\theta_0 = 0$ or if the impedance of the bridge is high compared with resistance shunting the galvanometer. Reversal of the current

is the most desirable method of testing for a balance, for then the impedance of the moving coil circuit remains constant and a permanent change in deflection can result only from an electromotive force which reverses when the test current is reversed. In case the balance depends upon the frequency, reversal may cause a sudden motion of the coil due to the differences in the rates of decay and growth of the currents furnishing the balancing electromotive forces. Care is also necessary if stray electric or magnetic fields are present.

In some measurements it is desirable that the detector be sensitive to currents of the fundamental frequency only. This is the case when sinusoidal supply is not available or when harmonics are introduced by the apparatus so that the comparatively large harmonic electromotive forces must not cause deflections which will mask the deflection produced by the fundamental

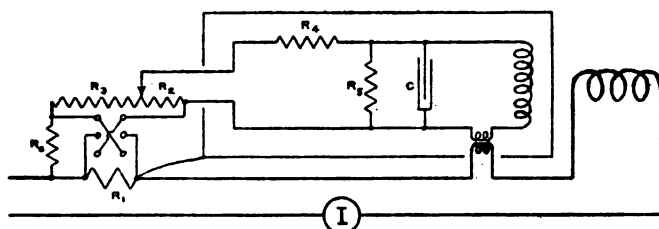


FIG. 9.—Diagram of connections used in calibration of galvanometer

unbalanced electromotive force. The use of electrical resonating systems in the circuits has been discussed above (pp. 34 and 43). These, together with care as to excitation, will usually be sufficient to give ample selectivity. Constancy of the frequency is essential.

2. OTHER USES

The electromagnet moving coil galvanometer is inherently an instrument of the galvanometer type inasmuch as it is not feasible to know with precision the constants depending upon the separate excitation, which data would be needed if the current in the moving coil were to be measured accurately. For approximate measurements the instrument can be calibrated by using connections as shown in Fig. 9. A small electromotive force approximately in phase with the magnetic field is obtained from the drop in potential across the low resistance r_1 by using the divided resistances r_2 and r_3 . The resistances must be sufficiently noninductive at the frequency used. With C , r_4 , r_6 , and r_8 adjusted to the values to

be used in the measurement the change in the deflection on reversing the electromotive force is noted. A simple calculation then gives the calibration factor. The resistance r_2 is very high in comparison with r_1 and is used to maintain the resistances r_1 and r_2 and hence the moving coil circuit at the potential of the shield when the switch is open. This is very necessary at high frequencies on account of the capacity current which would flow through the moving coil due to the capacity between the resistances r_1 and r_2 and the surroundings. If the apparatus containing the small electromotive force to be measured is available it should be inserted so that the moving coil circuit will be the same during the calibration as during the measurement.

The electromagnet moving coil galvanometer can be used to measure inductance in terms of capacity and vice versa by a resonance method in which the inductance and capacity are inserted in series in the moving coil circuit. For this measurement the moving coil is displaced from its position of zero induced electromotive force so that $\theta_0 = 0$. Now suppose that either the inductance or the capacity be adjusted until there is no deflection on closing the circuit. Then if the excitation be sinusoidal we must have from (11)

$$\frac{\sin \gamma}{Z} = 0 \quad (60)$$

which means that

$$L\omega = \frac{1}{\omega C}. \quad (61)$$

The sensitivity is determined by θ_0 , G , and the magnitude of the series resistance R' and from (11) can be shown to be determined by the equation

$$\theta = \theta_0 \frac{\omega G^2}{(R')^2 U} \left(L\omega - \frac{1}{\omega C} \right) = \theta_0 \frac{\omega G^2}{(R')^2 U} X. \quad (62)$$

By referring to the data on the instruments it is seen that a difference corresponding to 1 microhenry can be detected at 60 cycles per second. Therefore we see that a small reactance can be measured by itself, and equation (62) shows that the deflection is proportional to the reactance and is in one direction if it is inductive and is in the other if it is condensive reactance.

This study is part of a general investigation of galvanometers being carried on by Dr. Frank Wenner and the author, and the experience gained in the study of sensitive permanent magnet

moving coil galvanometers for direct current has been especially valuable. I am much indebted to Dr. Wenner for his hearty cooperation in this work.

V. SUMMARY

(1) The electromagnet moving coil galvanometer has been studied and the equation of motion obtained, from which its performance under various conditions can be determined. The performance as a detector is very similar to that of the direct-current moving coil galvanometer in that the motion of the coil may, in general, be of any degree of damping, depending upon the external circuit and in that the deflection is proportional to the impressed electromotive force. In the electromagnet moving coil galvanometer the period, as well as the damping, depends upon the constants of the external circuit being shortened by inductance and lengthened by capacity. Further, the deflection produced depends upon the component of the electromotive force in phase with the excitation.

(2) The relations between the operating and the intrinsic constants are given, so that the design and construction of instruments for any particular purpose are facilitated.

(3) A method of determining the intrinsic constants of electromagnet moving coil galvanometers has been devised. This gives from observations on the performance, under easily obtained working conditions, the moment of inertia, moment of damping, moment of restoration, etc., so that the performance of the instrument under other conditions can be predicted.

(4) An analysis of the disturbances has been made and it has been shown that the best procedure in the use of a sensitive electromagnet moving coil galvanometer is to surround the moving coil and moving coil circuit up to the testing apparatus by a metallic shield, which is maintained at a potential equal to that of the moving coil circuit. This can be done by connecting the shield to a point on the testing circuit having the same potential as the moving coil.

(5) The description and performance of four sensitive instruments are given and show results as expected from the theory. The sensitivity of these instruments is much greater than that of the telephone receiver at the lower frequencies. It is greater than that of the vibration galvanometer and is equal to that of the best direct-current moving coil galvanometers. The instru-

ments have been found to give very satisfactory performance at low frequencies. One instrument has been used at the frequency of 2100 cycles per second and, although very sensitive, many precautions have to be taken for satisfactory operation.

WASHINGTON, April 4, 1916.

STANDARD SUBSTANCES FOR THE CALIBRATION OF VISCOMETERS

By Eugene C. Bingham and Richard F. Jackson

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I. INTRODUCTION

In making measurements of viscosity in absolute units it is very desirable to have several substances available whose viscosities are accurately known in order that the accuracy of the method of measurement may be judged. Instruments whose results are expressed in terms of merely arbitrary numbers do not possess any advantage in this respect, since it is still necessary that the numbers obtained by two different instruments of the same type should agree, and in attaining this end the use of two or more substances of known viscosity is obviously of advantage.

Water is naturally the most important substance for this purpose since it can be so easily obtained in a pure condition and its viscosities at different temperatures have been very carefully

determined. But water is ill-suited for the calibration of the short-capillary technical viscometer, since water is so very much more fluid than most oils for which these instruments were intended. In spite of the oft-repeated statement to the contrary, the viscosities of two substances are by no means directly proportional to the times of flow of equal volumes through a given capillary under the same head. The chief cause of this lack of proportionality is the fact that the energy is not all expended in overcoming viscous resistance, a part of it being used up in imparting kinetic energy to the fluid entering the capillary. Thus in the flow of water through an Engler instrument only about 10 per cent¹ of the total energy expended is used in overcoming viscous resistance, the remaining 90 per cent being used in imparting kinetic energy to the liquid. The presence of so large a kinetic energy correction renders it manifestly desirable to have at hand some substance of high viscosity which can easily be obtained in a pure condition and whose viscosity is accurately known.

Castor oil and olive oil have been studied, but it has not been determined to what extent the viscosity may vary with the conditions of manufacture and exposure to light and air.

No pure hydrocarbon is readily available which possesses sufficiently high molecular weight. Monoacid alcohols of high molecular weight, like amyl alcohol, are not cheaply and easily obtained in the pure and anhydrous condition. A mixture of 45 per cent by volume of ethyl alcohol and water has a viscosity which is almost exactly four times that of water at 0° C. Since the viscosity of ethyl alcohol-water mixtures passes through a maximum for this concentration, the viscosity does not change rapidly with the concentration, which is a marked advantage. The viscosities of alcohol and water mixtures have been determined with care by several observers.

When more viscous substances are desired, the poly-acid alcohols are available, and of these glycerol is perhaps the best for the purpose. It is, however, hygroscopic and not readily obtainable in the pure anhydrous condition so that the preparation of a solution of predetermined concentration offers some difficulty.

The sugars are valuable substances for the purpose. They are not hygroscopic; they are crystallizable so that they may easily be obtained in a very pure condition. The concentration of sucrose

¹ Obtained by substituting the dimensions given in Ubbelohde's Tabellen zum Englerschen Viskosimeter, p. 24, in our equation (2) assuming the viscosity of water at 20° to be 0.01005 and the time of flow to be 51 seconds.

solutions may be determined by direct weighing of the constituents or from the density of the solution, or by means of the polariscope. Sucrose is very soluble in water, so that its solutions offer a wide range of viscosities. There is no concentration of sucrose in water which possesses the advantage of the alcoholic solution noted above, whose viscosity is independent of the exact concentration; hence it is necessary to guard the sucrose solutions against evaporation. Fortunately the technique of handling sucrose solutions has already been carefully worked out.³

The viscosity of sucrose solutions has repeatedly been the object of study, but the recent discovery⁴ of important sources of error in viscosity measurement which have hitherto remained undetected makes it desirable that these solutions be the object of still further research. Fortunately the data for water are sufficiently complete so that the more important corrections thus far recognized can all be made with sufficient certainty for our present purposes.

II. SUCROSE SOLUTIONS

1. PURIFICATION OF SUCROSE

The sucrose used in preparing the solutions was purified by crystallization from aqueous solution in the manner previously described by Bates and Jackson.⁴ Their procedure in outline was as follows: The material, a quantity of good granulated cane sugar of commerce, was dissolved in an equal weight of distilled water, clarified with a small quantity of washed "alumina cream," filtered and boiled in vacuo at a temperature of about 35° C until a concentration of 80 per cent was reached. The supersaturated sirup was seeded with a few crystals of sucrose and allowed to crystallize while in continuous motion. The crystals were separated from the mother liquor in a powerful centrifugal machine and were washed with aqueous alcohol. The crystallization was repeated until no evidence of impurity could be obtained.

The progress of the purification was studied and is described in the paper referred to. They found that sucrose prepared by this method contained less than 0.002 per cent of ash. The reducing substances, aside from sucrose itself, were of the order of 0.001 per cent if present at all. The optical rotary power of the material remained unchanged after fractional crystallization from aqueous

³ This Bulletin, 10, p. 537; 1914.

⁴ J. Am. Chem. Soc., 38, p. 27; 1916.

⁴ This Bulletin, 18, p. 75, 1916; Scientific Paper No. 268.

solution and after precipitation with ethyl and methyl alcohol. The specific rotation of the substance in the concentration of the normal⁵ solution was found for $\lambda = 5892.5 \text{ \AA}$ to be $66^\circ 529$, or slightly higher than the mean of the measurements of Tollens and of Nasini and Villavecchia who found for it $66^\circ 502$.⁶ The sugar used in the present investigation was prepared from the same source and possessed the same rotary power as that prepared by Bates and Jackson.

2. PREPARATION OF SOLUTIONS

In preparing the solutions for the viscosity measurements the constituents were weighed into a flask and the sugar dissolved. The solution in general was not completely free from dust particles. The amount of dust was too small to be weighable, but by accumulation in the capillary of the viscometer could readily have affected the time of flow. The solution was consequently poured on filters of hardened filter paper and repeatedly poured back to remove shreds acquired from the paper. The clear solution was finally poured through a funnel into a calibrated volumetric flask.

Three measurements of the concentrations of the solutions were made, two of which depended upon the solution density and one upon the rotary power.

The volumetric flasks which were used possessed graduated necks about 6 mm inside diameter. The graduations were 10 in number 0.02 ml apart. The interval between successive marks could be estimated to one-tenth of one division. The original solution was poured into the flask to some point on the scale, and flask and solution immersed in the water of a thermostat. When sufficient time had elapsed for the solution to assume the temperature of $20^\circ 00$, its volume was observed. From these data the density of the solution was calculated. Then by comparison with the tables of the Kaiserliche Normal Eichungs Kommission, the per cent composition of the solution was obtained.

The second measurement, made after the sample for viscometer measurement was taken, consisted of determining the density of the approximately normal solution taken for polarization. A portion of the original solution was poured into another weighed volumetric flask, and flask and solution weighed. Enough of the solution was taken to correspond to about 26 g of sucrose in 100

⁵ The normal concentration is 26 g in 100 ml of solution in accordance with the usage in sugar analysis.

⁶ This Bulletin, 18, p. 125; 1916.

ml. The sides of the flask were rinsed down into the solution and the latter diluted nearly to the capacity of the flask. The temperature was adjusted and the flask filled to the mark and weighed.

The third measurement consisted of a polarization of the approximately normal solution on a quartz-wedge saccharimeter. The thoroughly mixed solution was poured into polariscope tubes of known length and polarized at a fixed temperature. The reading of the saccharimeter was controlled by comparison with quartz plates No. 1 and No. 3, which are the primary standards of this Bureau referred to and described in the paper by Bates and Jackson. In determining the sugar value of these plates the conversion factor $34^{\circ}620$ was used instead of the erroneous $34^{\circ}657$.

In addition to the solutions prepared from purified sucrose one was prepared from a quantity of good granulated sugar of commerce in order to determine whether or not this generally available material would be suitable for standardization of instruments of ordinary precision. The substance after filtration contained but 0.012 per cent of ash. Its concentration was determined from the density of the solution by the second method described above, from its polarization, and from the reading of a standardized hydrometer. Since the effect of the impurities upon the fluidity was problematical, no attempt was made to apply corrections for them. The mean value of the three determinations was taken for the concentration. It is apparent that a calibrated hydrometer gives a satisfactory determination of the concentration.

The summary of the analyses is given in Table 1.

TABLE 1
Analytical Data on Sucrose Solutions

Solution	Percentage concentration by weight in vacuo				
	By density of original solution	By density of solution taken for polarization	By polarization	By standard hydrometer	Mean
No. 1.....	39.96	40.01	39.99		39.99
No. 2.....	40.00	39.97	39.99		39.99
No. 3.....	59.97	59.93	59.93		59.94
No. 4.....	20.003		20.011		20.007
No. 5 ^a		60.15	60.12	60.24	60.17

^a Granulated sugar of commerce.

III. MEASUREMENTS OF VISCOSITY

1. THE VISCOMETER

The viscometer used in this investigation is shown in Fig. 1, drawn to scale. The method of making a measurement was as follows: The clean and dry instrument was filled from *H* to *A* with the liquid to be measured, the surplus liquid overflowing into the trap at *A*. The liquid was introduced by means of a pipette drawn out into a fine tube. The left limb was connected with a tank filled with air under a pressure which could be measured on a water manometer, the right limb already having been connected with the air. The time which the meniscus required in falling from *B* to *D* was taken as the time of flow. The liquid was then in position for an observation of the time of flow in the opposite direction. When the temperature was raised, the volume was again adjusted by causing the surplus to run over into the trap.

2. VISCOSITY FORMULA

Knowing the time of flow, t , the pressure, p , and the two constants of the instrument, C and C' , it became possible to calculate the viscosity, η , of the liquid for the temperature of observation, using the formula

$$\eta = Cpt - C'\rho/t \quad (1)$$

The value of the density, ρ , does not need to be accurately known, since it appears only in the term which represents the kinetic energy correction, which in our experiments was purposely kept small in order that the slight uncertainties in regard to the value of the correction term might be rendered negligible.

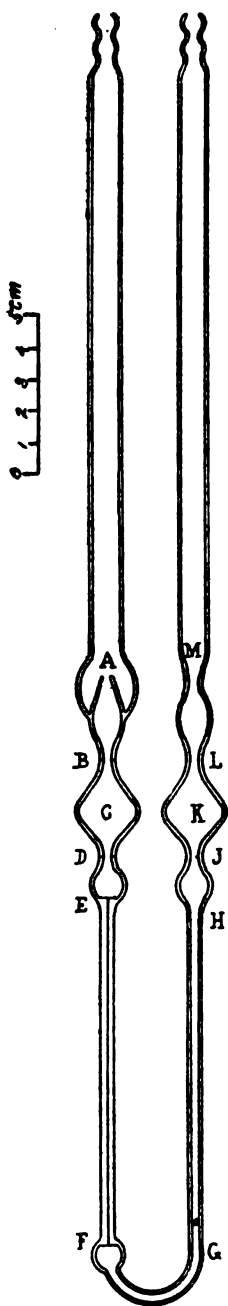


FIG. 1.—The viscometer.

The complete viscosity formula for the capillary tube method is

$$\eta = \frac{\pi g r^4 p t}{8 v (l + \lambda)} - \frac{m n \rho v}{8 \pi t (l + \lambda)} \quad (2)^7$$

where v is the volume of flow, r is the radius, and l the length of the capillary, λ is a correction to be made to the length on account of viscous resistance outside of the capillary and to the distortion of the stream lines just within the entrance to the capillary. According to all of the evidence at hand this correction is negligible when the capillary is very long in comparison with the radius of the tube. The number of capillaries in series is represented by n , while m is a constant whose value is being generally accepted to be about 1.12.⁷

3. CALCULATION OF CONSTANTS IN FORMULA

From the approximate dimensions $v = 4.00$ and $l = 7.50$ we can calculate the value of C'

$$C' = \frac{m n V}{8 \pi l} = 0.02376 \quad (3)$$

Observing the rate of flow of pure water at 20° C under a given pressure, and taking the absolute viscosity of water at this temperature to be 0.01005, we may calculate the value of C

$$C = \frac{\eta + C' \rho / t}{p t} \quad (4)$$

4. CALCULATION OF THE PRESSURE

The pressure used in the above formulas is expressed in grams per square centimeter. It is obtained as follows: If the height read on the manometer scale—corrected for scale error, if necessary—is h_0 , Fig. 2, and the density of the liquid in the manometer is ρ_0 , then the pressure is $h_0 \rho_0$. But this pressure is subject to several corrections which may be small but must be taken into consideration. (1) The correction for buoyancy of the air is $-h_0 \rho_{\text{air}}$. (2) The air in the closed limb of the manometer is under pressure and is therefore denser than the air outside. If the middle of the bulbs of the viscometer were at the level of the middle of the manometer

the correction for this cause would amount to $-\frac{h_0}{2} \rho_{\text{air}} \frac{h_0}{1033}$. (3)

⁷ Zs. physik. Chem., 80, p. 681; 1912.

If, however, the middle point of the manometer is at a distance h' below the middle of the viscometer, there is a further correction

$$-h'\rho_{\text{air}} \frac{h_0}{1033} \quad (4) \text{ There is a further correction for the hydrostatic}$$

head within the viscometer, arising from the fact that it is impossible to construct an instrument in which the two bulbs are of exactly the same shape and size or at the same height. If the

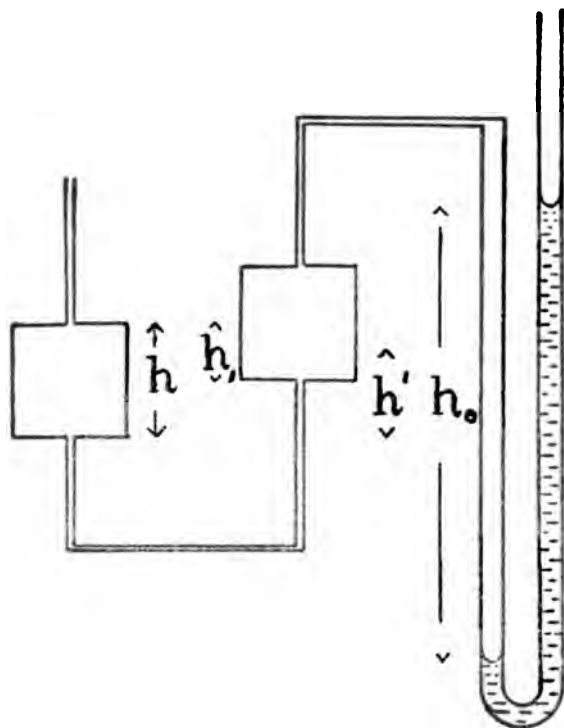


FIG. 2.—Diagram illustrating the method of estimating the pressure used in a viscosity determination.

hydrostatic head is h_1 , Fig. 2, obtained as will be described later, the pressure correction will be $h_1\rho$ and this may be either negative or positive, dependent upon the limb to which the pressure is being admitted; that is, whether the left limb is emptying or filling.

Making these corrections we have for the assumed constant pressure

$$p_0 = h_0\rho_0 - h_0\rho_{\text{air}} - \frac{h_0^2}{2066} \rho_{\text{air}} \pm h_1\rho_{\text{air}} \frac{h_0}{1033} \pm h_1\rho \quad (5)$$

We have calculated Tables 2 and 3 which simplify the use of the above formula, and cause the formula to take the form

$$p_o = h_o \pm h_1 \rho - K \pm L \quad (6)^*$$

In this formula h_o is the height in centimeters of the water column in the manometer, h_1 is the hydrostatic head, and ρ the density of the liquid within the viscometer, L is the correction for the difference of level between viscometer and manometer. This correction may usually be made negligible in the construction of the apparatus, but if necessary, the corrections corresponding to different values of h' and h_o may be obtained from Table 2. Table 3 contains the values of K , including the corrections for temperature, buoyancy, etc., for the different temperatures and pressures.

A single example will serve to show the method of using the tables. In our viscometer L was negligible, but the hydrostatic head was $h_1 = 0.2$ cm, the right bulb of the viscometer being higher than the left, so that, for a 40 per cent sugar solution at 20°C , $\rho = 1.176$, and at a pressure read on the manometer of $h_o = 269.5$ cm at 23°C , the correction is $(0.77 + 0.22 + 0.03) + 0.24 = 1.3$ cm[†] when the pressure is on the left limb, or $(0.77 + 0.22 + 0.03) - 0.24 = 0.8$ cm when the pressure is on the right limb of the viscometer.

TABLE 2
Values of L

h' in centimeters	h_o in centimeters		
	100	200	300
50.....	0.01	0.01	0.02
100.....	.01	.03	.04
200.....	.03	.05	.08
300.....	.04	.08	.11

(5) The applied pressure p_o is not necessarily the true average pressure to be used in the viscosity formula, hence a further correction may be necessary. Bingham, Schlesinger, and Coleman[‡], have shown that if the bulbs of the viscometer were cylindrical in shape and of the height h , the true average pressure $\bar{p}$, obtained by integration, would be

$$\bar{p} = \frac{0.8686 h p_o}{\log_{10} \frac{p_o + h \rho}{p_o - h \rho}} \quad (7)$$

* In obtaining (6) from (5) $K = h_o - h_o \rho_o + h_o \rho_{air} + \frac{h_o^2 \rho_{air}}{2066}$.

† The figures within the parenthesis are the interpolated value of L .

‡ J. Amer. Chem. Soc., 87, p. 27; 1916.

but the difference between p and p_0 becomes less than 0.05 per cent—that is, negligible for ordinary purposes—when the value of p_0 becomes as great as 30 times that of $h\rho$. They have shown how to obtain the value of h when this correction is not negligible and the bulbs of the viscometer are not true cylinders.

TABLE 3

Values of K

Temperature, °C	Manometer reading, h , mm											
	10	20	30	40	50	60	70	80	90	100	200	300
5.....	0.013	0.025	0.039	0.053	0.066	0.079	0.094	0.108	0.122	0.136	0.285	0.482
10.....	.016	.030	.046	.064	.078	.095	.112	.129	.145	.162	.337	.533
11.....	.017	.032	.050	.068	.083	.101	.119	.137	.154	.172	.357	.563
12.....	.018	.035	.053	.072	.089	.108	.126	.145	.163	.183	.379	.596
13.....	.019	.037	.057	.077	.095	.115	.135	.155	.175	.195	.403	.632
14.....	.020	.040	.061	.082	.102	.123	.144	.165	.187	.208	.429	.671
15.....	.022	.043	.065	.088	.110	.131	.154	.177	.199	.222	.457	.713
16.....	.023	.046	.070	.094	.118	.140	.165	.189	.212	.238	.489	.761
17.....	.025	.049	.075	.101	.126	.150	.176	.203	.228	.255	.523	.812
18.....	.027	.053	.080	.108	.135	.161	.189	.217	.245	.273	.559	.866
19.....	.029	.057	.086	.116	.144	.173	.203	.233	.262	.292	.597	.923
20.....	.031	.060	.092	.124	.154	.185	.217	.249	.280	.312	.637	.983
21.....	.033	.065	.098	.132	.165	.198	.232	.265	.299	.333	.679	1.046
22.....	.035	.069	.105	.141	.176	.211	.247	.282	.319	.355	.723	1.113
23.....	.037	.074	.112	.151	.188	.225	.264	.301	.341	.379	.770	1.184
24.....	.040	.079	.119	.160	.200	.240	.281	.321	.363	.403	.819	1.256
25.....	.042	.084	.127	.170	.212	.255	.298	.341	.385	.428	.869	1.331
26.....	.045	.089	.135	.181	.225	.270	.316	.362	.408	.454	.921	1.409
27.....	.048	.094	.143	.191	.239	.286	.335	.383	.432	.481	.975	1.490
28.....	.050	.100	.151	.202	.253	.303	.355	.405	.458	.509	1.031	1.574
29.....	.053	.105	.160	.214	.268	.321	.375	.429	.484	.538	1.089	1.661
30.....	.056	.111	.169	.226	.283	.339	.396	.453	.511	.568	1.149	1.751
31.....	.059	.117	.178	.239	.298	.357	.417	.478	.538	.599	1.210	1.842
32.....	.062	.124	.188	.251	.314	.376	.439	.503	.567	.630	1.273	1.937
33.....	.066	.130	.197	.264	.330	.395	.462	.529	.595	.662	1.337	2.033
34.....	.069	.137	.207	.277	.346	.415	.485	.555	.625	.695	1.403	2.132

To obtain the hydrostatic head h_1 , we determine the times of flow t_1 and t_2 for some substance, such as water at 20° C, for the right and left limbs of the instrument, respectively, with a given pressure, which is p_1 , corrected except for the hydrostatic head. We have the equations

$$p_1 + h_1\rho = \frac{\eta + C'\rho/t_1}{Ct_1}$$

and

$$p_1 - h_1\rho = \frac{\eta + C'\rho/t_2}{Ct_2}$$

hence

$$h_1 = \frac{\eta}{2C\rho} \left(\frac{1}{t_1} - \frac{1}{t_2} \right) + \frac{C'}{2C} \left(\frac{1}{t_1^3} - \frac{1}{t_2^3} \right). \quad (8)$$

In obtaining h_1 it is sufficient to use the approximate value of C , obtained by using p_1 in place of p in equation 4.

5. DETAILS IN REGARD TO INSTRUMENT AND MEASUREMENTS

The bulbs of the viscometer were made as short as practicable in order that the difference between the applied pressure and the true average pressure might be a minimum. The distance between the marks B and D was 3.0 cm.

The bulbs were made conical in shape in order to obtain the necessary volume while avoiding nearly horizontal surfaces which might cause faulty drainage. Drainage troubles were further obviated by having the part of the instrument directly above the point B in Fig. 1, similar in shape to the part above the point D . By always reading the volume of flow on the left limb, irrespective of whether the direction of flow is from left to right or vice versa, we measure the time of flow of the volume which the bulb C delivers in the former case but of the capacity of the bulb in the latter case. Thus any differences in the viscosity calculated from the times of filling and emptying of this bulb may serve as a test of the completeness of the drainage. A further test can of course be made by making observations of the time of flow at different pressures.¹⁰

The capillary tube was cut off squarely and sealed into the instrument so as to avoid a trumpet-shaped opening in order that there may be no doubt about the maximum value of the kinetic energy correction being applicable.

In order to prove that the corrections made are trustworthy, a series of observations were made on water at 25° C, using a considerable range of pressures. Table 4 shows that the calculated viscosity is satisfactorily constant.

¹⁰ By using an instrument similar to the one described in this Bulletin, 12, Scientific Paper No. 278, p. 309 (1916), all possibility of error due to bad drainage can be obviated, but in the present investigation that type of instrument is far less convenient than the one adopted and fortunately its use is unnecessary.

TABLE 4

Viscosity of Water at 25° C Calculated from the Constants Obtained from Observations on the Rate of Flow of Water at 20° C, Assuming the Viscosity at this Temperature to be 0.01005 ($C=0.00000014307$ and $C'=0.02376$)

Limb	t	p	P	η	Limb	t	p	P	η
	Secs.	g/cm ²	g/cm ²			Secs.	g/cm ²	g/cm ²	
R.....	570.9	110.04	109.53	0.008946	R.....	399.6	157.59	156.56	0.008951
L.....	572.9	109.65	109.14	.008946	L.....	400.2	157.22	156.19	.008943
R.....	570.6	110.04	109.53	.008942	R.....	306.5	205.43	203.67	.008931
L.....	572.9	109.65	109.14	.008946	L.....	307.6	205.04	203.28	.008946
R.....	571.3	110.04	109.53	.008952	R.....	306.5	205.44	203.68	.008932
L.....	572.4	109.65	109.14	.008938	L.....	308.1	205.05	203.29	.008961
R.....	819.2	76.76	76.51	.008968	R.....	274.8	229.64	227.45	.008943
L.....	821.2	76.37	76.12	.008944	L.....	276.8	228.08	225.92	.008947
R.....	818.2	76.81	76.56	.008963	R.....	274.4	229.48	227.29	.008923
L.....	820.6	76.42	76.17	.008943	L.....	274.8	229.58	227.39	.008940
R.....	399.4	157.56	156.53	.008944	R.....	273.4	230.45	228.24	.008928
L.....	400.6	157.19	156.16	.008950	L.....	275.1	229.72	227.53	.008955

In the above table, as well as in succeeding tables, we have given not merely the true average pressure p , but also the part of this pressure P , which is used up solely in overcoming viscous resistance. It is calculated by means of the formula

$$\eta = Cpt - C'p/t = CPt$$

whence we have that

$$P = p - \frac{C'p}{Cp} \quad (9)$$

The object in recording both p and P is in order to afford a measure of the kinetic energy correction.

The viscometer was attached to a brass frame which fitted in grooves on the side of the bath shown in Fig. 3, so that the viscometer was necessarily always in the same position. A part of the viscometer always projected above the bath, so that any vapor rising from the solution would tend to condense and run back. It may be remarked that the experiment was tried of repeating a measurement after heating a solution for several hours and then cooling, but without noteworthy effect.

The temperature was read by means of a telescope to one one-hundredth of a degree centigrade. As the thermometer had been calibrated at the Reichsanstalt, it was unnecessary to redetermine its corrections, but its ice point was determined before and after the investigation. At the two highest temperatures used a part of the stem of the thermometer was exposed, for which a correction was also made.

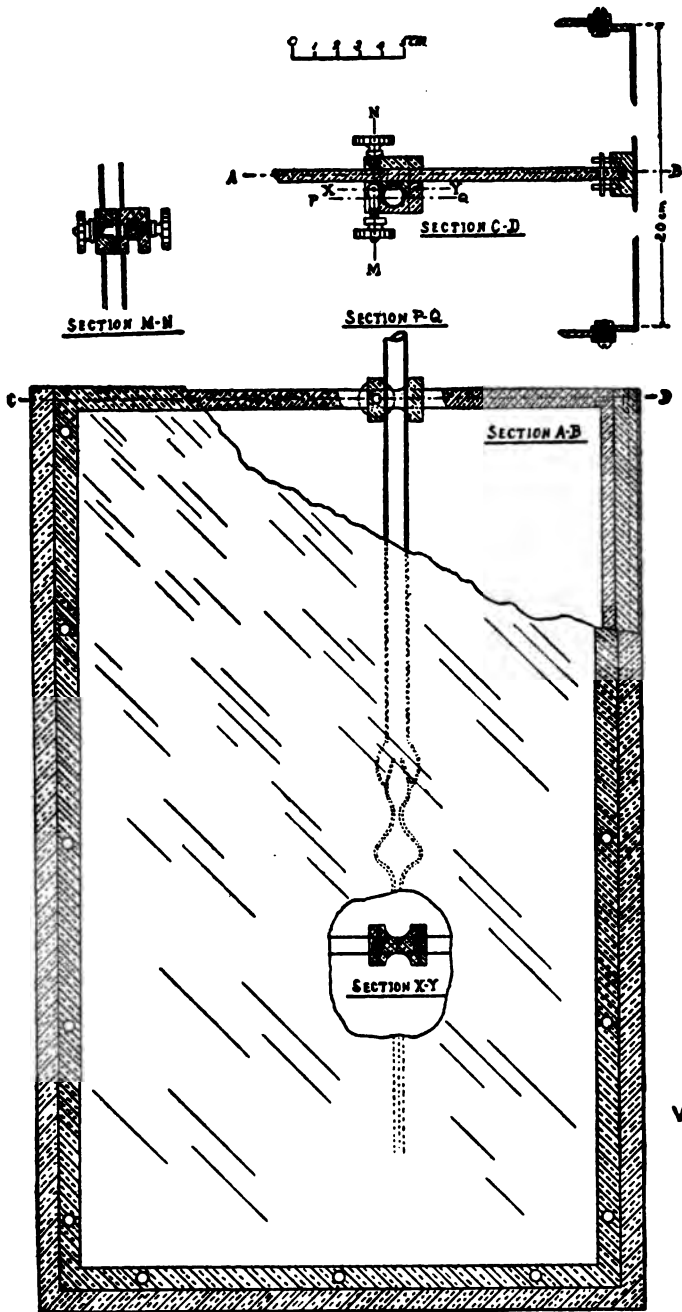


FIG. 3.—Detail of bath and frame with viscometer in position

The time was measured on an Agassiz stop watch which had been tested by this Bureau and given a rating of 56.2 in class A.¹¹ It was losing at the rate of a minute in a month during the time of the experiments.

IV. THE CENTIPOISE

It has been common practice to record viscosities in other than absolute units. There are at least three reasons for using specific viscosities rather than absolute viscosities. Absolute viscosities are often inconveniently small fractions, necessitating the use of many ciphers. We naturally compare the viscosity of any liquid with that of water, which is usually used as the standard, hence the specific viscosity makes an immediate impression upon the mind which the absolute viscosity does not until after considerable practice. Finally, the so-called specific viscosities are often arbitrary numbers which are not reducible, or at least not easily reducible, to absolute units. Thus in the use of most technical instruments such as those of Engler and Saybolt, the so-called viscosities are recorded in terms of arbitrary numbers known as Engler degrees or Saybolt seconds, respectively. These numbers are far from being proportional to the absolute viscosity, and the absolute viscosity is not in any way calculable without a supplementary determination of the density, the determination of which is often omitted.

It is quite evident that in studying the relation of viscosity to other properties it is the true absolute viscosity that is desired. This is the strongest possible argument in favor of giving up the use of purely arbitrary numbers and expressing all results in absolute measure. Moreover, whether the specific viscosities are reducible to absolute units or not it is awkward to make comparison between specific viscosities based upon different standards. Thus, when one worker uses water at 0° C as his standard it is not easy to compare his work with that of another worker who used as a standard water at 25° C, or either of these with results expressed in absolute units.

When two liquids having the same viscosity as measured on one instrument are measured in a different type of viscometer it has often been noted that the two liquids have quite different apparent specific viscosities. This arises from the fact that, in calculating the specific viscosity, important corrections are not taken into account which affect the two instruments differently. Thus, it is

¹¹ Circular No. 51, Bureau of Standards.

an objection to the use of specific viscosities that it has heretofore encouraged slovenly thinking in regard to the subject of viscosity measurement.

These views as to the relative merits of specific or absolute units are not as irreconcilable as may seem at first. It can probably be agreed that all viscosity measurements should be made under conditions such that the results can be expressed in absolute units. It is further desirable that, if specific viscosities be used, the same substance be taken as standard by all and that the absolute viscosity of the standard be definitely agreed upon, just as there is general agreement in the acceptance of atomic weights. If the suggestion of Deeley and Parr¹² is accepted, and the absolute cgs unit of viscosity be known as the "poise," then it is convenient to use the submultiple of this unit, which is one-hundredth as large, and which may therefore properly be called the centipoise (cp). It so happens that the centipoise is almost exactly the viscosity of water at 20° C, hence viscosities expressed as centipoises have the added advantage of being at the same time the specific viscosity of the substance referred to water as standard at almost exactly 20° C.

V. VISCOSITY AND FLUIDITY OF WATER

Previous determinations of the absolute viscosity of water and other substances have neglected to take into account the possible difference between the mean applied pressure and the true average pressure. It has therefore seemed desirable to go over the data available and make the correction where necessary and practicable.

Thorpe and Rodger¹³ calculated their viscosities on the assumption that $m = 1.00$ instead of the more generally accepted value¹⁴ $m = 1.12$. They also calculated the viscosity of water from the observations of Poiseuille,¹⁵ Sprung,¹⁶ and Slotte¹⁷ on the same assumption, hence it has seemed desirable to go over this data and make the needed correction. The error in the true average pressure and the error in the value of m are both in the same direction, both tending to make the substance appear to be more viscous than it really is. In a few instances the error amounts to as much as 0.3 per cent.

In arriving at the most probable values for the viscosity of water, it is important to observe that Poiseuille is usually credited

¹² Phil. Mag. [6], 26, p. 85; 1913.

¹³ Phil. Trans., 185A, p. 397; 1894.

¹⁴ See p. 65.

¹⁵ Mem. present. par divers Savants a l'Academie Roy. des Sciences de l'Inst. de France, 9, p. 433; 1846.

¹⁶ Pogg. Ann., 160, p. 1; 1876.

¹⁷ Wied. Ann., 20, p. 257; 1883.

with one series of observations from 0° to 45°, whereas he actually made observations over this range of temperatures with four different capillaries, and as there is no reason for supposing that his observations were any less accurate than were those of his successors, we have recalculated his data entire.

Hosking¹⁰ does not give sufficient data to permit a recalculation to be made, but as he adjusted the values of m and λ in such a way as to get concordant values of viscosities at different pressures, it seems proper to include his values as they are recorded by him.

TABLE 5
Viscosity of Water in Centipoises as Determined by Different Observers

Temperature, °C	Poiseuille				Sprung	Stolte	Therpe and Redger	Heaking	Bingham and White	Average	Calculated by formula
	A°	C	D°	E							
0.....	1.7755	1.7900		1.7944	1.777	1.807	1.7766	1.7928	1.7960	1.7887	1.7921
5.....	1.5108	1.5137	1.5143	1.5142	1.5089	1.523	1.5083	1.522	1.5241	1.5155	1.5188
10.....	1.3045	1.3078	1.3088	1.3088	1.2995	1.313	1.3014	1.3105	1.3082	1.3061	1.3077
15.....	1.1385	1.1464	1.1465	1.1456	1.1334	1.143	1.1324	1.142	1.1373	1.1406	1.1404
20.....	1.0028	1.0073	1.0063	1.0087	.9978	1.007	1.0005	1.006	1.0054	1.0046	1.0050
25.....	.8900	.8964	.8966	.8973	.8947	.895	.8900	.8926	.8940	.8941	.8937
30.....	.7958	.8016	.8011	.8027	.8183	.802	.7965	.800	.7991	.8019	.8007
35.....	.7154	.7194	.7190	.7207	.7216	.723	.7190	.724	.7223	.7205	.7225
40.....	.6466	.6523	.6508	.6531	.6558	.656	.6525	.657	.6557	.6533	.6560
45.....	.5867	.5934	.5937	.5932	.6001	.601	.5959	.600	.5984	.5958	.5988
50.....					.5512	.552	.5464	.5500	.5491	.5497	.5494
55.....						.509	.5044	.508	.5073	.5072	.5064
60.....						.471	.4676	.469	.4728	.4701	.4688
65.....						.437	.4343	.436	.4362	.4359	.4355
70.....						.407	.4048	.406	.4069	.4062	.4061
75.....						.380	.3782	.380	.3794	.3794	.3799
80.....						.356	.3547	.356	.3558	.3556	.3565
85.....						.334	.3336	.335	.3337	.3341	.3355
90.....						.315	.3140	.316	.3133	.3146	.3165
95.....						.297	.2970	.300	.2983	.2981	.2994
100.....						.281	.2814	.284		.2821	.2838

¹⁰ *Za. physik. Chem.*, 88, p. 647; 1913.

In next to the last column of Table 5 are given the averages of the values of the different observers. In the last column are given the values given by the formula

$$t = A(\phi + D) + C - \frac{B}{\phi + D} \quad (10)$$

which has been shown to be capable of expressing quite accurately the fluidity of liquids over a range of temperature, A , B , C , and D being arbitrary constants, and ϕ being the fluidity in cgs units. We

¹⁰ *Phil. Mag.* [6], 17, p. 502, 1909; 18, p. 260, 1909.

have taken $A = 0.23275$, $B = 8676.8$, $C = 8.435$, and $D = 120$. The calculated values are for the most part very close to the average observed values. This is particularly true between 5° and 80° . It seems probable, therefore, that in taking 1.005 cp as the viscosity of water at 20° C all of the figures are significant.

It is often desirable to know the viscosity of water at other than the 5-degree intervals given above, hence we have calculated the fluidity and viscosity of water for every degree between 0° and 100° , using equation (10) in the form

$$\phi = 2.1482 \left\{ (t - 8.435) + \sqrt{8078.4 + (t - 8.435)^2} \right\} - 120$$

TABLE 6

Fluidity and Viscosity of Water Calculated by Formula for Every Degree Between 0° and 100° C

Temperature, °C	Fluidity	Viscosity in cp	Temperature, °C	Fluidity	Viscosity in cp	Temperature, °C	Fluidity	Viscosity in cp
0.....	55.80	1.7921	33.....	132.93	0.7523	67.....	236.25	0.4233
1.....	57.76	1.7313	34.....	135.66	.7371	68.....	239.57	.4174
2.....	59.78	1.6728	35.....	138.40	.7225	69.....	242.91	.4117
3.....	61.76	1.6191	36.....	141.15	.7085	70.....	246.26	.4061
4.....	63.80	1.5674	37.....	143.95	.6947	71.....	249.63	.4006
5.....	65.84	1.5188	38.....	146.76	.6814	72.....	253.02	.3952
6.....	67.90	1.4728	39.....	149.60	.6685	73.....	256.42	.3900
7.....	70.01	1.4284	40.....	152.45	.6560	74.....	259.82	.3849
8.....	72.15	1.3860	41.....	155.30	.6439	75.....	263.25	.3799
9.....	74.28	1.3462	42.....	158.20	.6321	76.....	266.67	.3750
10.....	76.47	1.3077	43.....	161.11	.6207	77.....	270.12	.3702
11.....	78.66	1.2713	44.....	164.02	.6097	78.....	273.57	.3655
12.....	80.89	1.2363	45.....	167.00	.5988	79.....	277.04	.3610
13.....	83.14	1.2028	46.....	169.97	.5883	80.....	280.53	.3565
14.....	85.40	1.1709	47.....	172.95	.5782	81.....	284.03	.3521
15.....	87.69	1.1404	48.....	175.95	.5683	82.....	287.53	.3478
16.....	90.00	1.1111	49.....	178.95	.5588	83.....	291.03	.3436
17.....	92.35	1.0828	50.....	182.00	.5494	84.....	294.54	.3395
18.....	94.71	1.0559	51.....	185.05	.5404	85.....	298.06	.3355
19.....	97.10	1.0299	52.....	188.14	.5315	86.....	301.63	.3315
20.....	99.50	1.0050	53.....	191.23	.5229	87.....	305.21	.3276
20, 20.....	100.00	1.0000	54.....	194.34	.5146	88.....	308.78	.3239
21.....	101.94	.9810	55.....	197.45	.5064	89.....	312.35	.3202
22.....	104.40	.9579	56.....	200.62	.4985	90.....	315.92	.3165
23.....	106.86	.9358	57.....	203.78	.4907	91.....	319.53	.3130
24.....	109.38	.9142	58.....	206.95	.4832	92.....	323.13	.3095
25.....	111.91	.8937	59.....	210.13	.4759	93.....	326.74	.3060
26.....	114.45	.8737	60.....	213.33	.4688	94.....	330.38	.3027
27.....	117.03	.8545	61.....	216.54	.4618	95.....	334.01	.2994
28.....	119.62	.8360	62.....	219.80	.4550	96.....	337.65	.2962
29.....	122.25	.8180	63.....	223.07	.4483	97.....	341.30	.2930
30.....	124.89	.8007	64.....	226.34	.4418	98.....	344.96	.2899
31.....	127.54	.7840	65.....	229.64	.4355	99.....	348.63	.2868
32.....	130.22	.7679	66.....	232.94	.4293	100.....	352.30	.2838

VI. VISCOSITY AND FLUIDITY OF ETHYL ALCOHOL-WATER MIXTURES

It was stated above that ethyl alcohol-water mixtures possess certain advantages for the purpose of testing viscometers. They have already been used successfully for this purpose by Winslow H. Herschel, of this Bureau.

The fluidities of ethyl alcohol-water mixtures have already been determined by various observers, the data being brought together in a paper by Bingham, White, Thomas, and Cadwell.¹⁹ The older data is subject to some uncertainty on account of various circumstances, hence in getting the average some system of weighting seemed necessary. In obtaining the average values given in Tables 7 and 8 the data of Bingham and Thomas¹⁹ was given a weight of three, that of Noack²⁰ of two, and that of Traube,²¹ Pagliani, and Batelli,²² and Stephan²³ of one.

TABLE 7
Fluidity of Alcohol-Water Mixtures

Temperature, °C	Weight, percentage of ethyl alcohol												
	0	10	20	30	39	40	45	50	60	70	80	90	100
	Volume, percentage of ethyl alcohol at 25° C												
	0	12.36	24.09	35.23	44.92	45.83	50.94	55.93	65.56	74.80	83.59	92.01	100
0.....	55.8	30.2	18.8	14.4	13.8	14.0	14.4	15.2	17.4	21.0	27.1	36.6	56.4
5.....	65.8	38.8	24.6	18.9	17.8	17.9	18.2	19.0	21.6	25.6	32.0	43.3	61.6
10.....	76.5	45.9	31.6	24.7	22.8	22.8	23.0	23.9	26.5	30.6	36.9	47.6	68.2
15.....	87.7	55.8	38.2	30.7	28.4	28.3	28.5	29.1	31.8	36.1	43.3	55.5	75.1
20.....	99.5	65.0	45.8	36.9	34.7	34.4	34.7	34.8	37.4	42.2	49.8	62.1	83.3
25.....	111.9	75.6	55.1	45.9	42.5	42.5	41.9	41.7	44.6	49.1	57.2	70.2	91.2
30.....	124.9	86.2	64.4	53.4	50.0	49.4	49.5	49.6	51.9	56.6	65.3	78.2	99.7
35.....	138.4	99.4	75.1	63.3	58.6	58.3	57.7	58.0	60.1	65.4	73.8	87.2	109.4
40.....	152.4	110.2	86.2	73.1	67.9	67.5	66.9	66.7	69.1	74.4	83.1	96.6	119.9
45.....	167.0	123.2	98.5	84.1	77.9	77.6	76.5	77.3	78.7	84.1	92.5	106.5	130.8
50.....	182.0	136.3	110.2	95.2	89.0	88.3	87.1	86.6	88.7	94.2	103.3	117.9	142.5
55.....	197.4	150.9	122.9	107.6	100.7	100.2	98.4	98.0	100.3	106.0	115.3	130.8	155.2
60.....	213.3	164.3	135.8	119.9	113.0	112.0	110.3	109.5	110.8	116.8	126.7	142.1	168.9
65.....	229.6	180.5	150.1	133.0	125.3	124.7	122.6	122.3	124.1	130.6	140.7	156.0	181.5
70.....	246.3	194.5	164.5	146.4	138.0	137.5	135.2	135.1	137.2	143.9	153.9	169.9	198.6
75.....	263.2	210.2	178.8	160.3	151.5	150.8	148.9	148.7	150.8	157.1	166.6	183.0	212.5
80.....	280.5	232.7	198.1	176.4	167.1	166.5	164.1	163.5	165.7				

¹⁹ *Zs. physik. Chem.*, **88**, p. 644; 1913.

²⁰ *Wied. Ann.*, **27**, p. 289; 1886.

²¹ *Ber. d. deutsch. chem. Gesell.*, **19**, p. 871; 1886.

²² *Atti. d. R. Acc. di Torino*, **20**, p. 845; 1885.

²³ *Wien. Ber.*, **46** (2a), p. 495; 1862.

TABLE 8
Viscosity in Centipoises of Alcohol-Water Mixtures

Tem- perature, °C	Weight, percentage of ethyl alcohol												
	0	10	20	30	39	40	45	50	60	70	80	90	100
	Volume, percentage of ethyl alcohol at 25° C												
	0	12.36	24.09	35.23	44.92	45.83	50.94	55.93	65.56	74.80	83.59	92.01	100
0.....	1.792	3.311	5.319	6.94	7.25	7.14	6.94	6.58	5.75	4.762	3.690	2.732	1.773
5.....	1.519	2.577	4.065	5.29	5.62	5.59	5.50	5.26	4.63	3.906	3.125	2.309	1.623
10.....	1.308	2.179	3.165	4.05	4.39	4.39	4.35	4.18	3.77	3.268	2.710	2.101	1.466
15.....	1.140	1.792	2.618	3.26	3.52	3.53	3.51	3.44	3.14	2.770	2.309	1.802	1.332
20.....	1.005	1.538	2.183	2.71	2.88	2.91	2.88	2.87	2.67	2.370	2.008	1.610	1.200
25.....	.894	1.323	1.815	2.18	2.35	2.35	2.39	2.40	2.24	2.037	1.748	1.424	1.096
30.....	.801	1.160	1.553	1.87	2.00	2.02	2.02	2.02	1.93	1.767	1.531	1.279	1.003
35.....	.722	1.006	1.332	1.58	1.71	1.72	1.73	1.72	1.66	1.529	1.355	1.147	.914
40.....	.656	.907	1.160	1.368	1.473	1.482	1.495	1.499	1.447	1.344	1.203	1.035	.834
45.....	.599	.812	1.015	1.189	1.284	1.289	1.307	1.294	1.271	1.189	1.081	.939	.764
50.....	.549	.734	.907	1.050	1.124	1.132	1.148	1.155	1.127	1.062	.968	.848	.702
55.....	.507	.663	.814	.929	.993	.998	1.016	1.020	.997	.943	.867	.764	.644
60.....	.469	.609	.736	.834	.885	.893	.907	.913	.902	.856	.789	.704	.592
65.....	.436	.554	.666	.752	.798	.802	.816	.818	.806	.766	.711	.641	.551
70.....	.406	.514	.608	.683	.725	.727	.740	.740	.729	.695	.650	.589	.504
75.....	.380	.476	.559	.624	.660	.663	.672	.672	.663	.636	.600	.546	.471
80.....	.356	.430	.505	.567	.598	.601	.609	.612	.604				

VII. VISCOSITY OF SUCROSE SOLUTIONS

The first sucrose solution used was 39.99 per cent sucrose by weight in vacuo. The results obtained are given in Table 9 and plotted in Figs. 4 and 5. The first column shows whether the left or the right limb was emptying, the second column gives the corrected time of flow in seconds, the third column gives the corrected pressure, while the fourth column gives the pressure used up in overcoming the viscous resistance, and the fifth column gives the fluidity calculated for the measured temperature given in the sixth column. In the last column the temperatures²⁴ are calculated corresponding to these fluidities, using the formula

$$t = (\varphi + 20) 0.597 - \frac{1438.6}{\varphi + 20} + 38.24$$

The agreement between the observed and calculated values is good.

Were the fluidity concentration curves linear they would follow the dotted lines. That the observed curves depart so

²⁴ The temperatures are calculated instead of fluidities purely for the sake of convenience in the use of the formula.

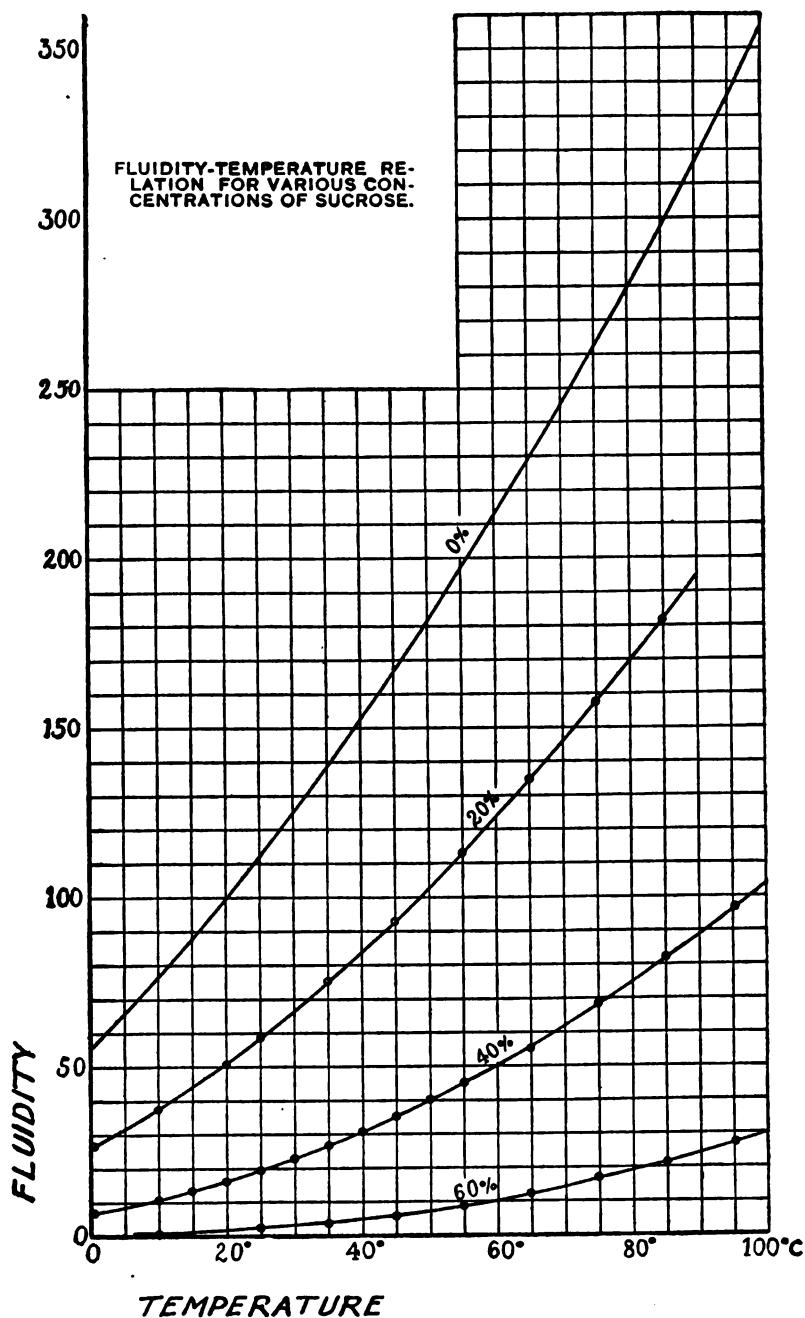


FIG. 4.—Showing the relation between fluidity and temperature for solutions of various sucrose content

widely from the linear is an indication of the chemical hydration of the sugar.

TABLE 9
Fluidity of a 39.99 Per Cent Sucrose Solution at Various Temperatures

Limb	Time, t	Pressure, p	Pressure, P	Fluidity, ν	Temperature measured	Temperature calculated
	Seconds				°C	°C
L.....	3,487.8	290.65	290.65	6.89	0.32	0.81
R.....	3,437.1	292.97	292.97	6.94	.46	.94
L.....	2,236.4	289.15	289.11	10.81	9.96	9.94
R.....	2,199.7	293.43	293.39	10.83	9.96	9.98
L.....	1,486.7	291.94	291.85	16.11	19.98	19.96
R.....	1,477.8	293.19	293.10	16.14	19.98	20.00
L.....	1,266.1	286.44	286.32	19.28	24.99	25.07
R.....	1,250.4	289.92	289.80	19.29	24.99	25.08
L.....	918.1	286.39	286.16	26.58	35.00	35.17
R.....	908.4	289.24	289.01	26.60	35.00	35.19
L.....	730.6	288.74	288.40	35.09	45.00	44.92
R.....	747.3	289.17	288.83	35.21	45.00	45.14
L.....	548.3	283.34	282.71	45.09	54.99	55.00
R.....	545.8	284.76	284.13	45.07	54.99	54.98
L.....	437.1	284.33	283.34	56.18	64.96	64.83
R.....	437.3	285.07	284.08	56.26	64.96	64.91
L.....	363.7	282.29	280.85	68.43	74.94	74.76
R.....	363.2	283.14	281.70	68.32	74.94	74.69
L.....	306.3	280.13	278.11	82.05	85.03	85.39
R.....	307.3	279.58	277.56	81.95	85.03	84.99
L.....	257.8	282.76	279.93	96.86	95.30	95.69
R.....	257.2	283.08	280.25	96.97	95.30	95.77

The values of the fluidity given above do not agree with the values obtained by other observers, as will be shown later, hence the viscometer was tested with pure water and another series of measurements were made with an entirely new solution, which, however, happened to have the same concentration as the former, viz, 39.99 per cent by weight.

TABLE 10
Fluidity of a Second 39.99 Per Cent Sucrose Solution at Various Temperatures

Limb	Time, t	Pressure, p	Pressure, P	Fluidity, ν	Temperature observed	Temperature calculated
	Seconds				°C	°C
L.....	2,024.4	257.97	257.92	13.39	14.97	15.10
L.....	1,256.4	288.16	288.04	19.31	24.99	25.12
R.....	1,232.4	293.15	293.02	19.36	24.99	25.18
L.....	1,045.3	293.24	293.04	22.82	30.00	30.10
R.....	1,039.8	294.43	294.27	22.84	30.00	30.24
L.....	770.6	295.00	294.65	30.78	40.00	40.23
R.....	768.4	295.41	295.10	30.82	40.00	40.27
L.....	594.0	294.30	293.73	40.06	50.00	50.14
R.....	593.1	294.63	294.10	40.07	50.00	50.15

The second sample gave values which correspond very well with the former values, hence we have additional reason for confidence in the reliability of our values. It should be noted in this connection that Bingham, Schlesinger, and A. B. Coleman,²⁵ using a viscometer of different construction, have already noted that the observations of Hosking on sugar solutions may be in error. This was confirmed by the measurements of C. Coleman.²⁶

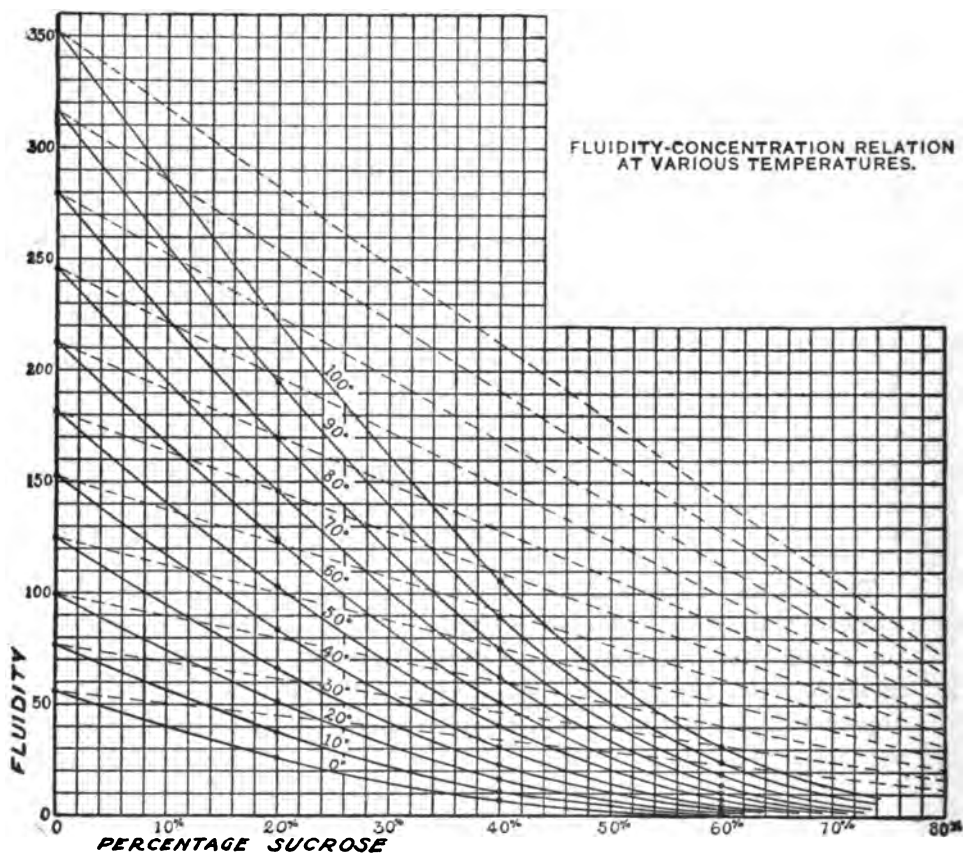


FIG. 5.—Showing relation between fluidity and sucrose content at various temperatures

Having found an appreciable difference between our own values and those of other observers, it seemed desirable to measure the fluidity of a 20 per cent solution, the concentration actually obtained being 20.007 per cent. The fluidities are given in Table II. The last observation recorded in the table was made several hours after the preceding observation, when the solution had cooled down from the high temperature. The fluidities are again considerably smaller than the values found by Hosking.

²⁵ Loc. cit.

TABLE 11
Fluidity of a 20.007 Per Cent Sucrose Solution at Various Temperatures

Limb	Time, t	Pressure, p	Pressure, P	Fluidity, φ	Temperature
	Seconds				°C
L.....	899.1	290.43	290.21	26.79	0.44
R.....	897.0	290.86	290.64	26.81	.44
L.....	643.4	289.26	288.83	37.61	9.96
R.....	644.2	288.77	288.34	37.63	9.96
L.....	414.7	288.42	287.38	58.65	24.99
R.....	415.5	287.77	286.78	58.67	24.99
L.....	263.4	288.91	286.35	75.10	35.00
R.....	261.4	288.89	286.29	75.09	35.00
L.....	324.5	288.52	286.82		45.00
R.....	324.3	288.71	287.01	98.40	45.00
L.....	217.8	287.93	284.20	112.92	54.99
R.....	215.8	289.86	286.06	113.22	54.99
L.....	201.3	262.51	258.17	124.50	64.96
R.....	200.5	262.86	258.48	124.89	64.96
L.....	213.1	212.40	208.55	157.28	74.94
R.....	212.4	212.84	208.96	157.48	74.94
L.....	248.4	157.68	154.86	181.72	85.03
R.....	247.4	158.13	155.29	181.93	85.03
L.....	1,783.8	76.92	76.86	50.98	19.98

Table 12 contains the results of measurements of the fluidity of a solution which contained 59.94 per cent sucrose by weight. This concentration has not been measured over a range of temperatures by previous observers.

TABLE 12
Fluidity of a 59.94 Per Cent Sucrose Solution at Various Temperatures

Limb	Time, t	Pressure, p	Pressure, P	Fluidity, φ	Temperature observed	Temperature calculated
	Seconds				°C	°C
L.....	26,023	295.87	295.87	0.908	9.96	11.61
R.....	10,310	296.61	296.61	2.286	24.99	24.99
L.....	6,222	295.76	295.75	3.798	35.00	34.84
R.....	4,029	295.50	295.49	5.871	45.00	44.90
L.....	2,736.6	295.89	295.86	8.633	54.99	54.99
R.....	1,937.2	296.22	296.17	12.058	64.96	64.79
L.....	1,392.0	295.67	295.56	[16.989]	74.94	76.30
R.....	1,111.3	295.65	295.48	21.286	85.03	85.03
L.....	870.6	295.84	295.57	27.162	95.30	95.92
L.....	1,508.4	289.45	289.34	16.015	74.94	74.18

On calculating out the above data it was seen that the measurement at 74.94° was manifestly in error, hence the last observation in the table was made on the same solution two days later, it

having remained at room temperature in the meantime. The temperatures in the last column were calculated by means of the formula

$$T = (\varphi + 5) 1.472 = \frac{323.2}{\varphi + 5} + 58.62$$

It was easy to keep the solution at a temperature which was constant within one-tenth of a degree, hence the formula does not serve to reproduce the observed values satisfactorily. We may add that in so viscous a solution the fluidity is greatly affected by the temperature, a rise of 1° at 45° causing an increase of 4.1 per cent in the fluidity.

A 60 per cent solution was now prepared from commercial sugar in order to learn whether any serious error would be made in using commercial sugar instead of specially prepared sugar. The solution proved to be 60.17 per cent sucrose. The fluidity is given in Table 13.

TABLE 13
Fluidity of a 60.17 Per Cent Solution of Commercial Sugar at 74.94° C

Limb	Time, t	Pressure, p	Pressure, P	Fluidity, φ	Temperature observed
	Seconds				°C
L.....	1,484.6	292.91	292.80	16.08	74.94

For convenience in comparison the observed fluidities were plotted to a large scale and interpolated for even concentrations and for every 5°. The resulting fluidities are given in Table 14 and the corresponding viscosities are given in Table 15.

TABLE 14
Fluidities of Sucrose Solutions Containing 0, 20, 40, and 60 Per Cent Sucrose by Weight at Every 5° C (Interpolated)

Temperature, °C	Percentage sucrose by weight				Temperature, °C	Percentage sucrose by weight			
	0*	20	40	60		0	20	40	60
0.....	55.91	26.29	6.77	0.42	55.....	197.16	113.12	45.06	8.57
5.....	65.99	31.71	8.65	.64	60.....	212.72	123.79	50.47	10.17
10.....	76.56	37.71	10.21	.91	65.....	229.41	134.81	56.24	11.99
15.....	87.67	44.11	13.39	1.34	70.....	246.18	145.97	62.17	13.98
20.....	99.54	51.02	16.13	1.77	75.....	263.57	157.56	68.41	16.12
25.....	111.84	58.69	19.28	2.28	80.....	281.21	169.53	74.96	18.51
30.....	124.70	66.51	22.82	2.96	85.....	299.31	181.80	81.92	21.14
35.....	138.79	75.12	26.58	3.77	90.....	317.87		89.06	24.07
40.....	153.07	83.82	30.78	4.70	95.....	335.46		96.41	26.85
45.....	167.84	93.42	35.13	5.82	100.....	354.49		104.11	29.96
50.....	181.92	103.07	40.05	7.14					

* These are the average fluidities calculated from Table 5.

TABLE 15

Viscosities in Centipoises of Sucrose Solutions Containing 0, 20, 40, and 60 Per Cent Sucrose by Weight at Every 5° C (Interpolated)

Temperature, °C	Percentage sucrose by weight				Temperature, °C	Percentage sucrose by weight			
	0 ^a	20	40	60		0	20	40	60
0.....	1.789	3.804	14.77	238	55.....	0.507	0.884	2.219	11.67
5.....	1.516	3.154	11.56	156	60.....	.470	.808	1.982	9.88
10.....	1.306	2.652	9.794	109.8	65.....	.436	.742	1.778	8.34
15.....	1.141	2.267	7.468	74.6	70.....	.406	.685	1.608	7.15
20.....	1.005	1.960	6.200	56.5	75.....	.379	.635	1.462	6.20
25.....	.894	1.704	5.187	43.86	80.....	.356	.590	1.334	5.40
30.....	.802	1.504	4.382	33.78	85.....	.334	.550	1.221	4.73
35.....	.720	1.331	3.762	26.52	90.....	.315		1.123	4.15
40.....	.653	1.193	3.249	21.28	95.....	.298		1.037	3.72
45.....	.596	1.070	2.847	17.18	100.....	.282		.960	3.34
50.....	.550	.970	2.497	14.01					

^a These are the average viscosities given in Table 5.

We reproduce here for the sake of comparison the viscosities of sugar solutions obtained by Hosking,²⁶ Table 16, and by Powell,²⁷ Table 17. Burkhardt,²⁸ Rudolf,²⁹ Grüneisen,³⁰ and Green³¹ have studied the viscosity of sugar solutions, but not over a range of temperatures. The viscosities determined by Hosking are generally considerably lower than our values, although it is to be noted that we agree with Hosking satisfactorily at the lowest temperature in the 40 per cent solution and at the highest temperature in the 20 per cent solution. The values of Powell, are, in general, intermediate between Hosking's values and our own, but agree better with the former. Bingham, Schlesinger, and Coleman obtained 1.731 cp for the viscosity of a 20 per cent sugar solution at 25° C using the Washburn viscometer. This is considerably higher than the present value.

²⁶ Phil. Mag. [5], 49, p. 274; 1900.

²⁷ Trans. (London) Chem. Soc., 106, p. 1; 1914.

²⁸ *Za. Rübenzuckerind.* 1874. Cf. Castell-Evans Physico-Chemical Tables, 2, p. 652; 1911.

²⁹ *Za. physik. Chem.*, 48, p. 281; 1903.

³⁰ *Wiss. Abh. d. Phys.-Tech. Reichsanstalt*, 4, p. 239; 1905.

³¹ Trans. (London) Chem. Soc., 98, p. 8023; 1906.

TABLE 16
Viscosity of Sucrose Solutions in Centipoises According to Hosking ²²

Temperature, ° C	Percentage sucrose by weight				
	1	5	10	20	40
0.....	1.810	2.048	2.436	3.720	14.76
5.....	1.537	1.729	2.050	3.042	11.33
10.....	1.331	1.488	1.754	2.578	8.95
15.....	1.168	1.292	1.518	2.212	7.30
20.....	1.031	1.139	1.328	1.910	6.07
25.....	.911	1.009	1.173	1.674	5.08
30.....	.812	.901	1.041	1.485	4.233
35.....	.737	.809	.933	1.319	3.618
40.....	.670	.732	.843	1.180	3.132
45.....	.609	.668	.763	1.059	2.728
50.....	.555	.611	.699	.961	2.410
55.....	.511	.564	.640	.872	2.140
60.....	.473	.521	.592	.799	1.908
65.....	.438	.487	.549	.732	1.722
70.....	.410	.455	.512	.676	1.553
75.....	.387	.427	.480	.629	1.414
80.....	.362	.399	.448	.586	1.288
85.....	.340	.377	.421	.548	1.182
90.....	.320	.349	.389	.511	1.093

TABLE 17
Viscosity of Sucrose Solutions in Centipoises According to Powell ²²

Temperature, ° C	Percentage sucrose by weight								
	1	5	10	15	20	25	30	35	40
25.....	0.92	1.01	1.17	1.40	1.70	2.11	2.75	3.67	5.12
30.....	.81	.90	1.06	1.25	1.50	1.85	2.39	3.14	4.28
35.....	.73	.80	.94	1.10	1.32	1.61	2.05	2.73	3.70
40.....	.67	.72	.84	1.00	1.18	1.43	1.83	2.34	3.20
45.....	.61	.65	.76	.90	1.06	1.28	1.62	2.08	2.75
50.....	.57	.61	.70	.80	.96	1.15	1.44	1.85	2.40

Same Solutions at 20° According to Burkhardt

20.....			1.362	1.601	1.934	2.405	3.140		
---------	-------	-------	-------	-------	-------	-------	-------	-------	-------

From the data of Green ²³ we calculate the viscosity of a 40 per cent solution to be 6.08 cp at 20° and 5.066 cp at 25°.

We have not attempted to apply any corrections to the above data, but the probable corrections would tend to decrease the

²² Loc. cit.

viscosity, hence they would not help to bring about agreement between our values and those of earlier observers.

In connection with the fact that our viscosities are in this case higher than those of most of our predecessors, it is of interest to note that when the viscosity of the commercial sugar solution used by us was reduced to the basis of a 60 per cent solution at 75° C, we obtain 6.07 cp, which is also lower than the 6.20 obtained by us for specially purified sugar.

In the appendixes are given tables of densities of sucrose and ethyl-alcohol solutions, reproduced here for the convenience of any who may desire to use the data given in this paper for the purposes of calibration.

VIII. SUMMARY

1. For the purpose of the calibration of viscometers, there is need for one or more liquids whose viscosity is greater than that of water, which can be easily obtained, and whose viscosity is known with a considerable degree of certainty.

2. Of the suitable substances ethyl alcohol-water mixtures and sucrose solutions each possess certain marked advantages. The viscosities of the former are well known, there existing data by several observers which agree as well as can be expected; but the correctness of the data for the latter has been questioned. Hence we have redetermined the viscosity of a 20 and a 40 per cent solution by weight and have in addition measured the viscosity of a 60 per cent solution from 10° to 95° C. The viscosities obtained by us are generally somewhat higher than the values obtained hitherto, but we have every reason to believe that our values are worthy of confidence.

3. The existing data on the viscosity of water has been reviewed in order to correct it so far as possible according to our present knowledge. The viscosity and fluidity of water for every degree centigrade from 0 to 100 has been calculated.

4. The advantages and disadvantages of expressing viscosity in absolute or specific units have been compared. The suggestion has been made that by expressing all data in terms of the centipoise (the one-hundredth part of the cgs unit), the absolute viscosity of substances would be practically also the specific viscosity, provided that we take water at 20° as the standard. We find the most probable value for the viscosity of water at 20° C to be 1.005 cp.

WASHINGTON, August 8, 1916.

APPENDIXES

Appendix A.—Density in Grams per Milliliter of Mixtures of Ethyl Alcohol and Water ²³

Per cent alcohol by weight	10° C	15° C	20° C	25° C	30° C	35° C	40° C
0.....	0.99973	0.99913	0.99823	0.99708	0.99568	0.99406	0.99225
5.....	9098	9032	8938	8817	8670	8501	8311
10.....	8393	8304	8187	8043	7875	7685	7475
15.....	7800	7669	7514	7334	7133	6911	6670
20.....	7252	7068	6864	6639	6395	6134	5856
25.....	6665	6424	6168	5895	5607	5306	4991
30.....	5977	5686	5382	5067	4741	4403	4055
35.....	5162	4832	4494	4146	3790	3425	3051
40.....	4238	3882	3518	3148	2770	2385	1992
45.....	3226	2852	2472	2085	1692	1291	0884
50.....	2162	1776	1384	0985	0580	0168	.89750
55.....	1055	0659	0258	.89850	.89437	.89016	8589
60.....	.89927	.89523	.89113	8699	8278	7851	7417
65.....	8774	8364	7948	7527	7100	6667	6227
70.....	7602	7187	6766	6340	5908	5470	5025
75.....	6408	5988	5564	5134	4698	4257	3809
80.....	5197	4772	4344	3911	3473	3029	2578
85.....	3951	3525	3095	2660	2220	1774	1322
90.....	2654	2227	1797	1362	0922	0478	0028
95.....	1278	0852	0424	.79991	.79555	.79114	.78670
100.....	.79784	.79360	.78934	8506	8075	7641	7203

²³ This Bulletin, 9, p. 327; 1913.

Appendix B.—Density in Grams per Milliliter of Sucrose Solution ²⁴

Per cent sucrose by weight	0° C	10° C	15° C	20° C	25° C	30° C	40° C	50° C	60° C
0.....	0.99987	0.99973	0.99913	0.99823	0.99707	0.99567	0.99232	0.98813	0.98330
5.....	1.02033	1.01960	1.01884	1.01784	1.01661	1.01518	1.01169	1.00735	1.00231
10.....	4135	4016	3925	3813	3679	3530	3165	2720	2198
15.....	6304	6146	6041	5916	5772	5612	5229	4772	4238
20.....	8546	8353	8233	8094	7940	7767	7366	6898	6358
25.....	1.10869	1.10642	1.10507	1.10354	1.10188	1.10005	9585	9106	8563
30.....	3274	3014	2863	2698	2517	2324	1.11888	1.11398	1.10850
35.....	5769	5473	5306	5127	4933	4730	4279	3779	3228
40.....	8349	8020	7837	7648	7439	7214	6759	6248	5693
45.....	1.21018	1.20657	1.20460	1.20257	1.20039	9812	9332	8811	8247
50.....	3775	3382	3173	2958	2732	1.22495	1.21996	1.21465	1.20891
55.....	6621	6203	5981	5753	5516	5271	4756	4211	3629
60.....	9560	9117	8884	8644	8399	8144	7615	7058	6468
65.....	1.32591	1.32125	1.31882	1.31631	1.31376	1.31113	1.30571	1.30002	9408
70.....	5719	5230	4976	4716	4452	4181	3625	3047	1.32447

²⁴ Plato, Abh. Norm. Eich.-Komm., 2, p. 140, 1900; Zs. Zuckerindustrie, 50, pp. 982 and 1079, 1900; cf. Landolt and Börnstein's Physikalisch-Chemische Tabellen, fourth ed., p. 317.

AN "AVERAGE EYE" FOR HETEROCHROMATIC PHOTOMETRY, AND A COMPARISON OF A FLICKER AND AN EQUALITY-OF-BRIGHTNESS PHOTOMETER

By E. C. Crittenden and F. K. Richtmyer

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I. INTRODUCTION

The work to be reported in this paper was undertaken in connection with the committee on research of the Illuminating Engineering Society. The reports of that committee¹ for 1914 give a general survey of photometric problems on which investigation is especially desirable. Following this general review, it was felt that the new committee appointed for 1915 might most effectively stimulate investigation by choosing a particular field and arranging for experimental work in it. The field chosen was that of heterochromatic photometry. In order to accomplish something definite within the time available for the investigation it appeared desirable to confine the work within rather narrow limits, and it was decided to give attention primarily to the question of the methods to be

¹ Trans. Ill. Eng. Soc., 9, pp. 307, 333, 345, 358, 505; 1914.

used in actual photometric comparisons involving a color difference and to such phases of this question as could be studied in a single laboratory.

As a solution for the whole problem of photometry with a color difference the use of a flicker photometer under certain specified conditions has been proposed in particular by H. E. Ives.² To strengthen the position of the flicker instrument there has been developed also a complete scheme³ for the choice of normal groups of observers, including the establishment of an "average eye." This proposed systematization of heterochromatic measurements appeared so definite and practical as to deserve a thorough trial.

The present work, therefore, was planned to show the difference to be expected between individuals and to include readings by a large number of observers so as to establish average or normal values for various measurements involving color differences. In general, similar measurements were to be made on a flicker photometer and on an equality-of-brightness photometer in order to establish the relation between results obtained by the two methods and the relative certainty of measurements made by the two types of instruments.

The experimental data to be presented were obtained in the laboratories of the Bureau of Standards during the summer of 1915. Besides extensive preliminary tests, the data obtained include (1) readings by 115 observers on the Ives-Kingsbury test solutions for choice of observers, (2) measurements by the same observers on blue glasses presenting a color difference equivalent to that involved in comparing carbon lamps with vacuum tungsten lamps, (3) a repetition of the above measurements by a selected group of observers, (4) sets on a blue solution corresponding to the color difference between a carbon lamp and a gas-filled tungsten lamp, (5) a direct comparison of lamps operated at the color of the pentane lamp flame with others run at 4 watts per candle, and (6) the calibration of a blue solution and measurements with it on lamps at various efficiencies.

With the exception of the solution used in testing observers, it may be noted that the work has dealt only with color differences of the type given by two incandescent bodies at different temperatures, such as two lamps operated at different efficiencies. Lights showing this type of color difference are, of course, much more

² *Phil. Mag.* (6), 24, p. 852, 1912; *Trans. Ill. Eng. Soc.*, 10, p. 317, 1915.

³ Ives and Kingsbury, *Trans. Ill. Eng. Soc.*, 10, p. 203; 1915.

easily compared than those showing a more nearly "saturated" hue, but the difficulties are sufficient to impair very seriously the accuracy of many practical photometric measurements required at the present day. It is highly desirable that a method of comparing the intensities of lights of different colors which can be used for all types of color difference should be agreed upon, but at present the field in which there is most urgent need of a high degree of accuracy in such comparisons is the rating of incandescent lamps. In the present investigation it has appeared desirable to make those tests which would have the most direct bearing upon the practical application of the instruments and methods involved.

II. APPARATUS

Two standard photometer bars were arranged as nearly as possible alike, on one of which a flicker photometer was used and on the other a Lummer-Brodhun photometer. In each case the photometer head was stationary and was illuminated on the left by a stationary lamp placed at such a distance as to give an effective brightness of about 2.5 millilamberts in the photometric field after allowing for all losses in the apparatus. A similar lamp on a carriage at the right was moved by turning a wheel beneath the photometer, and the cells and glasses referred to later were inserted on this side of the photometer so that a constant illumination was maintained. Settings of the movable lamp were printed on a record sheet and were measured from reference lines on the sheet, proper allowance being made for the optical thickness of the cells. This method of recording settings is much quicker than reading from the bar; it also has the advantage of giving a permanent record free from the errors which are likely to be made in transcribing numerical readings. The mean of the groups of points can be located very quickly with a sufficient degree of accuracy. In these tests each observer was asked to keep a tally of the settings, which required taking the hand from the wheel which moved the lamp, as well as turning away from the photometer.

The lamps used have double hairpin carbon filaments in one plane, and were operated at a voltage which made them match the color of the Bureau's 4 wpc standards. For the distances at which the lamps were used, the illumination given follows the inverse-square law with a sufficient degree of exactness so that no corrections were necessary. The voltage was controlled by potentiometers.

The flicker photometer used was a standard Lummer-Brodhun head with the rotating prism attachment described by E. F. Kingsbury.⁴ The particular instrument used was very kindly loaned by Mr. Kingsbury. The photometer head was modified by removing the original prisms and putting in a pair of which one has two quadrants cut away so as to make the duration of exposure to each light the same. (See Fig. 1, A.) In the flicker attachment as originally made the focal plane of the eyepiece fell considerably beyond the comparison prisms. By inserting a collar to extend the telescope the instrument could be made to focus on the face of the prisms, this arrangement being intended for use in making equality-of-brightness settings with the flicker prism at rest. A considerable number of trials indicated that for

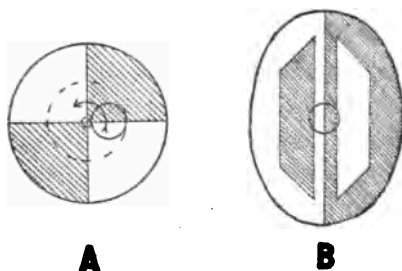


FIG. 1.—Photometric fields

A. Type of field used in the flicker photometer. The small circle which one observes travels over the field as indicated. B. Lummer-Brodhun field used in the equality-of-brightness measurements. (Contrast type, not indicated by sketch.) The circle shows the part used for the small field, the remainder being covered by an illuminated diaphragm

most observers this arrangement was decidedly better for flicker settings. Consequently it was used throughout the tests. Without a very good photometric field this could not be done, since any imperfections in the field would cause flicker, but the prism used was sufficiently good so that with a color and intensity match there was practically no flicker in the field even at low speeds.

A tachometer was attached to the flicker mechanism, and the speed was controlled by a rheostat in series with the motor. It was expected that each observer would have to choose a suitable speed for each color difference measured, but extensive trials with a number of observers showed that, over a limited range, change of speed had very little effect either on precision of setting or on the mean result. It was finally decided to adopt a moderate speed (12 light cycles per second) for all observers and all set-

⁴ Jour. Franklin Institute, 180, p. 215; 1915.

tings. Observers were then directed to set for a minimum of flicker. Although this speed was rather low for settings on the yellow solution and high for those with color match, few observers found serious difficulty in making definite settings. The constant speed was adhered to partly because in the preliminary measurements there had been indications of slight changes of results when speeds much higher or much lower were used, particularly with the yellow solution.

On the second photometer bar measurements were made under two conditions—(1) with the standard Lummer-Brodhun contrast field, (2) with an illuminated diaphragm which limited the field to about 2° , as in the flicker instrument. (See Fig. 1, *B*.) In the case of the contrast field, however, all observers were asked to make their settings by the middle strips, disregarding the contrast trapezoids. The small field was used because it gave equality-of-brightness measurements under conditions closely similar to those used with the flicker instrument. It was also thought that limiting the field so as to use only a fairly homogeneous part of the retina might reduce variations in judgment and give settings more truly characteristic of the observer's eye.

All the photometers used had pupil apertures 5 mm in diameter. Each photometer head was provided with holders for absorption glasses, as well as holders for absorption cells, especially constructed to prevent any diffusion of stray light into the photometric field.

Three pairs of 1 cm cells were provided. These were constructed practically like those described by Ives and Kingsbury,⁵ with removable sides of colorless optical glass. A small modification which was found to facilitate secure sealing of the sides was made by beveling slightly each edge of the cell blocks, thus making a groove to be filled by the paraffin seal. Although the sides were made of polished plate glass, it was found that there were appreciable differences between the transmission of different plates. Repolishing of the plates, which was necessary after using some solutions, also changed the transmissions perceptibly. The differences and the changes mentioned were less than 1 per cent, but were not negligible for precise work. So far as color is concerned, the glass used was satisfactory. It had been obtained for other work, requiring very clear glass, and its applicability for the present purpose was tested directly by measuring its transmission for

⁵ Trans. Ill. Eng. Soc., 9, p. 795; 1914.

the two extreme colors of light to be used—that is, the light transmitted by the two test solutions. Its transmission for the two was the same within 0.2 per cent.

Considerable time was given to experiments with the cells to determine how closely conditions could be reproduced. In brief, it may be said that results can be rather easily reproduced to within 1 per cent with them, but if an accuracy greater than one-half per cent is desired, extreme care is necessary. In any case thorough cleaning is essential, and it is desirable to compare the cells with each other after each cleaning.

The paraffin used for sealing the cells is somewhat difficult to remove completely from the plates, and a supply of hot, running water is almost a necessity for thorough cleansing. Xylol also is a convenient solvent of paraffin and is especially useful when the supply of hot water is not plentiful.

Cells should be filled well up to the neck, for if a meniscus of considerable size is left an appreciable amount of light may be reflected from it into the field.

III. MEASUREMENTS ON THE IVES-KINGSBURY SOLUTIONS FOR SELECTION OF OBSERVERS

The method of selecting observers, to which reference has been made,⁶ is based on determinations of the relative transmission of two solutions—one reddish yellow, the other blue-green. These are solutions, in water, of potassium bichromate and of copper sulphate, containing, respectively, 72 and 53 grams of the salt per liter of solution. While not definitely specified in the original proposal, it has been assumed that the solution is to be made up at 20° C, and that by copper sulphate is meant the crystals $\text{CuSO}_4 + 5\text{H}_2\text{O}$.

When measured at 20° C by the “average eye” with a flicker photometer conforming to specifications previously mentioned, 1 cm layers of these two solutions were intended to have equal transmissions for the light of a carbon lamp of the standard 4-watt-per-candle color. The average eye thus defined was originally established by measurements made by 61 observers on the transmission of a green solution.⁷ The two solutions above described were worked out later on the basis of measurements made by selected groups of observers. It is not at all clear that an average established by comparing the middle of the spectrum

⁶ Trans. Ill. Eng. Soc., 10, pp. 203–208; 1915. ⁷ Ives and Kingsbury, Phys. Rev. (2), 5, p. 230; 1915.

with the whole spectrum can be legitimately thus transferred by a few observers to the basis of a comparison of the two ends of the spectrum. For instance, it is known that an observer may be abnormally sensitive or nonsensitive in the middle of the spectrum and yet appear normal in the comparison of the two halves of the spectrum, or he may be normal according to the first test and not so by the second. In fact, it would appear that tests of both kinds should be included in choosing observers for measurements of illuminants which show marked selectivity in the visible spectrum. Ives and Kingsbury state, however, that groups of observers selected by one criterion were found to satisfy the other. For the types of color difference with which the present investigation has been most directly concerned the two-solution test appeared most significant, besides being more convenient than the earlier one. This method alone, therefore, has been used for testing observers, and the characteristics of a given observer will be supposed to be represented by the ratio of the transmission of the yellow solution to that of the blue solution, as measured by that observer, although it is recognized that this ratio is more strictly an index of the observer's sensitiveness to lights in which different proportions are contributed by the two ends of the spectrum.

1. PRELIMINARY MEASUREMENTS ON SOLUTIONS

The standard temperature for the solutions is 20° C. The greater part of the present work was done at temperatures ranging from 25° to 30°, and, consequently, it was necessary to determine the temperature coefficients of the transmission of the test solutions. Over the range considered the variation with temperature was found to be practically linear. The transmission of the potassium-bichromate solution decreased nearly 0.2 per cent per degree rise of temperature, while that of the copper-sulphate solution decreased about half as much. The differential correction to be applied to the ratio of the two transmissions was, therefore, practically 0.1 per cent per degree centigrade, the observed ratio (Y/B) being too small when the temperature was above 20°.

Other conditions which affect the value of the ratio obtained are the color of the light for which the transmissions are measured and the brightness of the photometric field. The color is supposed to be that of the standard 4 wpc carbon lamp and the effective brightness 2.5 millilamberts (equivalent to an illumination of 25

meter-candles on a perfectly diffusing and completely reflecting surface) after allowing for losses in the photometer. In the particular instruments used these losses aggregated nearly 50 per cent, so that the actual illumination necessary was about 50 meter-candles.

On account of the lack of a convenient nomenclature, as well as the difficulty of absolute determinations of diffuse reflectivities of surfaces, there has been some confusion as to the exact field brightness used in various investigations. In making the measurements recorded in this paper the effective brightness used was very close to 2.5 millilamberts; that is, the brightness produced by an illumination of 25 meter-candles on a perfectly diffusing and completely reflecting surface. The numerical values for the test ratios originally given before the Illuminating Engineering Society⁸ were reduced, however, to the basis of an illumination of 25 meter-candles on a white surface having a reflectivity of approximately 90 per cent. Since a specification of the absolute brightness is preferable, 2.5 millilamberts is here retained as the standard brightness, and test ratios are given in this paper on that basis. The difference is quite unimportant, since the correction previously applied to the ratios was only 0.003.⁹

Since the variation of results arising from changes in the efficiency of the lamp or in the illumination used is small, no very precise determination of the effects of such changes has been made. Some measurements were made, however, with a lamp operated at 3.1 and at 5 watts per candle and with effective illuminations of approximately 10 and 50 meter-candles. In accordance with the reversed Purkinje phenomenon shown by the flicker photometer,¹⁰ it was found that the ratio of transmissions (yellow + blue) was smaller at the higher illuminations. At 50 mc the average of three observers gave a ratio slightly over 1 per cent lower than the normal, while at 10 mc the ratio was 2 per cent higher than normal. A rise in the efficiency of the lamp naturally causes a decrease in the observed ratio, but the variation is so small that the effect of any error likely to occur in the rating of the lamp would be entirely negligible. Running the lamp at 3.1 wpc, or at 5 wpc, instead of 4, causes a departure from the normal ratio of only 1 to 2 per cent. It may be well to record that the funda-

⁸ Trans. Ill. Eng. Soc., 11, p. 331 (also note, p. 333); 1916.

⁹ Thus, in the earlier paper the average observed ratio 0.987 was "corrected" to 0.990, whereas in either case the round number 0.99 may as well be used for all practical purposes. Fig. 6, in which the change would be scarcely perceptible, has not been redrawn.

¹⁰ Ives, Trans. Ill. Eng. Soc., 5, p. 717, 1910; Phil. Mag. (6), 24, p. 170, 1912.

mental 4 wpc carbon standards have oval-anchored filaments, and since their average reduction factor is 0.825 the standard efficiency is 4.85 watts per spherical candle or 2.6 lumens per watt. The color of the light is practically the same as that given by a vacuum tungsten lamp at 3.1 wpc or 3.2 lumens per watt.

The composition of the test solutions is such that there is no reason to expect any difficulty in reproducing them or any change with time. Cells used for several months during the investigation showed no appreciable change in transmission.

2. MEASUREMENTS TO ESTABLISH A NORMAL CHARACTERISTIC RATIO

In order to obtain an independent check on the average eye as defined by the test solutions and to test the characteristics of observers to be used in later measurements, and at the same time to establish the relation between the characteristic ratio and some measurement met with in practical work, a series of measurements was made by 115 observers. On each photometer this series consisted of 8 sets of 10 readings each. On the flicker photometer sets were made on the two lamps at color match and on the same lamps with a blue-glass screen and with cells containing the two test solutions interposed in succession on one side of the photometer head. The four sets were immediately repeated in different order. The data given, therefore, are the mean results of 20 settings of the photometer on each condition.

The original plans were to include a similar series of measurements on the equality-of-brightness photometer, but with it only the most experienced observers could make any definite settings on the test solutions, and even they varied so greatly from day to day that the results were quite useless as an indication of the observer's color characteristics. For instance, ratios determined on one day by the mean of several hundred settings could not be repeated within 10 per cent on the next day, although the same observers could reproduce their ratios day after day on the flicker photometer with an average deviation of less than 1 per cent, only 20 settings being made on each solution. Consequently, the equality measurements were made only on the smaller color differences, such as that presented by the blue-glass screen, for which the results are given in the next section. One observer was unable even then to make settings definite enough to be used, and his results are consequently not included in the following data, although his flicker settings were good.

With very few exceptions the observers included in these measurements are men who have had some years of experience in physical or chemical observations, and nearly 30 of them have had considerable recent practice in some sort of photometric measurements. The distribution of 114 observers with respect to characteristic ratio ($Y + B$) is shown in Fig. 2, where the ordinates represent the number of individuals falling within a range of 1 per cent. For example, between 0.900 and 0.909, inclusive, there are 5 observers. Such frequency curves must be used with caution,

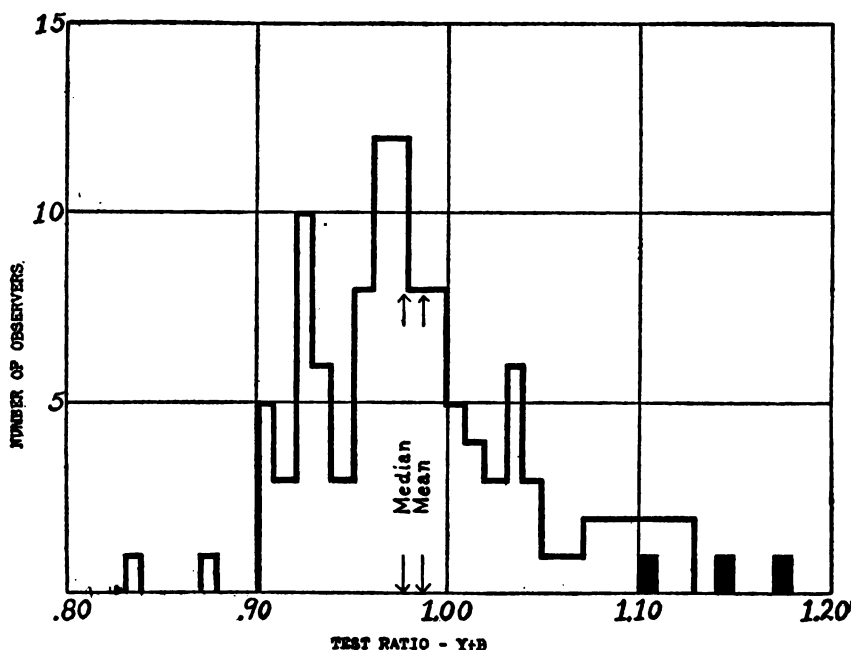


FIG. 2.—Distribution of 114 observers with respect to characteristic ratio (ratio of transmission of yellow test solution to that of the blue solution)

The ordinates show the number of observers falling in a range of 0.01 in ratio. The black rectangles represent "color blind" observers

since their shape can often be greatly changed by grouping in different ways. In this case the number of individuals must be considerably increased before it can be told with certainty whether the unsymmetrical shape of the curve is accidental, but there is a marked indication of the existence of a fairly definite type of eye which gives systematically high ratios in the neighborhood of 1.10 to 1.12. The solid black rectangles represent three men who were found to be definitely color blind in a test made by I. G. Priest with Nagel test cards. Some others in this group are known to have

peculiarities in their color perception, although they are able to pass color-vision tests.

The arithmetical average of the 114 ratios is 0.987. If the three observers classed as color blind are omitted, the average is 0.983, but actually the line between normal and abnormal color vision is very difficult to draw. Moreover, the relation between color and luminosity is not very definite, and in making up an average luminosity scale there is little justification for ignoring that percentage of the people who are color blind. On the other hand, taking an average may give undue weight to the abnormal observers, and there is some advantage in taking instead of the average the median value; that is, a value such that there are equal numbers of observers above and below it. In this case the median for the 114 ratios is 0.977, and the effect of omitting the three extreme observers mentioned is only to make the median fall between 0.976 and 0.977. To show how closely groups selected at random might be expected to agree on the ratio, these 114 observers were arranged alphabetically and for each half of the list the mean and the median values were found. The two means were 0.982 and 0.992; that is, 1 per cent different. The two medians were 0.975 and 0.979, or 0.4 per cent different. In other words, in reproducibility the median appears to be somewhat better than the mean. Even if the three color-blind observers are omitted, the means of the two groups still differ by 0.8 per cent, being 0.979 and 0.987.

In order to test the constancy of the characteristic ratio of individuals, 20 observers repeated their measurements at the end of the test. The average deviation of an individual from his first value was 1 per cent, and the mean of the 20 differed by 0.2 per cent in the two sets. Repeated measurements during the several months occupied by other parts of the work have indicated that this is the amount of variation to be expected in successive sets. Consequently, the average deviation of an experienced observer from his mean value is usually well below 1 per cent. No definite indications of any important change in an individual's ratio have been found.

IV. MEASUREMENTS ON BLUE GLASSES

The blue glasses mentioned presented a color difference equivalent to that between a 4 wpc carbon lamp and a vacuum tungsten lamp at about 1.2 wpc (8.2 lumens per watt). In other words,

when the glass was placed in front of a carbon lamp the light transmitted was similar in color to that from a tungsten lamp, and it was compared with unmodified light from another carbon lamp. On such glasses measurements were made by all the observers with the flicker photometer and with the two forms of equality field. The latter both showed large variations in results, and the general result is perhaps better shown by averaging out some of the individual errors. In order to do this the observers

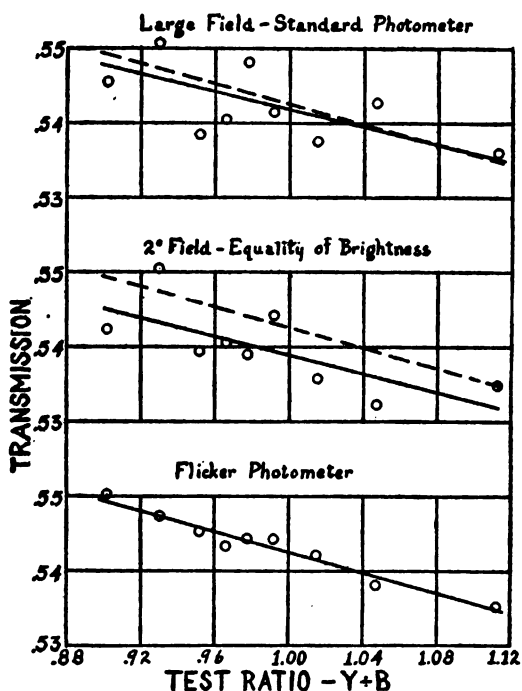


FIG. 3.—Relation between observed transmission of blue glass (3G) and characteristic ratio of observer

The color difference involved is equivalent to that between 4 wpc carbon lamps and 1.2 wpc tungsten. Each point is the mean of 12 or 13 observers. The solid lines represent the least square solutions for 114 observers. The flicker photometer curve is drawn (dashed) with the others for comparison

were arranged in the order of their characteristic ratios and averaged in groups. Three of the extreme groups consist of 12 observers each, the others of 13. The results are shown in Fig. 3. The flicker photometer data are plotted at the bottom, and for comparison the curve drawn to represent them is reproduced in a broken line along with the data for the two equality fields. The curves drawn represent the least square solutions for the whole 114 observers, assuming a linear relation between characteristic

ratios and observed transmissions. In the flicker measurements the change in transmission corresponding to 1 per cent difference in ratio is 0.13 per cent, while the other curves show corresponding changes of 0.11 and 0.12 per cent.

The observers were somewhat arbitrarily classified by inspection of their settings on the equality photometer with regard to the consistency of settings (not their accuracy), and in Table 1 are given the mean results of the three classes, "a" meaning good, "b" medium, and "c" poor sets. The transmissions are all reduced to the basis of the mean ratio (0.987) so as to be strictly comparable, and the residuals are departures from the curves of Fig. 3; that is, the systematic errors due to individual characteristics have so far as possible been eliminated and the residuals represent largely the accidental errors in judgment. All transmissions given in this paper are for light of the quality given by a 4 wpc carbon lamp.

TABLE 1

Transmission of Blue Glass (3G) and Residual Errors (114 Observers)

Class	Number of observers	Transmission			Mean residuals (per cent)		
		Flicker photometer	Equality photometer		Flicker photometer	Equality photometer	
			Large field	Small field		Large field	Small field
a.....	31	0.5430	0.5426	0.5404	0.5	1.2	1.2
b.....	58	.5436	.5429	.5369	.6	1.7	1.5
c.....	25	.5437	.5414	.5454	.7	3.2	3.5
Mean.....	114	.5434	.5425	.5396	.6	1.9	1.9

To test the reproducibility of results 20 observers well distributed with respect to ratio, 12 of whom had had considerable photometric experience, were selected to repeat these measurements. The individual observations are shown in Fig. 4, in which, as before, the flicker curve is drawn through the other data for comparison. The average values for the transmission are given in Table 2.

TABLE 2
Transmission of Blue Glass (3G)—Two Sets by 20 Observers

	Flicker photometer	Equality photometer	
		Large field	Small field
First set.....	0.5429	0.5431	0.5447
Second set.....	.5422	.5472	.5456
Mean.....	.5426	.5452	.5452
Average difference between two sets by each observer (per cent)....	.4	1.4	1.2
Mean obtained by 14 more consistent observers.....	.5427	.5472	.5436

Both groups of better observers in Table 1 find the transmission of the blue glass smaller with the small field than with the large,

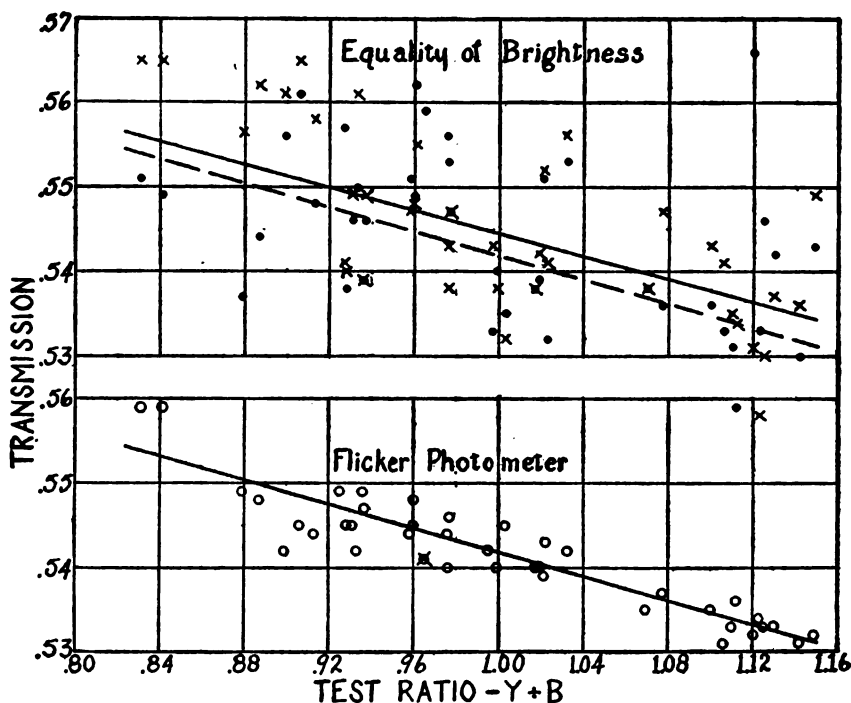


FIG. 4.—Transmission of blue glass (3G)—two sets by each of 20 observers

The crosses represent measurements with a standard Lummer-Brodhun photometer, the dots those with a small field. For this group of measurements the average results with these two fields were the same. (See Table 2.) The dashed line is the flicker curve

which is to be expected, but obtain higher values with the flicker than with either form of equality field. The 20 who were chosen to repeat the measurement happen, however, to read, on the aver-

age, higher on the equality fields. The number of sets made is by no means sufficient to establish standard values with the equality photometer, and a better approximation to the real "average" transmission can perhaps be obtained by omitting the observers whose sets were not consistent. Six of the 20 observers in the second set obtained results on one or both of the equality fields which differed by 2 per cent or more from their first sets. If these six are omitted, the means of the remaining 28 measurements are as shown in the last line of Table 2.

With practice most observers tend to fix upon a more definite concept with regard to what constitutes equality of brightness between two different colors, and the question has been asked whether experienced observers obtain the same result as inexperienced ones. Naturally the practiced observer can make more precise settings. In other words, to reproduce results with a given percentage of accuracy the unpracticed observer must make more measurements, but it does not appear that this fact gives any reason for expecting a systematic difference between results obtained by experienced observers and those obtained by inexperienced persons. However, of these 114 observers the ones who might be classed as experienced in photometry did get a result slightly higher than the others. Twenty-two observers were selected as having had such experience as would justify giving extra weight to their sets if the purpose of the paper was to establish normal values for the Lummer-Brodhun photometer, and their results follow in comparison with the mean of all the observers.

TABLE 3
Transmission of Blue Glass (3G)

Observers	Flicker photometer	Equality photometers	
		Large field	Small field
114.....	0.5434	0.5425	0.5396
22 "experienced".....	.5437	.5462	.5434

Among these 22 experienced observers are included half of the group whose results are given in Table 2, but the agreement between the two groups would not be materially affected if the observers common to the two were omitted.

Table 1 by itself would indicate a high degree of certainty in the transmission as determined by the large-field equality photometer,

but this certainty is reduced by the failure of the 20 observers to repeat their results or to agree with the larger group. Nearly half of the change shown by the 20 observers is due to one very poor set, but of these 20 observers the 14 who repeated results most consistently got a value 0.7 per cent above the mean of all the 134 observations, and this higher value is corroborated by the other experienced observers.

As the result of the 134 measurements, the unweighted average values for the transmission of this glass are 0.543 for both flicker and large-field equality and 0.541 for the small-field equality photometer, but in general the observations here recorded leave an uncertainty of the order of 1 per cent in the values to be assigned for measurements by the equality photometer. Further measurements, discussed in a later section, indicate that the values obtained by the more consistent and the more experienced observers (Tables 2 and 3) are nearer the result which would be obtained by increasing greatly the number of observations. The 22 observers whose Lummer-Brodhun settings should have most weight on the score of experience obtain practically the same result with the two small fields (flicker and equality) and about one-half per cent higher with the large field. In searching for such small differences, however, it is sometimes misleading to take averages of a small number of observers without considering the individual observations, for if most of the observers get small differences, a few erratic observers may throw the average results to one side or the other. Comparing the two small fields of these 22 observers 11 are higher on the flicker and 11 higher on the equality. As between the large-field equality and the flicker, 13 read higher on the former and 9 higher on the latter. While there is a definite preponderance in the one direction, it is evident that the magnitude of the difference is not established with much accuracy.

Middlekauff and Skogland¹¹ have already called attention to the relation between the above values and those obtained in several other laboratories. This glass has a transmission 1.6 per cent greater than the glass 3B included in their comparative measurements, and if the values assigned for 3B are correspondingly increased for comparison with the preceding (0.543), the results in different laboratories are 0.538 with a flicker photometer and 0.546, 0.551, and 0.552 with Lummer-Brodhun photometers.

¹¹ *Trans. Illum. Eng. Soc.*, 11, p. 164, 1916; this *Bulletin*, 18, p. 287, *Sci. Paper No. 277*.

There are two differences in conditions which may in part account for the fact that the result obtained in the present work is below all the other Lummer-Brodhun values. In this work the photometer was used as an "equality" rather than a "contrast" field, and the illumination was much higher than that used in the other measurements. The difference probably arises, however, more from the fundamental uncertainty of equality-of-brightness measurements than from any of these systematic differences in conditions.

Whether or not there is a systematic difference between the results obtained by contrast settings and those given by strictly equality settings can not be told except by testing a large number of observers, for on changing the method of judgment each observer must to a considerable extent reestablish his "definite concept" of what constitutes a setting, and some observers change in one direction, some in the other. In general, the mean result obtained by a group of observers does not seem to be changed definitely in either direction by changing the method of judgment. When there is a considerable color difference in the field, some observers can set more definitely if the contrast strips are removed. In this case the strips were not removed, but observers were asked to disregard them so far as possible and to set for equality of brightness in the central strips of the field. It is to be noted that the 22 "experienced" observers setting this way obtain exactly the same average result as Middlekauff and Skogland's group setting by contrast.

The greater certainty of the flicker values is shown by a comparison of the mean residuals in Table 1 and the differences between sets in Table 2. It may be remarked that some observers with practice develop the ability to repeat values very closely on the equality photometer, and such observers would make a much better showing for that photometer in a comparison like that of Table 2. Unfortunately, however, such observers are comparatively rare and do not in all cases settle on a value in agreement with their characteristics as indicated by the flicker method.

Of the equality-of-brightness photometers the small field shows no material superiority over the large one. Table 2 would seem to indicate that it gave more reproducible results, but in Table 1 it will be seen that the residuals average the same for the two forms, while the small field shows the poorest agreement between groups of observers. The use of the small field was discontinued after this test, because it gave no promise of any practical advan-

tage. The data obtained are sufficient to indicate that if the small field had been carried through the measurements on the larger color differences the results would almost certainly have agreed with the flicker values more closely than the large-field measurements did. Practically, however, with these larger differences equality-of-brightness settings show such large variations from time to time and such wide differences between individuals that the results have very little significance. The use of the large field was carried as far as practicable because the tendency has been to use that type of photometer for all sorts of measurements, and it was desired to correlate that instrument with the flicker, particularly with respect to differences between individuals as well as relative average values for the two instruments.

V. OTHER MEASUREMENTS BY SELECTED OBSERVERS

The preceding sections cover the primary purposes of this work, which was to establish an average eye and to test the usefulness of the proposed method of selecting observers by having a large number make measurements on the test solutions and on some typical color difference. The following sections give the results of measurements by a few observers with several other degrees of color difference. The observers were selected from those conveniently available, the principal purpose being to secure a wide range of individual characteristics in order to illustrate the application of the method of selection. Only a few measurements were made, and the results are intended to be qualitative demonstrations of the method, not precise determinations of the things measured.

1. MEASUREMENTS ON A BLUE SOLUTION REPRESENTING THE COLOR DIFFERENCE OF CARBON AND GAS-FILLED TUNGSTEN LAMPS

This color difference is so great that most observers showed marked fluctuation in results from day to day with the equality photometer, although the majority made fairly good settings at any one time. Eleven observers made two sets on both photometers, using only the large field on the equality of brightness. On it each set consisted of three separate groups of readings, while on the flicker only two groups were made for each set, but the equality results were so erratic that the whole series of measurements with it were repeated. The individual sets are shown in Fig. 5. The circles crossed by a line indicate sets in which means of groups showed marked discrepancies among themselves. The mean of the equality-of-brightness sets is about 5 per cent above

the flicker curve, but the only observers who repeated values at all consistently from day to day fall closer to that curve, and the most consistent observer has all four sets below it. Nevertheless, even if the more erratic sets are discarded, the equality values are definitely higher than those obtained by the flicker method.

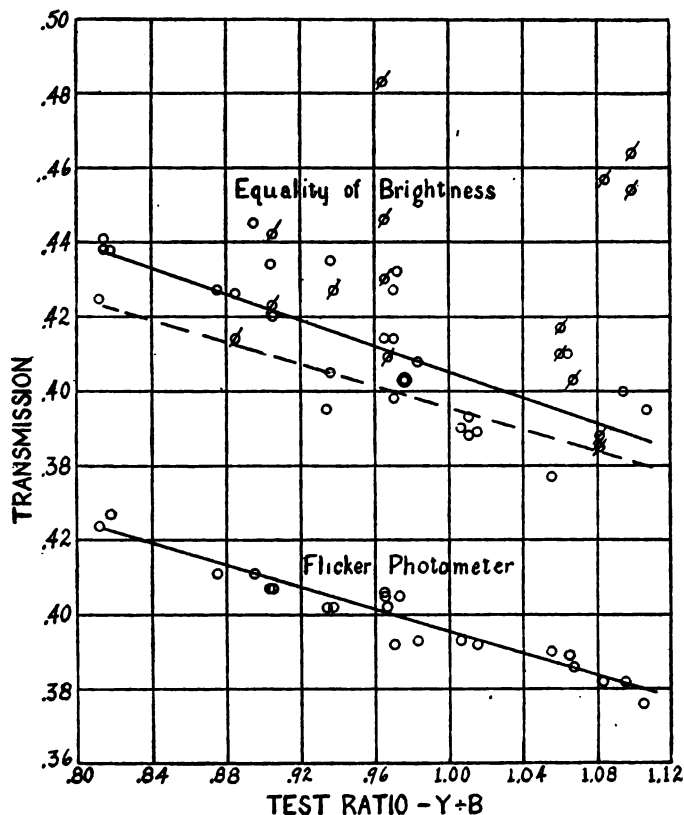


FIG. 5.—Transmission of 1 cm cell of 75 per cent concentration Ives-Kingsbury nickel-ammonium sulphate solution

The crossed circles indicate measurements in which the groups of readings differed greatly. These are disregarded in drawing the curve. The published equation for transmission of this solution gives a value of 0.397 for this concentration

For a ratio of 0.99 the mean transmission by the flicker photometer is 0.397; by the equality (including all sets) 0.417, the mean residuals being 0.8 per cent and 3.5 per cent. The slope of the flicker curve indicates a change in transmission of 0.38 per cent for 1 per cent difference in ratio.

It may be noted that this solution is the 75 per cent concentration of Ives and Kingsbury's blue working solution,¹² for which

¹² Trans. Ill. Eng. Soc., 10, p. 253; 1915.

the transmission calculated from the published equation is 0.397, as determined for a characteristic ratio of unity, which should give 0.399 for a ratio of 0.99. The difference between the original calibration and the present check on this one point with the flicker photometer is therefore one-half per cent. There are no other equality-of-brightness measurements available for comparison. Some further measurements on this solution are given in a later section of this paper.

2. MEASUREMENTS ON A LAMP GIVING THE COLOR OF A PENTANE STANDARD AGAINST 4 WPC CARBON LAMPS

Fig. 6 shows similar data for the comparison of a carbon lamp with a standard operated to match the pentane lamp flame in

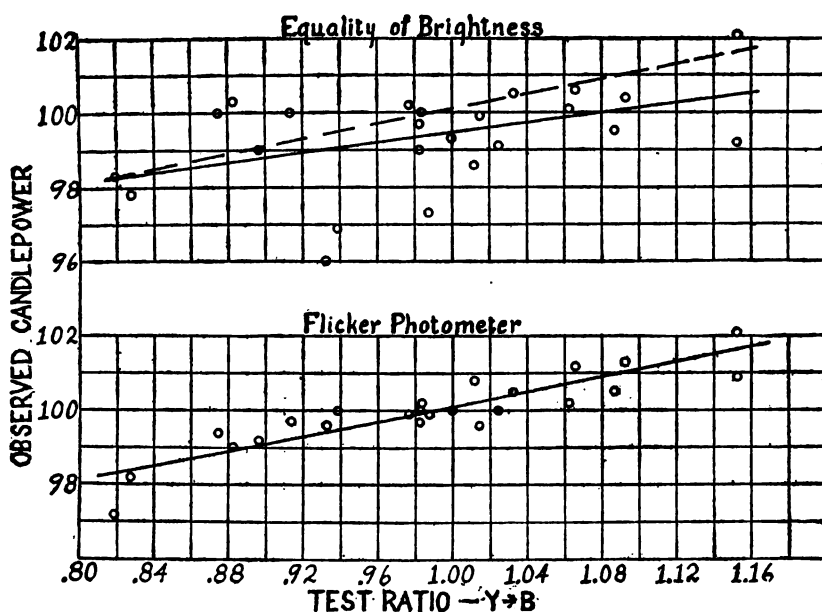


FIG. 6.—Measurements of the candlepower of a lamp matching the pentane flame in color, using a 4 wpc carbon standard

The candlepower scale is chosen to make the average flicker photometer value 100

color. This is in the neighborhood of 7.5 wpc. Here the slope is reversed because the light measured is redder than that of the carbon lamp, and the change in observed candlepower corresponding to 1 per cent in ratio is 0.1 per cent.

Each observer made two sets on each photometer, a set on the equality consisting of three groups of readings, but on the flicker of only two groups. The average difference between the two sets on the flicker was 0.6 per cent, on the equality 1.2 per cent, the

average residuals (departures from the curve) being 0.4 per cent and 0.9 per cent. In the two measurements of the characteristic ratio made by each observer the average difference was as usual 1 per cent.

The mean value obtained on the equality is 0.6 per cent below that given by the flicker photometer, and that this is not entirely due to accidental variation is indicated by the fact that 10 of the 12 observers varied in this direction.

The actual candlepowers found were 8.75 by the flicker and 8.70 by the equality photometer for a lamp whose value had previously been established as 8.70 by repeated calibrations in comparison with 4 wpc standards on the standard Lummer-Brodhun photometer.

3. MEASUREMENTS ON A BLUE SOLUTION AND ON MULTIVOLTAGE STANDARDS

At the completion of this work the lamps which had been measured at various laboratories, as reported by Middlekauff and Skogland,¹² were available, and in order to obtain a direct comparison of further flicker values with the results of those measurements a few sets were made on the lamps at different voltages. As a convenient means of making the measurements with the standard illumination on the flicker photometer, the values for the lamps were obtained indirectly by first calibrating, with the flicker photometer, cells filled with properly chosen concentrations of the Ives-Kingsbury blue solution, which has already been mentioned, and then measuring the lamp on the Lummer-Brodhun photometer with approximate color match obtained by the cells. This procedure also gave a check on the calibration of that solution as published. The measurements made on each concentration of the solution were four sets by each of three observers, with slight corrections to put the results on the basis of the average eye giving a characteristic ratio of 0.99. The results are given in Table 4, the transmissions being percentages of the transmission of a similar cell filled with water. The values in column 4 are calculated from Ives and Kingsbury's equation. These values for transmission all refer to the transmission for light similar in quality to that of a 4 wpc carbon lamp.

¹² Trans. Ill. Eng. Soc., 11, p. 164, 1916; this Bulletin, 13, p. 287, Sci. Paper No. 277.

TABLE 4
Transmission of 1 cm Cells with Ives-Kingsbury Blue Solution

Equivalent watts per candle	Concentration of solution	Observed transmission	Calculated transmission	Calculated transmission ratio, 0.99
0.65.....	0.75	39.5	39.7	39.9
0.85.....	.47	56.4	56.5	56.7
1.00.....	.42	60.0	60.2	60.3
1.2.....	.34	66.7	66.5	66.5
1.4.....	.28	71.8	71.6	71.6

Since the original calibration was supposedly based on an average eye corresponding to a characteristic ratio of 1.00, the degree of concordance obtained should be judged, perhaps, by comparison of columns 3 and 5, but as absolute calibrations of the solution columns 3 and 4 should be taken, and within the uncertainty of the present determinations made by the small number of observers mentioned, the equation from which column 4 is calculated ($\log_{10} T = -0.539C^{1.09}$) is correct.

The last four of these solutions were used in measuring the lamps. The 75 per cent concentration is very much bluer than is required by any of the voltages at which these lamps were measured in the comparison between different laboratories, and it was included only for a more complete comparison with the calibration curve and with the previous measurements by a larger number of observers. (See Fig. 5).

The results on the lamps are best shown, perhaps, by comparison with those calculated from the Middlekauff-Skogland equations,¹⁴ which are based on measurements made with the standard Lummer-Brodhun photometer. The following table shows the percentage departure of the values found at the different efficiencies from the calculated values for each lamp. These measurements are equivalent to the direct comparison of lamps at the different efficiencies with a 4 wpc carbon lamp. The differences given are positive when the flicker values are lower.

TABLE 5
Differences Between Standard Lummer-Brodhun and Flicker Photometer Values

Approximate wpc.....	1.4	1.2	1.0	0.85
Lamp 1.....	+0.15	+0.5		+1.45
Lamp 2.....	-.05	+.35	+1.7	+1.35

¹⁴ This Bulletin, 11, p. 483, Sci. Paper No. 238; Trans. Illum. Eng. Soc., 9, p. 734, 1914.

The large departure at 1 wpc is quite inconsistent with all other measurements, and unfortunately the other lamp was not measured at this efficiency. The course of the variation probably should be considered independently of this point. The color match obtainable with this solution is far from being perfect, but the variations arising from this cause are not sufficient to explain the departure of this point from the other measurements. The significance of these measurements can best be seen by referring to the more recent paper by Middlekauff and Skogland, which includes them with results from other laboratories. It may be said that they agree very well with values obtained by Ives using a luminosity scale based on the flicker photometer but fall considerably below all the results obtained with the usual Lummer-Brodhun photometer. The measurements already reported on the 75 per cent concentration of the solution indicate that this difference probably will continue to increase as comparisons are carried to higher efficiencies.

VI. CONCLUSION

1. EFFECT OF INDIVIDUAL CHARACTERISTICS

The preceding results emphasize the fact that for accurate heterochromatic measurements a systematic choice of observers is essential. The system proposed by Ives and Kingsbury appears to be practical and reliable at least for color differences of the type dealt with in these tests. The average eye established is represented by a value of approximately 0.99 for the ratio of the transmission of the yellow test solution to that of the blue solution under the specified conditions. The agreement with the original ratio assigned (1.00) is very good, especially since the latter was largely based on an indirect derivation of the average eye. It is suggested, however, that if the lack of symmetry in the distribution of these observers (as indicated by Fig. 2) is found to persist when larger numbers of observers are included greater reproducibility of the normal ratio might be obtained by choosing as the normal not the average value but the median, which in this case is approximately 0.98.

The differences in observed values arising from individual peculiarities of course increase as the color difference increases. When comparisons are made directly with a 4 wpc carbon lamp, a difference of 1 per cent in the characteristic ratios of observers should result in approximately the following differences in observed

candlepower of a tungsten lamp at the various specific consumptions given:

Watts per candle	Per cent candle-power difference
3.1	0.0
1.4	.1
1.0	.2
.75	.3
.6	.4

When plotted in terms of lumens per watt, these data give nearly a straight line.

Fig. 7 shows the percentage deviation from normal values as a function of the characteristic ratio for the several color differences

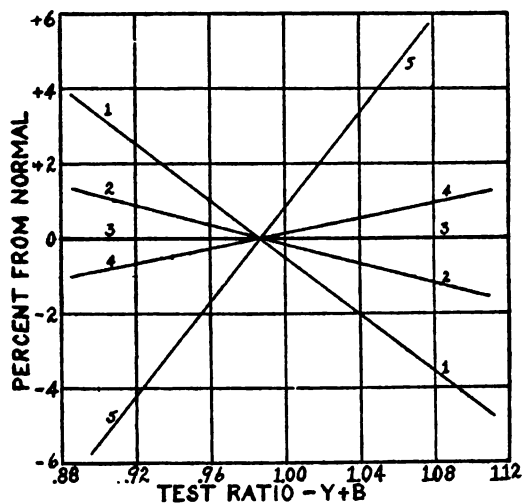


FIG. 7.—Deviation from normal values corresponding to different test ratios when the light indicated is compared with that of a 4 wpc carbon lamp

1. 0.75 concentration blue solution with carbon lamp, equivalent to tungsten lamp at about 0.65 wpc. The curve for the blue test solution practically coincides with this. 2. 1.2 wpc tungsten. 3. Color match, 4 wpc carbon or 3.1 wpc tungsten. 4. Pentane lamp. 5. Yellow test solution

indicated. Thus, an observer whose characteristic ratio $\frac{Y}{B}$ is 0.90, in measuring the candlepower of a 1.2 wpc tungsten lamp against a 4 wpc carbon lamp (3.1 wpc tungsten) would assign to the tungsten lamp a value 1.2 per cent too high. To a 0.65 wpc tungsten lamp he would assign a value 3.5 per cent too high. Of the 114 observers 19 (17 per cent) would have obtained values for a 1.2 wpc tungsten lamp differing 1 per cent or more from the

normal value; 70 (61 per cent) would have departed 1 per cent or more from the normal value in measuring a 0.65 wpc tungsten lamp.

For measurements on color differences of this type it is not necessary to have a group of observers whose average ratio is normal; any observers may be used and their results corrected to the normal by use of curves similar to Fig. 7. These curves can be summarized by the equation

$$I_n = \frac{I}{1 + m(R - R_n)}$$

where I_n is the normal value of a photometric quantity (such as candlepower, illumination, or transmission), R_n is the normal characteristic or "test" ratio, while I and R are the values of these quantities found by a particular observer. m is an empirical constant, which may be either positive or negative and which depends on the color difference involved in the observation. It is the slope of the curves in Fig. 7. There may be slight systematic errors in the corrected values; that is, some observers may be always off the curve in the same direction, and consequently a number of observers should be used for highly accurate results. Not enough repeated measurements are at hand to show with certainty the average magnitude of these systematic errors, but they are certainly small in most cases.

On the average, measurements made with an equality-of-brightness photometer will show practically the same differences due to individual characteristics as those made with the flicker, but the erratic variations are often so great as to overshadow these systematic differences. It has been remarked that some observers develop the ability to make very consistent settings on the equality photometer. In the majority of cases such observers have been found to read close to the value indicated by their test ratio, but this is not always the case.

2. COMPARISON OF FLICKER AND EQUALITY-OF-BRIGHTNESS PHOTOMETERS

With regard to certainty of measurement the flicker photometer shows a decided advantage even with small color differences. With more experienced observers, specially selected, this advantage would probably be materially reduced, but would not be entirely lost, because even when an observer makes consistent settings on the equality photometer the relation of his settings to those of the normal observer is uncertain.

Trained observers are needed with either photometer, but with the flicker any observer of fair ability can make definite sets even with large color differences, whereas on the Lummer-Brodhun photometer it is only the exceptional observer who can do so. Extensive investigations at the English National Physical Laboratory¹⁶ indicate that the final certainty of results is not increased by using the laborious "cascade" or step-by-step method of measurements to avoid sets with large color difference. Little is gained by such a procedure, unless the results of successive steps are agreed upon and made a practically independent standard for future use. This is the tendency of the present practice, and it appears that by this method fairly satisfactorily standards of successively higher and higher temperature may eventually be agreed upon by interlaboratory and international comparisons. No one can say with what degree of accuracy the values of these standards can be reproduced from the fundamental standards a few years hence, and of course, this method applies only to those color differences which can be thus built up step by step with concrete standards to preserve the values at each step.

The flicker photometer, on the other hand, affords a means of relatively precise comparison between lights of all degrees of color difference and makes possible the use of test readings for which average values, which should be highly reproducible, can be established.

In regard to relative results there appears to be no room for doubt that for sources having relatively high intensity at the blue end of the spectrum the values given by the flicker photometer as here used depart appreciably from those obtained with the Lummer-Brodhun as used in common practice, the difference probably being of the order of 3 per cent at the higher efficiencies reached by the present gas-filled lamps. It is, however, hardly proper to assume that the results obtained by either photometer are "right" and anything different is "wrong." The equality-of-brightness method of measurement is undoubtedly more closely related to the way in which the light is used, but it is by no means established that that method correctly indicates the relative usefulness of two kinds of light. It must be recognized that there is no one definite "correct" ratio between the intensities of two lights of different color. The relative candlepowers assigned to a carbon and a tungsten lamp, for example, depend to some extent

¹⁶ Paterson and Dudding, *Proc. Phys. Soc., London*, **27**, p. 263, 1915; and *Phil. Mag.* (6), **80**, p. 63, 1915.

on the conditions under which the measurements are made. The specification of conditions of measurement must be more or less arbitrary, and the results obtained can not be expected to be an exact indication of the value of different kinds of light under different conditions. Before we shall know much about the relative usefulness of different kinds of light much more experimental work must be done. An important prerequisite for such investigations or any others involving the comparison of the intensity of lights of very different color is a method which will enable different experimenters to make consistent measurements of the quantity which must serve as a basis for the comparison of their results. The usual equality-of-brightness method of comparison certainly does not fulfill this requirement. The flicker photometer at present furnishes the most promising method available.

For the standardizing laboratory, which is expected to reproduce results after the lapse of years when the observers available may be entirely different, the flicker photometer (used under definitely specified conditions), with a systematic method of determining the relation of each observer to the normal, promises to give to heterochromatic photometry a certainty which has appeared quite unattainable with other instruments. Besides giving this probable increased certainty in future reproduction of values, it reduces considerably the labor necessary to attain a given accuracy at the present time. Comparison of actual tests made in the routine work of the laboratory shows that even with relatively small color differences a given accuracy of reproduction of results requires several times as many measurements with the equality-of-brightness or the contrast photometer as with the flicker.

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WASHINGTON, January 15, 1916.

EMISSIVITY OF STRAIGHT AND HELICAL FILAMENTS OF TUNGSTEN

By W. W. Coblentz

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I. INTRODUCTION

The emissive properties of incandescent tungsten have been the subject of numerous investigations by various observers, the object being to find some law connecting the partition of spectral energy with wave length, similar to the Wien-Planck law of spectral radiation of a black body.

The present investigation was begun in January, 1914, but was interrupted, and not until during the past year was it possible to resume this work.

In a previous paper¹ on the radiation constants of metals the Wien equation, $E_{\lambda} = c_{\lambda} \lambda^{-5} e^{-c_2/\lambda T}$, was tentatively assumed for a working basis. The results obtained indicated quite conclusively that the distribution of energy in the emission spectrum of a metal can not be represented by an equation which is as simple as the Wien formula.²

In a subsequent investigation of the reflecting power of tungsten and other metals³ it was shown that a spectral-radiation formula

¹ This Bulletin, 8, p. 339; 1908.

² In spite of the specific statement that the assumption of the Wien equation was tentative and in spite of the conclusions that this equation is unsatisfactory, subsequent writers (e. g., Stubbs and Pridoux, Proc. Roy. Soc., A 87, p. 451, 1912) quoting this paper say that the Wien equation was "assumed to hold for metallic radiation."

³ This Bulletin, 7, p. 224; 1910. A recent investigation of the reflecting power of tungsten shows a band of selective reflection in the region of 0.7 μ with a conspicuous minimum at 0.8 μ . This depression in the reflection curve corresponds with a well-defined elevation in the emission spectrum, at 0.8 μ . These data will be published in a forthcoming paper.

of metals must contain factors which take into consideration the variation in emissivity (1—reflecting power) which is a function of the wave length and the temperature. For it is well established that in the infra-red the variation (decrease) in reflecting power⁴ with rise in temperature becomes perceptible at a wave length of about 2μ and is very marked for the region of the spectrum beyond 4μ . On the other hand, in the visible and ultra-violet region the most recent work⁵ seems to indicate that the emissivity decreases (reflecting power increases) with rise in temperature and decrease in wave length.

The reflecting-power curve obtained by the writer (*loc. cit.*) does not show elevations in the infra-red at 1.5μ and 2μ . Hence, it is highly improbable that the depressions in the spectral-energy curves, at these two points in the spectrum, observed by Nyswander,⁶ are due to selective emission, but are to be attributed to atmospheric absorption bands and to improper factors for reduction to the normal spectrum. Similarly, the unusual results of McCauley⁷ are open to question. For example, he found that at high temperatures the maximum emissivity of tantalum occurs "at a longer wave length than does a black body" (at the same temperature), which from a consideration of the reflecting-power data does not appear to be possible. He found that the metals investigated (tantalum, platinum, and palladium) acquired a minimum reflecting power in the region of 1.2μ , which become more marked with rise in temperature. In view of the fact that these data were obtained indirectly by computation, it seems more probable that the reflection minimum at 1.2μ is to be attributed to incomplete knowledge of the radiation constants used in the calculations. Such reflection minima would give rise to emission maxima in the spectral-energy curve, but no maxima have yet been observed. The point of interest in this investigation was the verification of the previous work just mentioned indicating that the Wien equation can not be applied to the spectral radiation from metals.

In a paper on the reflecting power of tungsten⁸ physical data were given showing why tungsten (and in fact all metals which have a lower reflecting power in the blue than in the red) radiates

⁴ Hagen and Rubens, *Sitzber. Akad. Wiss.*, XVI, p. 478, 1909; XXIII, p. 467, 1910.

⁵ Worthing, *Phys. Rev.*, 7, p. 497, 1916; Hulburt, *Jour. Franklin Inst.*, 182, p. 695, 1916; Weniger and Pfund, *Jour. Franklin Inst.*, 183, p. 354, 1917.

⁶ Nyswander, *Phys. Rev.*, 28, p. 438, 1909.

⁷ McCauley, *Astrophys. Jour.*, 27, p. 164, 1913.

⁸ This Bulletin, 7, p. 197, 1910. See Fig. 2, which, unfortunately, is drawn upon such a small scale that a small indentation at 0.8μ is not shown. In a recent investigation the slope of the reflecting-power curve in the region from 0.7μ to 1.5μ was found to vary for different samples of tungsten.

selectively in the visible portion of the spectrum in such a way as to make the amount of radiation, relative to that of a black body, greater in the blue than in the red.

II. RADIATION CONSTANTS OF A NITROGEN-FILLED TUNGSTEN LAMP

The appearance upon the market about three years ago of the gas-filled tungsten lamp afforded an opportunity for further investigation of the radiation constants of metals at much higher temperatures than were heretofore attainable.

The first measurements on a spiraled tungsten filament (20-mil wire) were made in January, 1914. The lamp was set to the same emissivity (color match) as a vacuum tungsten lamp operated at 1.2 w. p. m. h. c. The spectral-energy curves of these two types of lamps were observed with a vacuum spectrophotometer and fluorite prism, and the radiation constants were computed by the methods previously described.⁹

The spectral-energy curve of the helical filament superposed so closely upon the energy curve of the vacuum tungsten lamp that the lack of coincidence was then attributed to a difference in absorption of the glass bulbs,¹⁰ which is very marked for wave lengths greater than 2μ . However, from the recent work it appears that a more probable explanation is that the spirals, which were suspended in V-shaped loops, were so situated as to prevent much radiation, emitted from within the loop, from entering the spectrometer slit. For this reason the wave length of maximum emission ($\lambda = 1.18\mu$) of the helical filament was the same as that ($\lambda_m = 1.2\mu$) of the vacuum tungsten lamp when operated at a color match with the latter, and the radiation constant, α , mentioned below, was the same as that of the straight filament.

The data obtained in this test indicated, as in previous work, that no simple formula like the Wien equation can apply to the spectral radiation from metals. The wave length of maximum emission of the helical filament under normal energy input (968 watts; 110 volts) was $\lambda_m = 1.06\mu$.

The constant α ($\alpha - 1 = 4$ for a black body) was computed as in previous communications. The values of α for different points on the energy curve did not fluctuate so much as in previous work, varying from $\alpha = 6.56$ to 6.92 . The predominating values were of the order of $\alpha = 6.8$, which means that the emissivity of tung-

⁹ This Bulletin, 5, p. 339; 1908. In a future paper it is intended to give these constants of radiation at the true temperatures of tungsten, which from the recent work of Worthing appear to be determinable with a reasonable degree of accuracy.

¹⁰ Coblentz, *Lighting Journal*, 2, p. 35; February, 1914.

sten is proportional to about the 5.8 power ($\alpha - 1$) of the temperature. This value is closely the same as previously observed¹¹ by the writer on a straight filament, which may not have been made of as pure material as the present filament. In addition to the value ($\alpha = 6.8$) obtained on the commercial lamp, further data were computed from observations on the special lamps described in the present paper. Of these special lamps the one (No. 61) with the hairpin filament gave values varying from $\alpha = 6.9$ to 7.2 with an average value of about $\alpha = 7$. The lamp (No. 59) having the same kind of wire, but wound into a helix, gave values ranging from $\alpha = 6.7$ to 7.2 with an average value of about $\alpha = 6.8$. Lamp No. 53, having a spiraled filament, was operated on a 15 per cent overload ($\lambda_m = 1.01\mu$). It gave an average value of $\alpha = 6.5$. This is distinctly lower than the values obtained at the lower temperatures ($\lambda_m = 1.06\mu$), but it is in agreement with the previous investigation, and it is entirely in agreement with theory, which indicates that with rise in temperature the value of α should decrease and approach that of a black body, viz, $\alpha = 5$. It is to be understood, of course, that these computed data are qualitative in character and that the actual change in value with temperature is probably not so rapid as indicated by the computations. In fact, the experiments of Worthing¹² show but little variation in α for the temperature range between 1500° and 2500° abs. His value is $\alpha = 6.35$ (i. e., $\beta = 5.35$ as compared with $\beta = 4$ for a black body in the equation $E = \sigma T^\beta$), which is in remarkably close agreement with the values computed upon the basis of a spectral-energy (distribution) equation, the constants of which had to be assumed. In view of these facts, it is hoped to obtain more complete infra-red spectral-energy curves of tungsten, which will make it possible to compare the experimental with the theoretical spectral-energy distribution.

III. APPEARANCE OF AN INCANDESCENT HELIX OF TUNGSTEN WIRE

The appearance of an incandescent spiraled filament of tungsten (in a nitrogen-filled lamp) is very interesting and illustrates several novel problems in radiation. As shown in the photograph, Fig. 1, the inside of the turn of the helix is much brighter than the outside of the turn of the wire. In the present case measurements were made with a physical photometer¹³ on the same areas (indicated by the rectangular black spots in Fig. 2, A) on the inside and the outside of the turn of the wire.

¹¹ This Bulletin, 5, p. 372; 1908.

¹² Worthing, Phys. Rev., (2), 4, p. 535; 1914.

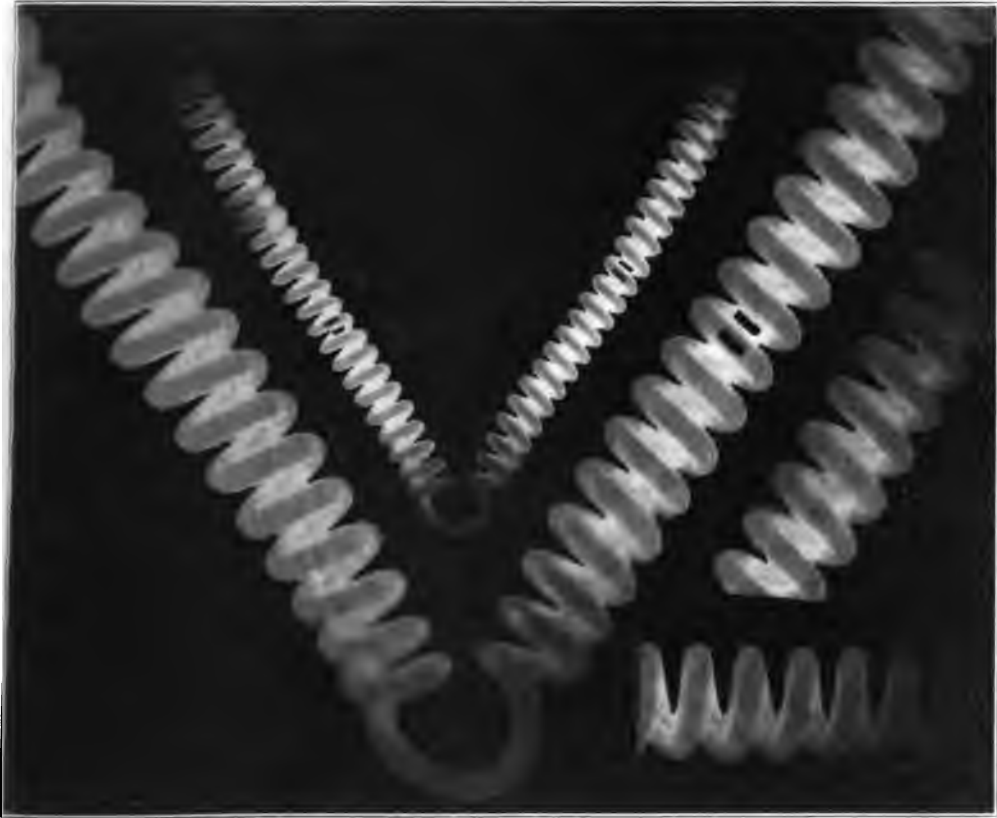


FIG. 1.—*Photograph of an incandescent helical filament of tungsten in a gas-filled lamp*

1. RATIO OF BRIGHTNESS

Using a mirror of 50 cm focal length, an enlarged image of the inside and the outside of the turn was focused upon a slit, 1 by 0.5 mm, placed so as to have the edges parallel with the turn of wire. Back of the slit was placed the physical photometer. Several series of measurements on the inside and the outside of the turn indicated that the light from within the helix was 1.87 times as bright as from the outside of the wire.

This increased brightness is produced no doubt by the emission and multiple reflection of light from adjacent turns of wire. For

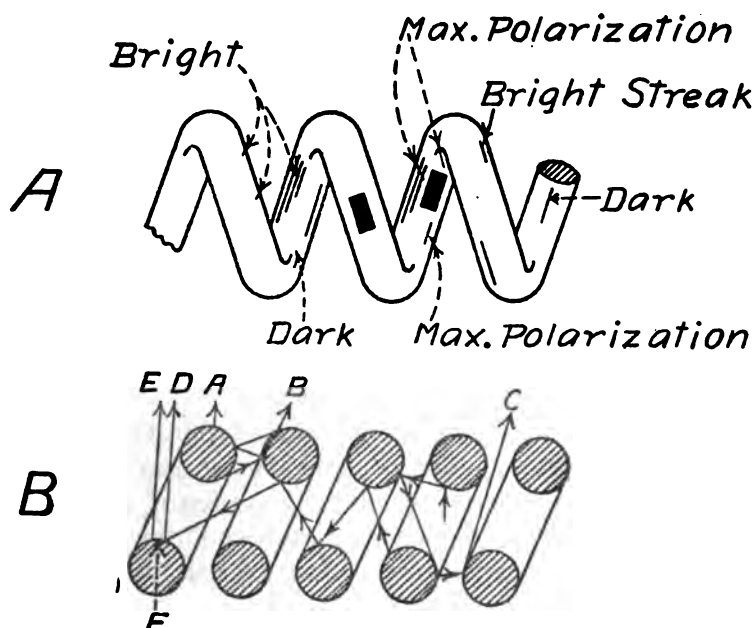


FIG. 2.—Illustrating the manner in which multiple reflection may occur within an incandescent helical filament

example, if the intensity of the light emitted directly by the inside of the turn (E in Fig. 2) be taken as unity (the same as the outside of the turn, A in Fig. 2), then the intensity of the image D would be $1 + R$ for the light emitted by the adjacent turn, $+R^2$ for the light emitted by the second turn, etc. Hence, the increased brilliancy may arise from reflections, $D = 1 + R + R^2 + R^3 + \dots$, as shown at C and D in Fig. 2. The increased brightness along the edge of the helical filament as compared with the appearance of the edge of a straight filament is no doubt due to multiple

¹² Ives and Kingsbury, *Phys. Rev.*, 6, p. 319; 1915. Coblentz, *Jour. Franklin Inst.*, 180, p. 335, 1915; 181, p. 233, 1916.

reflections as indicated at *B* in Fig. 2. Along the highest part of the inside of the turn *F*, in Fig. 2 (see also Fig. 1) and the regions marked "Dark" in Fig. 2 (multiple), reflections can not occur in the direction viewed, and the intensity of the light is quite as low as on the outside of the turn.¹⁴

On the assumption that the reflecting power of tungsten is $R=0.51$ at $\lambda=0.55\mu$, the increased brightness for three reflections is $(1+R+R^2+R^3)=1.90$ as compared with the observed value of 1.87. As will be noticed presently, making similar computations for the infra-red, the observed and calculated energy curves are found to be in remarkably close agreement.

2. TEMPERATURE RISE VERSUS BLACKENING OF RADIATION WITHIN THE HELIX¹⁵

In view of the fact that opinions differed as to whether the increased brightness within the helix was due to a higher temperature or to internal reflection, it is relevant to discuss this question and give some criteria to distinguish between these two possible explanations.

When this work was first undertaken the preliminary data on the heat conductivity of tungsten at high temperatures, by Worthing,¹⁶ had just been published, and from his data it seemed quite evident that the increased brightness could not be explained solely on the basis of a higher temperature within the helix. On the basis of these data the difference in temperature between the inside and outside of the turn would be less than 1° for a 0.5 mm filament, while under the most unfavorable condition Langmuir¹⁷ estimates that the temperature difference can not exceed 5° at 2900° Abs.

Furthermore, by differentiating the Wien equation we have $\frac{dE}{dT} = \frac{C_2 E}{\lambda T^2}$. Using $C_2=14\,350$, $\lambda=0.6\mu$ and temperature $T=2300^\circ$ Abs., it can easily be shown that $\frac{dE}{dT} = \left(\frac{14\,350\,E}{0.6 \times 2300^2} \right) = 0.0045\,E$, or a change of 1° corresponds to only 0.45 per cent in brightness. It would therefore require an increase in temperature of almost 200° in order to account for the observed increased

¹⁴ See also Langmuir, *Phys. Rev.*, **6**, p. 138, 1915, and Coblenz, *Elect. World*, **64**, p. 1048, 1914, for further illustrations and similar conclusions.

¹⁵ By "blackening" of radiation is meant a modification of the quality of the radiation from an incandescent substance, so that it has more nearly the properties of the radiation from a uniformly heated inclosure, or so-called "black body."

¹⁶ Worthing, *Phys. Rev.*, **8**, p. 67; Jan., 1914. Since then more complete data have been published, *Phys. Rev.*, **4**, p. 535; 1914.

¹⁷ Langmuir, *Phys. Rev.*, **6**, p. 150; 1915.

brightness of about 90 per cent within the helix. This would shift the maximum emission by the amount of 0.06μ to 0.09μ toward the short wave lengths as compared with the maximum of the radiation from the outside of the turn. Such a large shift has not been observed.

In view of the fact that the reflecting power of tungsten¹⁸ is lower in the blue than in the red, the intensity of the light emitted by one turn of the helix and reflected from another turn is more depleted in the blue than in the red. Hence, if the increased intensity of the light from within the helix is caused by multiple reflections, then its color should be redder; that is to say, the light from the outside of the turn should be relatively more intense in the blue than the radiation from within the helix. The difference is so small, however, that it requires special care in observing this effect.¹⁹

If the increased brightness observed within the helix is due to blackening of the radiation, caused by reflections, then the spectral-energy curve should be much higher and unsymmetrical on the long wave-length side of the maximum emission, as compared with the spectral radiation from the outside of the turn of the helical filament, for it is well established (both experimentally and theoretically) that in metals the spectral-energy curve in the infra-red falls very much below the spectral-energy curve of a black body at the same temperature.

On the other hand, as already stated, if the increased brightness, to be observed within the helix, is due to a higher temperature, then the wave length of maximum emission should be shorter than that of the radiation from the outside of the turn; and the energy curves, reduced to equal ordinates at a given wave length, say 0.6μ , should intersect unsymmetrically. If the temperature within the helix is but a few degrees higher than the outside of the turn, then the energy curves will superpose and intersect quite symmetrically.

The probable reason why, in the earlier part of this work on nitrogen-filled tungsten lamps containing helical filaments, it was

¹⁸ This bulletin, 7, p. 197; 1910.

¹⁹ For this reason the writer feels justified in adhering to the statement in the preliminary report (*Electrical World*, 64, p. 1048, 1914), viz, that "the quality of the light coming from within the helix is not appreciably modified." For, if the quality of the light was very markedly changed, it would not require great efforts in order to establish the fact. In a recent paper Shackelford (*Phys. Rev.*, 8, p. 470, November, 1916), by means of a special color-match test, was able to show that the light from within the helix is redder than that emitted from the outside of the turn. His ratio of diameter of helix to diameter of wire was much greater (five times greater) than that of the sample examined by the writer. This would reduce the crimping within the turn and lessen the blackening of the radiation from this cause.

not possible to settle the questions just enumerated is attributable in part to the fact that all the measurements were not made on the same lamp, as was done in the later work. However, to make the infra-red measurements upon the inside and outside of a single turn of the helix involves considerable difficulties which it was hoped could be avoided.

In the lower part of the V-shaped loop, Fig. 1, it may be noticed that the outside of several of the turns of the helix show bright spots indicating reflection of light emitted by the turns of wire which are on the opposite sides of the loop.

IV. COMPARISON OF THE RELATIVE EMISSIVITIES OF STRAIGHT AND HELICAL FILAMENTS

In order to carry out more fully the investigations just described, an opportunity was presented through the courtesy of the General Electric Research Laboratory, Schenectady, N. Y. (in February, 1914), to determine the relative energy distribution in the spectrum of straight ("hairpin") and helical filaments of tungsten in nitrogen-filled bulbs of the same kind and closely the same thickness of glass.

The tungsten wire used was 0.5 mm (20 mils) in diameter. In the helical filament the pitch was twice the diameter of the wire, while the inside diameter of the helix was two times the diameter of the wire. This was a rather small diameter and caused a noticeable crimping on the inside of the turn which might cause a blackening of the radiation.

One series of spectral-radiation measurements was made (with a vacuum spectrophotometer and fluorite prism) upon the helically wound filament and upon the hairpin filament set to the same emissivity, in the visible spectrum, as the outside of the turn of the helical filament. For this purpose the lamps were calibrated by operating the hairpin filament at its normal temperature ($\lambda_m = 1.06\mu$) and adjusting the current through the spiraled filament so that the outside of the surface of the turn had the same emissivity (the same intrinsic brilliancy) in the visible spectrum as the straight wire.

1. CALIBRATION OF LAMPS

The apparatus for calibrating the lamps is shown in Fig. 3. The nitrogen-filled lamps were placed (one at a time) at *L* and the image of the filament was viewed by means of a microscope, *O*, consisting of a 48 mm focal length objective and a low-power

(Zeiss No. 2) eyepiece, *E*, the combination having a magnification of about 20. A higher power gave poor definition owing to imperfections in the lamp bulbs. An iris diaphragm, at *I*, was useful in adjusting the focus on and within the spiral, and, when using the device as a pyrometer, for limiting the cone of rays in the field of view. A water-cooled screen was placed at *W*. An image of the filament *L* was brought to focus at *M* by means of two Zeiss achromatic telescope objectives *T*, having a high aperture (18 cm focal length, 6 cm diameter). These lenses give the best definition when used in pairs, with the source placed at the principal focus so that the light is parallel on leaving the first objective and on entering the second objective. An absorption screen of Schott's black glass was placed at *A* to reduce the

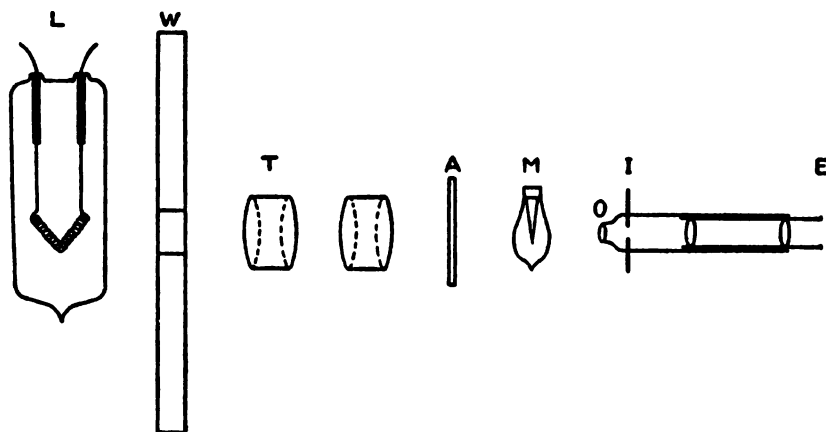


FIG. 3.—Arrangement of apparatus for calibrating the gas-filled lamps

intensity to that of the comparison lamp *M*, which was a 40-watt vacuum tungsten lamp.

The intensity of the light from *L* might have been reduced by means of a rotating sector or a nitrogen-filled lamp with hairpin filament might have been used at *M*, if one having homogeneous glass walls had been available. The lamp at *M* served, of course, merely as an intermediary in the substitution method of calibrating the lamps placed at *L*. The glass screen *A* is, of course, slightly selective in its transmission, so that when the image of the filament is viewed through red, green, and blue glass at *E* one experiences the same difficulties that are found in any optical pyrometer. This, however, does not interfere with the calibration so long as use is made of the same kind of wire in all the lamps.

2. POLARIZATION TESTS

An examination was made of the light from within the helix by using a Nicol prism in the eyepiece *E*, Fig. 3. As shown in Fig. 2 (section *A*), the regions of greatest brightness show marked polarization of the emergent light. The outside of the turn shows but little polarization. Hence, while some of the polarized light from within the filament might be due to emission, there is strong evidence that it is due primarily to reflection of light coming from adjacent filaments.

The highly polarized condition of the light is a further test, showing that the quality of the radiation from within the helix is quite different from that of a black body.

3. PYROMETRIC TESTS

It was found that when the hairpin filament (nitrogen-filled lamp No. 61) at *L* was operated at a predetermined temperature (giving a maximum emission at $\lambda_m = 1.06\mu$) the comparison lamp *M* disappeared against the image of the hairpin filament when operated on 41.3 (to 41.4) volts when viewed through red glass, on 43.6 (to 43.7) volts when viewed through green glass, and on 45.1 (to 45.2) volts for blue glass. The spiraled filament was substituted for the hairpin filament at *L* and its temperature raised until the comparison lamp on 41.3 volts (red glass) disappeared against the image of the outside surface of a turn of the spiraled filament. It was then found that there was a disappearance of the filament on 43.6 volts for green glass and 45.1 volts for blue glass, as in the case of the hairpin filament. Making similar settings on the brightest part of the inside surface of a turn of the helical filament, the current being held constant, the voltage of the comparison lamp *M* had to be raised to 48.6 volts for disappearance in the red light and to 52.4 volts in blue light. In both sets of measurements, on the inside and on the outside surface of the turn, the difference in voltage for disappearance when viewed through red and blue glass happens to be close to 3.8 volts.²⁰ By actual test the candlepower (data kindly furnished by the photometric division of this Bureau) of the comparison lamp was found to have doubled (1.6 to 3.2 cp) by this increase in voltage (7.3 volts) in setting on the inside and outside of the turn, using red glass. In increasing the voltage when com-

²⁰ In a preliminary report in the *Electrical World*, 64, p. 2048, 1914, these tests were discussed in the terminology appropriate to pyrometers, viz, "apparent temperature," but it was not intended to convey the idea that the increased brightness is due "solely to increased temperature." See further discussion in *Electrical World*, 69, p. 328, 1917.

paring the radiation from the inside of the filament, the spectral-energy distribution in the lamp *M* becomes relatively stronger in the blue. This method of attacking the problem was useful only in showing the difference in selective emission of different samples of tungsten wire, when the comparison was made in red and in blue light.

4. COMPARISON OF SPECTRAL-ENERGY CURVES (DATA OF 1914)

In spite of the care taken in the calibration of the lamps the results obtained with the spectral-energy curves did not prove to be very satisfactory. The shape of the spectral-energy curve of the hairpin filament differed but little from that of the composite radiation coming from within and from the outside of the spiraled filament. After reducing the energy curves to a common ordinate in the visible spectrum it was found that the area of the energy curve of the spiraled filament was only 7 per cent larger than that of the energy curve of the hairpin filament, due to "blackening" of the radiation in the region from 1 to 3μ , as already explained. While this proved, of course, the presence of some reflected light from within the helix, it did not account for the 90 per cent increase in the visible spectrum.

It was evident that in order to reach definite conclusions in this matter it would be necessary to determine the spectral-energy curves of a small area on the inside and outside of the turn of the helix. This is a much more difficult problem than the one in which observations are made on the radiation from a long filament because of the reduction in energy and difficulty of projecting a satisfactory image of the spiral upon the spectrometer slit. However, during the past year the work was undertaken anew, using a mirror spectrometer, fluorite prism, and bismuth-silver thermopile.²¹ The helical filament stood at a distance of about 2 m from the spectrometer slit, and for the investigation of the visible spectrum an enlarged image of the spiral was projected upon the spectrometer slit by means of the lens *T*, as shown in Fig. 3. For determining the infra-red energy curves the lens *T* was replaced by a concave silvered mirror of 50 cm focal length and a silvered plane mirror. This combination permitted the projection of an image of the spiraled filament upon the spectrometer slit (at *E* in Fig. 3) without distortion of the image. The spectrometer slit was 0.55 mm wide and its length was reduced to 1 mm by covering the upper and lower portions

²¹ This Bulletin, 9, p. 7, 1912; 11, p. 131, 1914.

with pieces of thick white cardboard, which were adjustable and which had sharp beveled edges. By means of an adjusting screw a uniformly illuminated image of a small portion of the inside or outside of the turn (shown diagrammatically by the dark rectangles in Fig. 2 (section A; also Fig. 1) could be projected upon the spectrometer slit.

5. ENERGY MEASUREMENTS IN THE VISIBLE SPECTRUM (DATA OF 1916)

The data for the visible spectrum are given in Fig. 4. Curve *I* gives the distribution of energy radiated from the inside of the

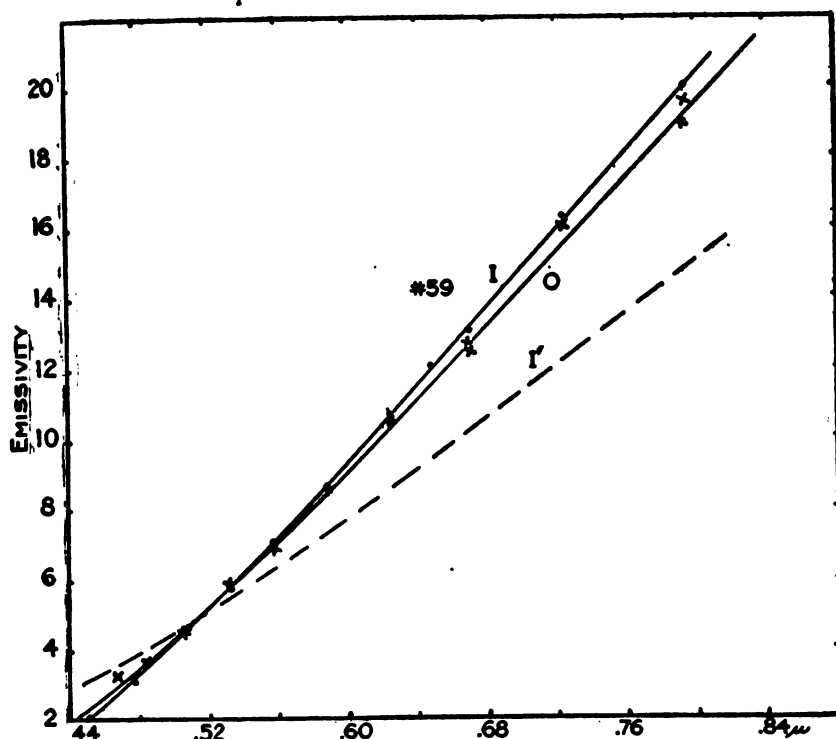


FIG. 4.—Distribution of energy radiated from the inside, *I*, and outside, *O*, of an incandescent helical filament of tungsten

turn. Similarly, curve *O* gives the energy distribution of the radiation from the outside of the turn of the helical filament of the tungsten. In the latter the ordinates were magnified 2.2 times in order to superpose the curves, and hence the errors are magnified as compared with curve *I*. In each curve the data represent two distinct series of measurements. It is to be noticed that the two curves intersect at a very acute angle, which is so

small that, in the earlier tests, the difference in the slope of the two curves was attributed to experimental errors.

The curve *O* indicates that the light emitted from the outside of the turn is relatively stronger in the blue than the light emanating from within the turn of the helix. That is to say, the light from within the helix is redder than that from the outside of the turn. As already explained, this indicates that the increased brightness of the light from within the helix is caused by reflections which deplete the blue as compared with the red.

If the temperature of the interior of the helix were enough higher than the outside of the turn, then the curves *I* and *O* would be interchanged. As already mentioned, the temperature within the helix would have to be 200° higher in order to account for the observed increased brightness of 90 per cent. This would produce a spectral-energy distribution, as shown by the dotted curve *I'* in Fig. 4. The depletion of the blue light by reflection within the helix produces an energy distribution corresponding to an apparently lower temperature (of 1° to 30°) than that of the outside of the turn. The actual difference in temperature, which, under the most unfavorable condition of plotting the curve *I* in Fig. 4, can not be higher than 37°, is not sufficient to account for the increased brightness of 90 per cent observed within the helix.

Using two wave lengths, λ_1 and λ_2 , in the visible spectrum the temperature *T*, of a black body may be computed from the Wien equation in the form

$$T = c_2 \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \log \epsilon + \log \left[\left(\frac{\lambda_2}{\lambda_1} \right)^5 \frac{E_2}{E_1} \right]$$

in which E_1 and E_2 are the observed emissivities at λ_1 and λ_2 , respectively.

This indicates a black-body temperature of 2550° C, using $c_2 = 14,350$. This is somewhat higher than the observed black-body temperature computed from the value of λ_m , to be discussed presently.

6. INFRA-RED SPECTRAL-ENERGY MEASUREMENTS (DATA OF 1916)

The distribution of the energy radiated from the inside of the turn of the spiral is shown in curve *I* of Fig. 5. Similarly, curve *O* gives the distribution of spectral energy radiated from the outside, *O*, of the turn of the wire. Curve *C* gives the energy distribution of the hairpin filament when set to the same bril-

liancy as the outside of the turn of the helical filament. The lack of coincidence of curves *O* and *C* may be due to a slightly higher temperature (perhaps 10°) of the hairpin filament.

The wave length of the maximum emission of the radiation from the outside of the turn of the helix is $\lambda_m = 1.06\mu$. The wave

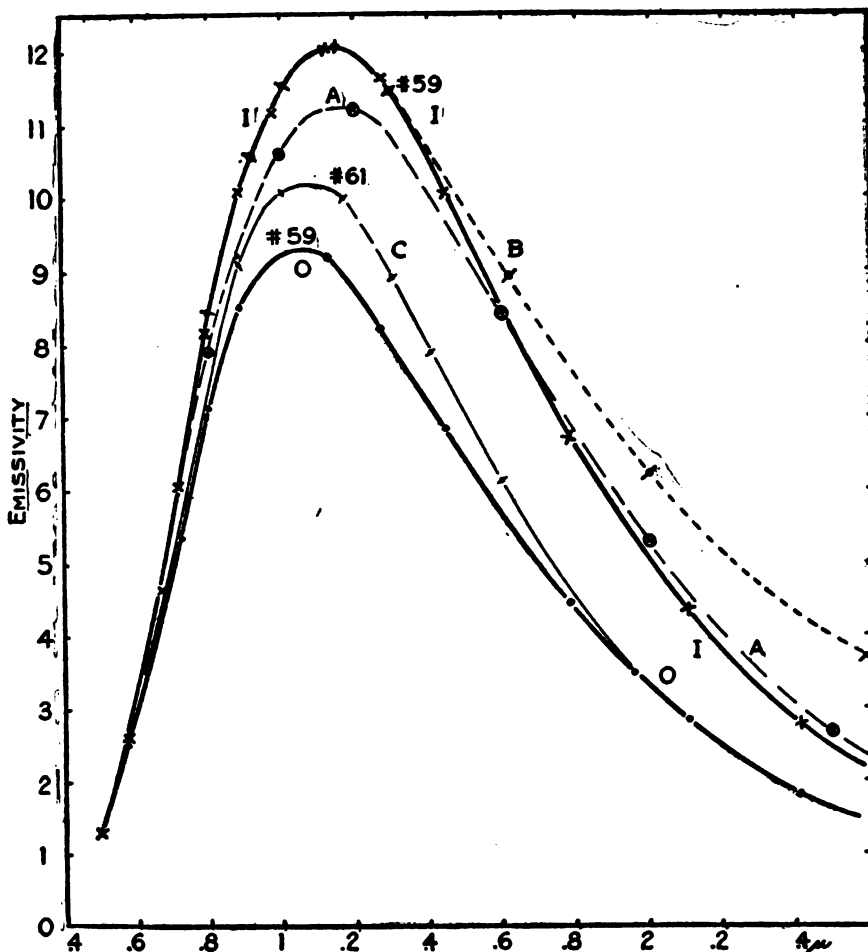


FIG. 5.—Distribution of energy radiated from the inside *I*, and outside, *O*, of the turn of a helical filament of tungsten

length of the maximum emission of the radiation from within the helix is $\lambda_m = 1.115\mu$. Using $\lambda_m T = 2894$ for a black body, which, of course, is not exactly the condition for curve *I*, Fig. 5, the value of the temperature is about 2596° to $2600^\circ T$ or, roughly, $2325^\circ C$, which is close to the estimated black-body temperature.

The most interesting application of the data illustrated in Fig. 5 is in the possibility of accounting for the increased brightness within the helix on the basis of multiple reflection of light. As already stated, the increased intensity in the visible spectrum of the light from within the helix can be accounted for on the assumption of three reflected images, which would increase the brightness by 90 per cent. A similar application of the reflecting-power data,²² in the infra-red, to the spectral-energy curve *O* gives curve *A*, which is in remarkably close coincidence with the observed curve *I* for the radiation from within the helix. This is especially marked in view of the fact that no correction has been applied to the reflecting-power data, which are for room temperature. Now, it is well known that in the infra-red the reflecting power of metals decreases with rise in temperature. The data of Hagen and Rubens²³ indicate that in the region of 2 to 3 μ the temperature coefficient of reflection is already perceptible, while beyond 4 μ it is quite large, so that the emissivity (100 reflectivity) of platinum increases about 40 per cent in a temperature range of about 1400° C.

Hence, an application of a temperature correction to the observed reflecting-power data would bring the computed curve closer to the observed curve. For example, at 2.5 μ , where a value of $R=0.84$ (20° C) was used, if the reflecting power decreased to $R=0.80$ at 2300° C, which is not improbable (in fact, it seems a very conservative estimate), then the observed curve *I* and the computed curve *A* would coincide within 2 per cent, which is as close as the experimental data are known at this point.

These data appear to furnish conclusive proof that the phenomenon of increased brightness within the helical filament can be accounted for by multiple reflections within the helix. A slight blackening of the radiation may be produced by crimping in case the filament is wound upon a small mandrel, but there is little or no evidence that the temperature within the helix is higher than on the outside of the turn of the wire.

While the increased radiation from within the helix can be accounted for on the basis of multiple reflection, the quality of the radiation in the infra-red is quite different from that of a black body at the same temperature. This is illustrated in Fig. 5, curve

²² This Bulletin, 7, p. 201; 1911. A redetermination of these reflecting-power data is in progress.

²³ Hagen and Rubens, Sitzber. Akad. Wiss., XXIII, p. 467; 1910. See also a paper by Weniger and Pfund, Jour. Franklin Inst., March, 1917, in which a decrease of 12 per cent in reflecting power, at 2 to 3 μ , has been published since the completion of this paper.

B , which gives the distribution of energy in the spectrum of a black body at the temperature ($T = 2600^\circ$) which gives the same $\lambda_m (=1.115\mu)$ as observed for the radiation from within the helix. The intensity of the radiation at 2.5μ is about 40 per cent higher (allowing 15 per cent for absorption of the glass) than the observed value. In other words, the apparent emissivity of the inside of the helix at 2.5μ is only 60 per cent that of a black body.

The ratio of the areas of the curves $I + O$ is 1.36, which indicates that "blackening" of the radiation within the helix has reduced the luminous efficiency by 36 per cent. The filament in a gas-filled lamp is wound helically in order to reduce the loss of energy by conduction and convection. To the writer the most striking observation is the increase in light efficiency introduced by actually increasing black-body conditions, when, as is well understood, an increase in light efficiency is to be sought by decreasing the infra-red radiation. In the present case the helical filament acts as a very thick, short filament as regards elimination of convection losses; and in spite of the fact that the luminous efficiency is reduced by blackening of the radiation from within the helix, there is a net gain in light efficiency due to the prevention of dissipation of energy by convection and conduction.

V. SUMMARY

In the present investigation data are given on the radiation from the inside and the outside of the turn of a helically wound tungsten filament in an atmosphere of nitrogen.

The intensity of the radiation from within the turn of the helix is from 90 to 100 per cent greater than from a similar area on the outside of the turn. This is accounted for on the basis of multiple reflection within the helix. This modifies the quality of the light so that it is redder than the light from the outside of the turn.

The observed infra-red measurements on the radiation from within the helix and the computed values (obtained on the basis of multiple reflection and the reflectivity of tungsten) are in close agreement. This is further evidence that the phenomenon is the result of multiple reflection.

Tests were made with a Nicol prism, which showed that the light from some parts of the inside of the filament is highly polarized, indicating that the quality of the light is quite different from that of a black body. The infra-red energy measurements also indicate that the quality of the radiation differs from that of a black body.

Computations of the radiation constant of tungsten indicate, as found in previous investigations, that the Wien equation is not applicable to the radiation from tungsten.

There is no indication that the temperature within the helix is higher than on the outside of the turn. A difference in temperature of 200° would be required to account for the observed difference in brightness of 90 per cent, whereas pyrometric, thermal conductivity and other measurements place this temperature difference at less than 5° .

Although the quality of the radiation has been modified by multiple reflection within the helix, it is not sufficiently similar to that of black-body radiation to permit its use in exact temperature measurements by sighting an optical pyrometer within the turns of the helix.

In conclusion, especial acknowledgment is due to W. B. Emerson for able assistance in this investigation.

WASHINGTON, December 27, 1916.

AN ANEROID CALORIMETER FOR SPECIFIC AND LATENT HEATS

By Nathan S. Osborne

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I. INTRODUCTION

The calorimeter here described has been designed primarily for the measurement of the specific heats and the latent heats of a certain class of materials adapted for use in the production of artificial refrigeration. These materials include ammonia, carbon dioxide, sulphur dioxide, methyl chloride, and ethyl chloride. At temperatures where the thermal properties are of importance to the engineer the vapor pressures of these materials range from less than 1 atmosphere to over 70 atmospheres.

Calorimetric measurements upon such active substances require special provision for mechanical isolation in addition to the means for measuring the heat-absorbing ability, because these thermal properties are indefinite unless the substance is kept under control as regards pressure and volume. The instrument must accomplish the thermal isolation of a system composed of the material, its container, and special parts for measuring quantities of heat exchanged, as well as the mechanical isolation and control of the material.

The principle of the unstirred or "aneroid" type of calorimeter, as developed by Eücken,¹ Nernst,² and associates, Harper,³ Dickinson,⁴ and others, utilizing instead of the convection of a stirred liquid the conduction of the parts for distributing heat developed electrically, offers an elegant means of fulfilling these requirements. A calorimeter of this type, which has been previously described,⁴ was used to measure the specific heat and latent heat of ice, and subsequently after being modified so as to adapt it to the material, it was used to determine the specific heat and heat of vaporization of ammonia.

While the results of these latter experiments would perhaps have met the immediate needs of the engineering profession, they did not promise to yield final values which would bear strictest scientific scrutiny, owing to certain instrumental characteristics which, though causing no difficulty in the experiments with ice, yet under the new conditions interfered with efficient completion of the intended experimental program and so left doubt as to the actual order of accuracy attained. In order to dispel all such doubt and establish values of these two important properties which should be authoritative, it was decided to construct another calorimeter embodying as a result of the experience gained, certain new features calculated to add to the convenience of operation, precision of measurement, and facility in executing an adequate experimental program.

The features in which the newly designed calorimeter was expected to excel its predecessor, briefly enumerated, are as follows:

1. Hermetically sealed calorimeter jacket to prevent ingress of the liquid of the thermostat bath. Long continued operation of the aforementioned aneroid calorimeter at widely varied tempera-

¹ Eücken, *Physikalische Zeitschrift*, 10, p. 586; 1909.

² Nernst, Korf, and Lindemann, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, p. 247; 1910.

³ Harper, *Physical Review* (2), 1, p. 469; 1913.

⁴ Dickinson and Osborne, *this Bulletin*, 12, p. 23; 1915.

tures (-40° to $+40^{\circ}$ C) invariably developed leaks in the submerged joint where the jacket was closed.

2. Location of the heating coil in the interior portion of the instrument instead of in the outer wall, thereby avoiding so extreme temperature gradients in the surface and favoring the control of thermal leakage.

3. Installation in the calorimeter of a strain-free type of platinum resistance thermometer. The performance of the built-in thermometer of the older calorimeter was unsatisfactory as to precision, it having hysteretic characteristics attributable to mechanical strain.

4. Provision of a mechanically operated device for cooling the calorimeter without introducing variability in its heat capacity.

5. Vertical instead of horizontal arrangement of heat-distributing vanes in the interior calorimetric space to promote uniformity in rate of evaporation. This arrangement was supplemented by a series of baffle plates in the upper part to insure dryness of the vapor.

6. Distribution of connections between calorimeter and jacket so as to render the lead conduction independent either of influences external to the jacket or of the nature of the temperature distribution within the calorimeter.

The instrument has been constructed and determinations have been made of the specific heat, heat of evaporation as well as the latent heat of compression of liquid ammonia. It is expected that similar measurements will be made on other refrigerating media in the near future.

II. GENERAL DESCRIPTION

The method by which the calorimeter was to be used is similar to that described in the previous work on the specific and latent heats⁵ of ice.

The material to be investigated is inclosed in a calorimeter of known heat capacity and a measured amount of energy supplied electrically. The initial and final temperatures being measured and the experiment being performed with minimum thermal leakage by the method of keeping the envelope at approximately the same temperature as the calorimeter surface, the data for specific heat of the material are obtained; or, the initial mass in the calorimeter and the amount of vapor removed without changing the temperature being determined, the data are obtained for heat of vaporization.

⁵ Dickinson and Osborne, this Bulletin, 12, p. 49; 1915.

A cylindrical metal shell of sufficient strength to withstand the vapor pressure of the material in the temperature range to be covered, provided with an electric heating coil and platinum resistance thermometer located in the central axis, is suspended axially within a metal jacket which is immersed in a thermally controlled liquid bath. An air space between the polished nickel surfaces of calorimeter and jacket furnishes thermal insulation. Two tubes extend from the top of the calorimeter through the jacket and liquid to the outside air, terminating in valves with unions. One of these tubes is intended for connection to pressure-indicating apparatus and the other for the introduction and removal of the material to be investigated.

By avoiding heavy metal connections across the air space and by suitably distributing upon the calorimeter surface those connections which are necessary, the part of the thermal leakage due to lead conduction is not only minimized, but also rendered less dependent upon disturbances of thermal equilibrium during experiments than would be the case if the connections were grouped indiscriminately. Thermoelements indicate relative surface temperatures of jacket and calorimeter, 10 junctions being distributed upon each surface. This permits close control of thermal leakage and correction for such as it not avoided.

Thermojunctions placed upon the connecting tubes indicate the temperature of these tubes at several points relative to a point on the jacket, for the purpose of ascertaining the temperature of the vapor expelled during vaporization experiments.*

III. DETAILED DESCRIPTION

1. CALORIMETER SHELL

The construction of the calorimeter may be seen by reference to the illustrations. Fig. 1 shows a section through the calorimeter and envelope, Fig. 2 the appearance of the principal parts before assembling, Fig. 3 the same parts partially assembled, while in Fig. 4 the assembled calorimeter is shown ready to be incased in the envelope.

The foundation of this shell consists of steel parts, united by threaded joints put together with tin.

The inside of the central tube is accessible at the bottom for the introduction of the heating coil and thermometer. Upon

* The detailed description which immediately follows, up to page 146, is devoted to a more minute description of the construction, and unless eager for such intimate knowledge, the reader may use his own discretion in omitting the text.

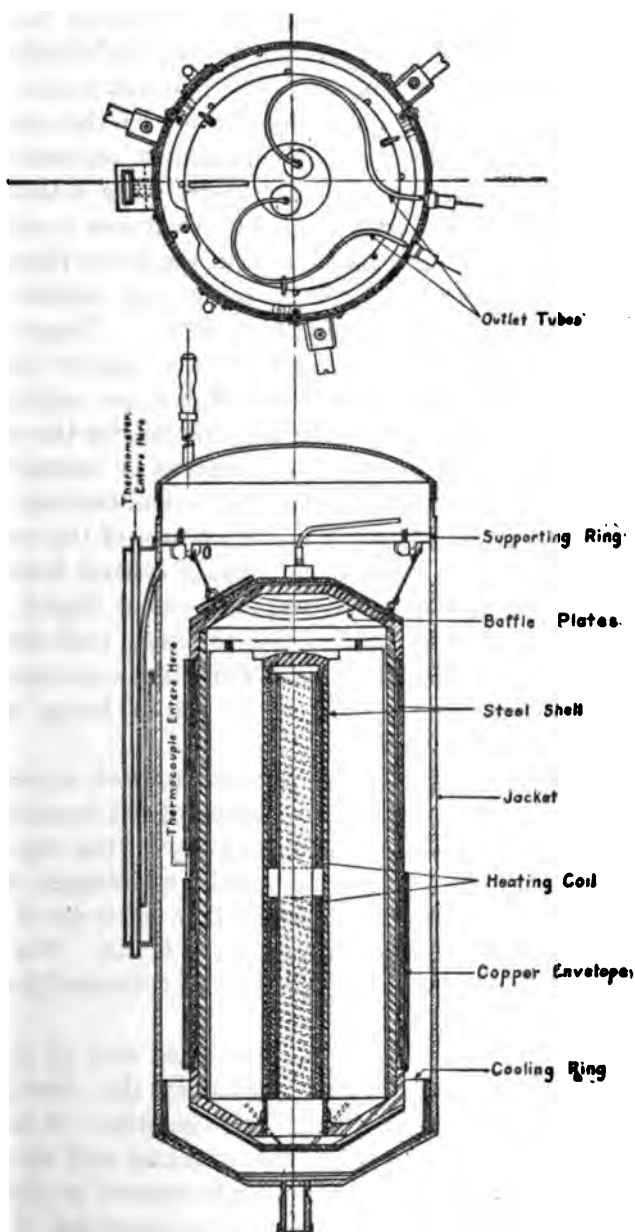


FIG. 1.—Sectional view of calorimeter and jacket. ($\frac{1}{2}$ size)

the outside of this tube, fastened in vertical slots with tin, are 12 radial vanes of tinned iron about 0.3 mm thick, extending to within about 1 mm of the surrounding cylindrical wall. These vanes are for the purpose of promoting the distribution of heat within the annular space containing the material under investigation. The vanes extend just above the top of the central tube. At this place are two flat circular baffle plates, separated about 2 mm by three small steel studs. The lower plate is united to the tops of the radial vanes with tin. The clearance to the outside wall is about 0.5 mm. A central hole in the lower plate and several holes in the upper one between center and outside furnish a tortuous passage for vapor coming from below. These two plates are intended to intercept any large drops of liquid which might be thrown up by vigorous boiling, should it occur, and also act as a thermal shield for the top of the calorimeter, for the prevention of excessive local heating either by radiation or convection.

A second set of four baffle plates of spherical contour separated about 2 mm are attached to the inside surface of the conical part of the calorimeter top. Each plate has a central hole and four slots at the edge so as to avoid trapping gas or liquid, but these passages are so sized and spaced that the main path through the plates is very tortuous, so as to make difficult the passage of liquid particles from below in a current of vapor being withdrawn through the outlets in the top.

In using the older calorimeter for evaporation experiments it was found that the system of horizontal plates installed for the double purpose of heat distribution and drying the vapor did not permit a uniform rate of evaporation to be maintained on account of the sudden increase of evaporating surface when one of the plates emerged from the descending surface of the liquid. The combination of vertical heat distributing plates and system of baffle plates avoids this fault.

The entire inner surface of the steel shell and of the various plates within were all tinned, using pure block tin. Much care and time were devoted to this process with the object of making the tin coating perfect, but the best result obtained still showed a few minute rust spots when the parts were immersed in water. It is possible that these spots were due to fine particles of the steel imbedded in the tin and not to actual holes through the tin coating.

A cylindrical shell of copper about 2 mm thick envelopes the steel shell between the shoulders formed by the steel end caps.



Copper wall sheath Steel top cap with copper sheath Baffle plates for top Steel shell Heating element Pt. thermometer Central tube with vanes and baffles Cu. sheath for bottom Steel bottom cap

FIG. 2.—Principal parts of calorimeter before assembling. ($\frac{1}{3}$ size)



FIG. 3.—Partially assembled calorimeter. ($\frac{2}{5}$ size)

The copper is separated from the steel by an air space of about 0.5 mm except for a narrow zone at either end where it is united to the steel shell by tin. Conical copper caps likewise envelop the ends, being held in place by screws. The upper cap is separated from the steel top by an air space except at the outer edge. The lower one fits the entire conical surface except for eight slots to accommodate electrical leads. The central space between the lower cap and the shell is large enough to permit electrical connections to be made. The purpose of these copper envelopes about the steel shell is to diminish irregularities in the distribution of the surface temperature such as result from localized heat exchanges inside the shell during experiments. It is apparent that the less abrupt the temperature variation upon the surface, the more perfectly can the surface temperature be ascertained with a given number of thermojunctions.

Three rings attached to the top steel cap and passing through slots in the edge of the copper cap provide for the suspension of the calorimeter in its envelope by fine steel wires.

The vertical ducts through which the electric conductors pass in leading from the heater and thermometer to the points where they leave the calorimeter surface consist of small copper tubes fastened with tin upon the surface of the copper sheath. A similar vertical tube is cut away for a length of 1 cm at the middle to admit the two branches of the integrating multiple thermocouple described elsewhere. A tube soldered on the top and a slot in the lower cap are provided for the two end junctions of the multiple thermocouple. (See Fig. 3.)

2. HEATING ELEMENT

The heating element consists of two parts mounted on a common support, each containing a 5-ohm noninductive resistance of insulated constantan wire so embedded in a copper cylinder as to promote the transfer of heat to the calorimeter. A distance of 1 cm between the inner ends of the mounted coils permits the heat developed to be transmitted either to the upper or lower part of the calorimeter by using the coils independently. By joining two of the terminals only three current leads to the outside are required.

The complete heating element fits closely into the central tube of the calorimeter, and is held in place by a small brass clip fastened to the steel bottom cap. The center of the heater forms a receptacle for the platinum resistance thermometer which is next described.

3. CALORIMETER THERMOMETER

The thermometer, Fig. 2, consists of a coil of the strain-free type previously described by Waidner and Burgess.⁷ The coil of Heraeus highest purity platinum wire 0.1 mm in diameter is about 8 cm long and 7 mm in diameter. The mica frame supporting this coil fits inside a thin brass tube. The thermometer is of the four-lead potential terminal type, the four copper leads being brought out through porcelain tubes fixed in a plug which is threaded into the brass containing tube. The entire thermometer coil is readily removable from its case and the incased thermometer fits as a unit into its receptacle in the center of the calorimeter.

4. JACKET

The jacket consists of a brass cylinder fitted with end caps to close top and bottom when soldered in place. The surfaces are nickel-plated and the interior is polished. It is shown in section in Fig. 1 and exterior view in Figs. 4 and 5.

5. SUPPORTING RING

The supporting ring for the calorimeter, Fig. 4, furnishes a means of handling the calorimeter with its various attached parts as a unit for assembling in the jacket. The ring slides closely into the jacket resting on three adjustable supports. The calorimeter is suspended from this ring by links of steel piano wire 0.25 mm in diameter and 2 cm long. A metal strip fastened to the ring extends downward in close contact with the surface of the jacket, being held in place by four guides. Attached longitudinally to this strip are seven copper tubular ducts for the electric leads. The object of these ducts is to bring the leads to the temperature of the jacket, thus minimizing the influence upon the thermal leakage, of the difference in temperature between the calorimeter and the room. By this device heat conducted by the leads from without the envelope is shunted to the jacket instead of reaching the calorimeter.

6. COOLING DEVICE

In use of a previous instrument of somewhat similar construction much time was consumed in cooling the calorimeter by thermal leakage across the insulating air space. The experiment was tried of fitting a small carbon dioxide refrigerating coil to the calorimeter, making connections to the outside by means of small

⁷ This Bulletin, 6, p. 155; 1910 (Scientific Paper No. 124).



- | | | |
|------------------------------|---|--------------------------------|
| 1. Supporting ring | 5. Outflow tubes with thermocouples | 8. Conduits for electric leads |
| 2. Electric leads | 6. Receptacle for t. c. junctions and auxiliary thermometer | 9. Lifter for cooling ring |
| 3. Integrating thermocouples | 7. Casing for receptacle | 10. Casing for ring lifter |
| 4. Cooling ring | | |

FIG. 4.—Calorimeter, cooling ring, and partly assembled jacket. ($\frac{1}{3}$ size)

thin tubes of low conductivity. This coil was a success in respect to satisfactory refrigeration, but seemed to reduce the calorimetric precision.

The cooling device for the present instrument consists of a copper ring (Fig. 4) having a cylindrical surface shaped to the inside of the jacket and a conical surface shaped to the calorimeter bottom, which can be raised to make contact with both calorimeter and jacket making a thermal short circuit of the insulation to permit rapid thermal leakage or lowered to the bottom of the jacket, leaving the calorimeter thermally insulated. The mechanism for moving the ring consists of three rods extending vertically through tubes to the top of the lagging outside the bath, where they enter a branched handle. The apertures at the top are protected from ingress of water vapor by telescoping joints. At the bottom the rods are threaded into lugs, which extend outward from the ring through slots in the jacket into the casings provided for this connection.

The operation of this device to cool the calorimeter consists in raising the ring into contact with the calorimeter bottom, then reducing the temperature of the jacket and when the calorimeter temperature has fallen sufficiently dropping the ring to the bottom of the jacket of which it then forms a part.

The arrangement has proven satisfactory in every way. The possibility of trouble from condensation of water from the atmosphere in the telescope joints of the lifting rods when operating at low temperatures was anticipated, but it was found that except in very humid conditions these joints remained above the dew point, and that the condensation of water within the tubes could always be avoided by maintaining a very slow stream of dry air entering the jacket through the two electric lead conduits and departing through the three lifter tubes.

7. ELECTRIC LEADS

Two tubular metal conduits bring the electric lead wires from within the jacket through the bath to the outside, the thermocouple leads passing through one and the thermometer and heating coil leads through the other. Outside the envelope each set of wires passes through a flexible rubber tube to a glass *T*, in which it is sealed with cement, leaving an air passage into the jacket through each conduit past the wires. Inside the jacket the ends of the conductors are spread out to permit soldered connections to be made with the leads from the calorimeter. The four ther-

monometer and three heater current leads extend downward through the tubes on the strip attached to the supporting ring (previously described) to the points where they leave the jacket to cross the air space. The conductors crossing this space are made of such a length as to permit a thermally symmetrical distribution of contacts with the calorimeter (see p. 136), keeping the conductors midway of the air space (Fig. 4). The potential leads of the heating coils join the current leads halfway between the points of contact of the current leads with the calorimeter and jacket.

On account of the failure in the heater of insulation which had been tested only at low voltage in the initial assembling of the calorimeter, all the connections to the thermometer and heater were, after repairs, required to show insulation resistance of several thousand megohms when tested at 500 volts. Sufficiently good insulation to withstand this test where the wires pass through the close-fitting copper tubes on the surfaces was obtained by wrapping the wires with several layers of thin, tough, shellac-coated paper well baked in place.

8. CONNECTIONS TO INTERIOR

Two tubes are provided for access to the interior of the calorimeter, one for filling or emptying and the other for connecting with a manometer.

It would have been advantageous to have located the valves in these connections at the boundary wall of the calorimeter, but this would have involved rather serious mechanical difficulties. Instead the tubes extend through the envelope to a point outside the lagging, where they terminate in accessible valves. This construction necessitates special precautions to avoid errors due to condensation of vapor or accumulation of liquid within the tubes.

The tubes where they cross the air space between the calorimeter and jacket are of steel 0.1 mm thick, 1.2 mm inside diameter, and 10 cm long, tin coated inside and out. They are made light in order that they may quickly acquire the temperature of vapor flowing out and also to avoid too great thermal conduction. The tubes are symmetrical and slope so as to drain toward the calorimeter.

The temperature of the outflow tube was found to be influenced by the thermocouple leads from without the jacket, owing to their nearness in the air space. The central portion of this tube was observed to be either warmer or cooler than the adjacent ends, depending on whether the thermocouple leads were heated or

cooled by conduction from the room. When this tube was thus cooled below the saturation temperature of the free surface in the calorimeter, which occurred only when the calorimeter was above room temperature, condensation would occur unless the tube were already full of liquid. This peculiarity was discovered after the determinations of the specific heat at saturation conditions were completed, and its effect is considered in discussing those results. A method of eliminating error from this cause was found and applied in the determinations of latent heat of vaporization. A heating coil placed on the thermocouple-lead conduit outside the bath enabled the temperature of the leads within the jacket to be kept either at or above the calorimeter temperature. Any liquid in the tube could thus be expelled before the beginning of an experiment and its withdrawal unevaporated be thus avoided.

The thermocouples for indicating the temperature conditions of the fluid in the tubes are seven in number, with a common reference junction, the branches being made of equal resistance. To minimize the lag, small (0.127 mm diameter; B. & S. gage No. 36) silk and shellac-covered copper and constantan wire were used, the junctions being hard soldered. One junction was placed on the center of the calorimeter top and three on each tube at points dividing the length into four equal parts. For electrical insulation the tubes were first wrapped with a thin layer of shellacked silk fiber at the places where the bare junctions were to be put, and then, to insure good thermal contact, the thermocouple wires laid longitudinally were bound in place with silk fiber and shellacked. The reference junction when in place is in good thermal contact with the jacket near the outflow tube.

Beyond the jacket, tubes are used of smaller internal diameter (about 0.6 mm). To allow these tubes to be kept at a temperature above that of the liquid bath through which they pass, a heating element is bound longitudinally on each and the whole sheathed with a copper tube, leaving a small air space. A thermoelement placed on each tube indicates the rise in temperature above the bath.

9. INTEGRATING THERMOCOUPLES

To provide for the indication of the relative surface temperatures of calorimeter and jacket at any instant, two integrating multiple thermocouples are employed. Each of these when used alone indicates the mean temperature of one of the surfaces relative to the temperature of the third body, and therefore when

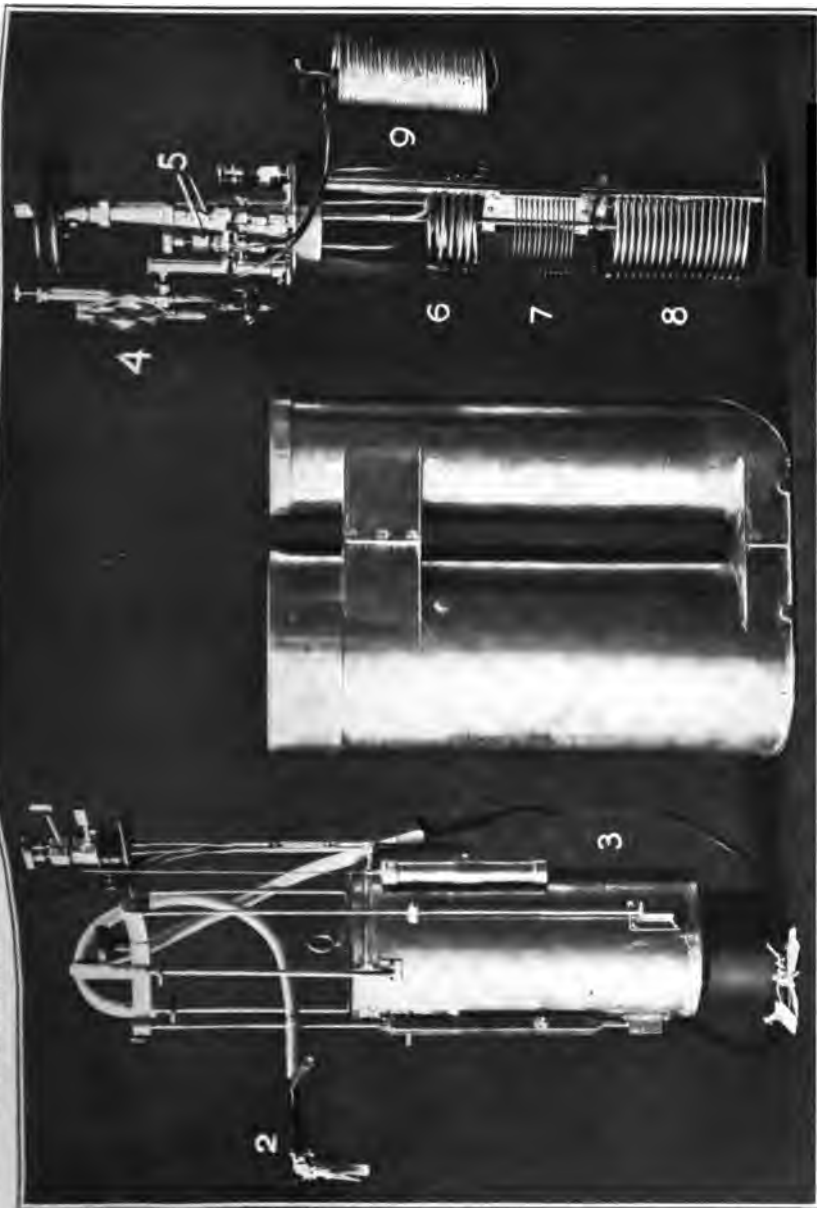
joined in series in the proper sense the combination indicates the temperature difference between the two surfaces. Each multiple couple consists of 10 elements joined in series. The elements are made of copper and constantan wires 0.2 mm in diameter with hard-soldered junctions. The 10 elements of each set are bound together into a unit which branches where it enters the vertical receiving tube on the calorimeter or jacket surface. Each of these branches bears 5 junctions; thus, 10 junctions are applied to the calorimeter surface and 10 to the jacket. Upon the calorimeter each of the 10 junctions is so located as to be influenced by one of the 10 equal elements of the total surface. Upon the jacket the same is true with the exception that the 2 junctions which properly belong on the top and bottom caps are, for convenience, placed in the receiving tube near the top and bottom. This is considered permissible on account of the uniformity of the jacket temperature, and is done to obviate the difficulties which would attend the soldering of these caps in place without doing injury to junctions upon their surfaces.

It is assumed that the symmetry of the cross section of the calorimeter and the nature of the stirring of the bath obviate the necessity for horizontal integration.

The two complementary sets of 10 junctions each enter the pair of adjacent copper tubes which extend through the jacket wall into a metal box which by means of an air space partially insulates the tubes from the bath. The two multiple thermocouples are made and assembled to be as nearly as possible similar, so as to avoid error from lag.

Extending through the ends of the box mentioned above and connected intimately with the two thermocouple tubes is a flat tube just large enough to receive the bulb of a platinum resistance thermometer of the standard calorimetric type inserted from without the bath so as to bring the bulb into close thermal union with the complementary junctions. The air space damps out the small fluctuations of the bath and enables the reference junctions and the thermometer to acquire the same temperature, thus making it possible by means of the thermocouples to make a thermometric transfer from the calorimeter to a second thermometer outside which is thus available as an independent calorimeter thermometer.

The method of experimenting is such that the temperature differences measured by the thermocouples are sufficiently small to per-



1. Valves terminating outlets
2. Electric leads
3. Thermocouple leads
4. Operating device for thermoregulator
5. Control valves for cooling coil
6. Cooling coil
7. Heating coil
8. Thermoregulator bulb
9. Heat interchanger

FIG. 5.—Calorimeter in jacket, bath container, and thermoregulating unit, in the order named. ($\frac{1}{5}$ size)

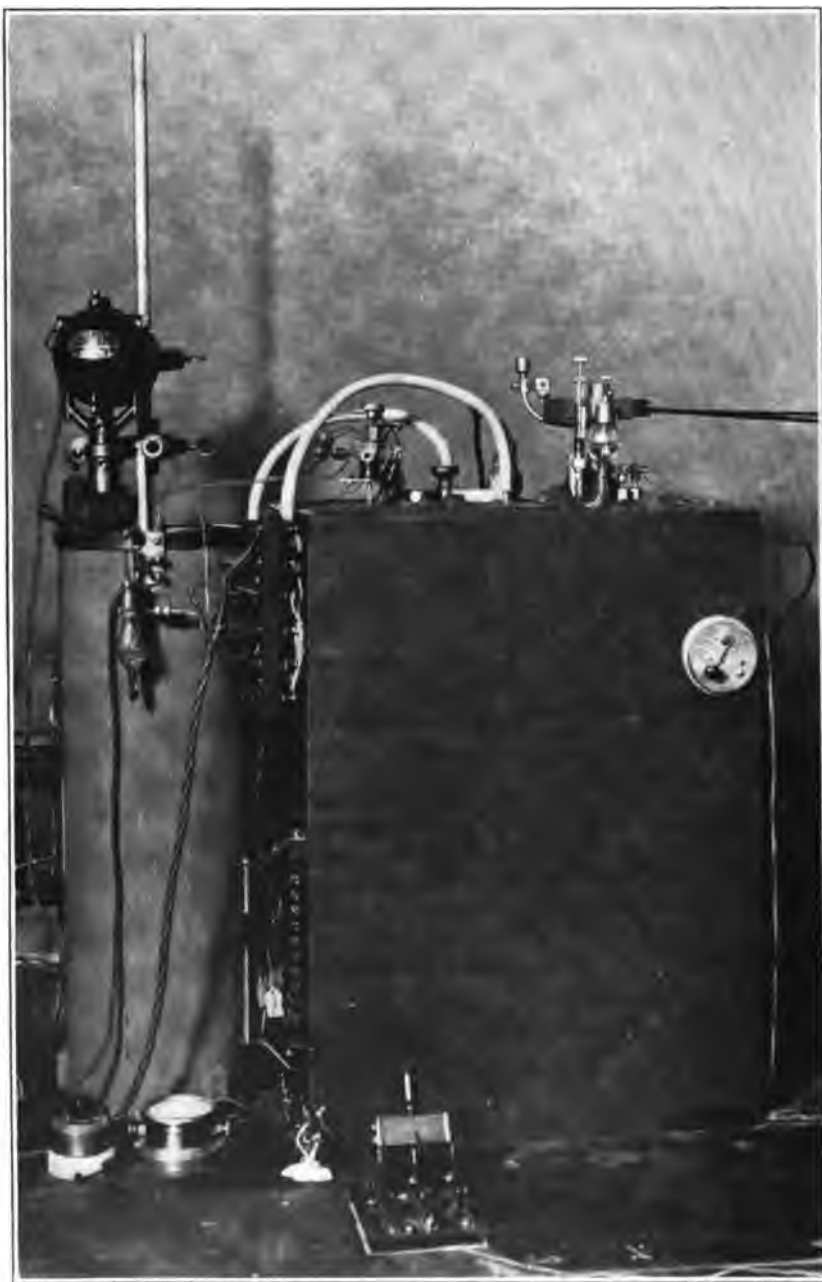


FIG. 6.—Complete calorimeter with condenser attached. ($\frac{1}{6}$ size)

mit their observation to the requisite degree of precision by direct galvanometer deflection. A somewhat complicated arrangement of switch connections is provided to enable the observer with a single galvanometer to shift quickly from any one of the numerous thermoelectric circuits to the Wheatstone bridge thermometric circuit. A thermoelectrically neutral coil of manganin wire with copper leads, wound on a heavy copper spool with close-fitting copper sheath and having a resistance equal to the thermocouple circuit is used to facilitate the resetting of the galvanometer zero. A reversing switch between the galvanometer and couple circuits enables this zero adjustment to be checked. A set of resistances connected through switches in the galvanometer circuit series and parallel give four different critically damped sensitivities on the thermocouples. A sliding multiple-point switch, designed for interchange of the thermocouples leading from the outflow tubes, permits the rapid reading of these couples in succession during evaporation experiments. The contact pieces are all made of copper to avoid thermoelectromotive forces.

10. THERMOREGULATED BATH

The function of the stirred bath is to control the jacket temperature. The container for this bath is a brass vessel consisting of two cylindrical vertical tubes connected at bottom and near the top by rectangular ports as shown in Fig. 5. The principal difference from previous models for a similar purpose consists in the separation of the well in which the thermoregulating element is located from the main tube which accommodates the part to be controlled. This prevents local irregularities due to direct conduction from the vicinity of the heating or cooling elements. The diameter of the container is about 4 cm larger than that of the jacket. A ring fixed inside just below the upper port supports the jacket in position by means of three studs, which lock in place beneath lugs. The cover for the main part is cut into three parts for convenience in assembling, the parts fitting together about the emerging tubes. The liquid used in the range -40 to $+40^{\circ}$ C is ordinary commercial gasoline.

The thermal control is effected by the unit shown in Fig. 5, which fits the adjacent tube or well in the container. The elements of this unit are a screw propeller driven by shaft and pulley; carbon dioxide refrigerating coil with control valves and heat exchanger; heating coil of 10 ohms (diameter, 0.25 mm; B. & S.

gage No. 26) asbestos covered constantan sheathed with copper, hermetically sealed to the binding posts; and thermoregulator consisting of tubular coil for bulb and contact-making head.

The lagging is contained in a wooden box shown in Fig. 6, and consists of cork board at bottom and top and granulated cork elsewhere. The minimum thickness of insulating material at any place exceeds 5 cm.

IV. ACCESSORIES

1. CONDENSER

As reservoirs for the material three containers, Fig. 7, were used. Each is fitted with a tube and valve, which connects to the valve giving access to the calorimeter. The manner of attachment is shown in Fig. 6. The three containers are made of steel, tin coated inside and out. The two smaller ones are used when weighing small samples where the large mass of the larger container would be inconvenient. They have a capacity each of about 180 cm³. The capacity of the large container is about 725 cm³.

The containers are interchangeable, and when used in connection with a second thermoregulated bath, as shown in Fig. 5, constitute a condenser and furnish a means of transferring material to or from the calorimeter by distillation.

This thermoregulated bath is contained in a cylindrical Dewar vessel within a metal sheath. A motor-driven stirrer furnishes circulation of the liquid. A carbon dioxide refrigerating coil, an electric-heating coil of 10 ohms resistance, and a thermoregulator with relay, etc., constitute the means of thermal control. The fluctuations of temperature in this bath sometimes reached 0.1°, but this regulation was sufficiently good for condenser temperature.

The container is suspended at the axis of the Dewar vessel and is immersed in the circulating liquid.

2. MANOMETER

For obtaining an approximate measure of the pressure of the vapor withdrawn in determinations of heat of vaporization a closed manometer, Fig. 8, was used.

The manometer is connected, through its valve, with one of the tubes which lead to the calorimeter, just as the condensers are connected. The frame carrying the manometer tube is mounted upon the flat front of the calorimeter box. The peculiar shape of



FIG. 7.—Steel containers for ammonia, etc. ($\frac{1}{8}$ size)



FIG. 8.—Manometer. ($\frac{1}{6}$ size)

the manometer tube results from disposition of the length of tubes used to the available space. The manometer is calibrated at the time of experiments by simultaneous observations of the temperature of the saturated fluid in the calorimeter and the position of the mercury column. As shown, the manometer is adapted for measurement of pressure of saturated vapor only up to the temperature of the room as condensation in the manometer tube would preclude its use for higher calorimeter temperatures.

V. AUXILIARY APPARATUS

1. WHEATSTONE BRIDGE

The Wheatstone bridge (B. S. No. 7481) used in connection with this calorimeter for observing the platinum thermometer resistances has been described in an earlier publication.⁸

2. ENERGY-MEASURING APPARATUS

A Leeds & Northrup type K potentiometer was used in conjunction with standard 0.1 ohm manganin coil and standard 100:1 volt box to measure the current and potential drop in the calorimeter heating coil. The time was recorded chronographically, using the second beats of the Riefler clock by use of the special-recording quick-throw switch described by Dickinson.⁹

VI. CALIBRATION

1. CALIBRATION OF RESISTANCE THERMOMETER

The ice point resistance R_0 and fundamental interval $R_{100} - R_0$ were determined by direct observations in ice and in steam, and the δ in the Callendar equation was determined by comparison at -50°C with a standard resistance thermometer (Bureau of Standards C 28) which was calibrated in ice, steam, and sulphur vapor. The scale defined¹⁰ by the thermometer is therefore the centigrade scale for Heraeus platinum of highest purity according to the equation

$$\theta = \frac{R_\theta - R_0}{R_{100} - R_0} 100 + \delta \left(\frac{\theta}{100} - 1 \right) \frac{\theta}{100}$$

when the sulphur boiling point at normal pressure is taken as 444.6° . The value of δ was found to be 1.48.

⁸ This Bulletin, 11, p. 571; 1914 (Scientific Paper No. 241).

⁹ This Bulletin, 11, p. 189; 1914 (Scientific Paper No. 230).

¹⁰ The departure from the ideal gas scale of the temperature scale so defined, down to -50°C , is not more than the limit of accuracy of existing gas thermometer data.

Subsequent to installing in the calorimeter, and again after reassembling on April 18, 1916, following repairs to the heating coil, the constants depending on the lead resistance were determined, which permit the thermometer to be used as a three-lead compensated type. The values of the ice point resistance R_0 and fundamental interval $R_{100} - R_0$ are given in Table 1.

TABLE 1
Constants of Calorimeter Thermometer No. 15046

Constants	Fundamental calibration as 4-lead potential terminal type	When used as 3-lead compensated type using battery lead designated C	
		Previous to Mar. 30, 1916	After Apr. 18, 1916
R_0	24.9688	24.9337	24.9325
$R_{100} - R_0$	9.7572	9.7431	9.7434

The auxiliary resistance thermometer used for check observations by thermocouple transfer was designated "Pt.3, B. S. No. 4725," and is one of the standard calorimetric platinum thermometers which have been in use at the Bureau for several years.

2. RATE OF THERMAL LEAKAGE

The rate of thermal leakage between the calorimeter and jacket was determined throughout the range of temperature in which the calorimeter was used and was checked at various times throughout the progress of the experimental work.

The method of determining this rate was to observe the change in temperature of the calorimeter by means of the resistance thermometer at regular intervals of time while the temperature difference between calorimeter and jacket was kept approximately constant, as indicated by readings of the integrating thermocouples. Determinations were made both with jacket temperature above and below the calorimeter temperature to ascertain whether influences external to the envelope—that is, "lead conduction from the room"—affected the leakage rate. No such effect was detected even at the temperatures furthest from room temperature, showing that the means adopted for intercepting external leakage were adequate. In observing, the readings were continued long enough to insure that a steady state

was reached. The rate of thermal leakage may be expressed by means of the following equations:

$$B = 0.146 + 0.000152\theta$$

$$A = 10.4 + 0.032\theta$$

B = coefficient of thermal leakage expressed in joules per minute per millimeter deflection of the galvanometer scale as used with integrating thermocouples.

A = coefficient of thermal leakage expressed in joules per minute per degree difference between calorimeter and jacket.

Experiments made to test whether the thermal leakage is correctly determined by the indications of the integrating thermocouples are of two kinds, one of which deals with the leakage occurring when the thermal equilibrium of the calorimeter itself is disturbed but little. In this method the jacket is so manipulated as to induce thermal leakage for a given period, and the actual heat lost or gained by the calorimeter is determined by the change in temperature of the calorimeter observed when in equilibrium before and after this leakage period. The increment so determined was in agreement with that calculated from periodic observations of the integrating thermocouples. The other test consisted of the comparison of the separate determinations of the heat capacity of the calorimeter using first one of the 5-ohm coils of the heater and then the other. Owing to the location of these coils, one in the upper and the other in the lower half of the calorimeter, the distribution of the surface temperature in the two measurements was different in the two cases. No difference in the measured heat capacity could be detected, showing that for these variations in surface conditions the error in the determination of the thermal leakage by the thermocouples was not appreciable.

3. THERMOCOUPLE CALIBRATION

Since the thermocouples were used only for the measurement of small differences of temperature, the necessity for precise calibration was obviated.

The galvanometer sensitivity was determined when in series with a resistance equal to that of the entire thermocouple circuit and used in connection with the known constants for the thermoelectric power of the thermocouple wire. Direct determinations of the combined thermocouple and galvanometer sensitivity agreed with the indirect within the limit of observational error, and the

indirect method was therefore used for the periodic checking as being the more convenient. The sensitivity when used with the integrating thermocouple corresponded at 0° to about 0.014 per millimeter deflection. When used to transfer from calorimeter to auxiliary thermometer at 0° , 0.0008 gave 1 mm deflection.

4. HEAT-CAPACITY DETERMINATIONS

The method of determining the heat capacity of the empty calorimeter or of the calorimeter containing a specimen for obtaining the specific heat is the same.

The first operation is the cooling of the calorimeter to the initial temperature of the first experiment of a series. This is accomplished by use of the cooling device previously described. The jacket is cooled as rapidly as possible by means of the refrigerating coil, and the cooling ring lifted to permit the transfer of the heat from calorimeter to jacket. When the calorimeter temperature is as low as desired the ring is lowered and then the jacket is brought into thermal equilibrium with the calorimeter, using the bath heating coil. Attainment of equilibrium in the calorimeter was ascertained by observing the constancy of temperature indicated by the resistance thermometer when the surfaces of calorimeter and jacket were at the same temperatures as shown by the integrating thermocouples. The empty calorimeter was somewhat slow in reaching equilibrium. Usually 50 minutes were sufficient to reach equilibrium to within 0.001 after a change in temperature. When the calorimeter contained liquid half this time sufficed.

When in equilibrium the initial temperature of the calorimeter is observed by measuring the resistance of the platinum resistance thermometer with the Wheatstone bridge. In addition, or as alternative, the temperature is determined by the outside resistance thermometer in conjunction with the thermocouple reaching from thermometer receptacle to the calorimeter.

The calorimeter heating current is furnished by a large storage battery. This current is set to the proper value in an auxiliary coil, equal in resistance to the calorimeter coil, so connected to a quick throw switch that the current may be instantaneously transferred to the calorimeter coil without any considerable change in value. The potentiometer used to measure the current and potential drop is balanced against the standard cell.

The heating current is thrown on the calorimeter heater at the beginning of a minute, the instant being recorded automatically on a chronograph. Throughout this period of the experiment, and until equilibrium is again established, the integrating surface thermocouples are observed at equal periods of one minute. The current in the jacket heater is controlled by hand regulation to keep the jacket temperature equal to that of the calorimeter. The thermocouple readings are recorded to the nearest millimeter and the deviations from zero summed at once. It is found to be very easy to make the sum of these deviations for an entire experiment equal to zero, thus eliminating from the computations any correction for thermal leakage, and this is done in practically all heat-capacity experiments. The galvanometer zero is periodically set to the nearest tenth millimeter. Potentiometer readings of current and potential drop in the calorimeter heating coil are taken alternately at equal intervals throughout the period of heating. At the end of a predetermined time interval the current is thrown off and the time again automatically recorded. The jacket is again brought under control of the thermostat at the temperature of the calorimeter and after waiting for the calorimeter to reach equilibrium its final temperature is observed as before. This observation may be taken as the initial temperature of the next experiment and the procedure repeated until completion of the series. The data obtained are:

- (a) Initial and final resistance of the thermometer giving temperature change.
- (b) Series of values of current and potential drop giving the rate of energy supplied electrically to heat the calorimeter.
- (c) Duration of the energy supply.
- (d) Record of thermocouple readings of temperature difference between calorimeter and jacket surfaces from which the amount of thermal leakage during the experiment is determined, usually made zero by manipulation.

5. EXPERIMENTAL RESULTS

The experimental data corrected for instrumental errors are given, together with the reductions to obtain the heat capacity, in Table 2. The determinations are divided into two series corresponding to the period before and after repairs to the heating coil.

TABLE 2
Heat Capacity of Calorimeter
BEFORE REPAIRS

Date	Exp. No.	Mean R. Cal. Therm.	Mean θ	$\Delta\theta$ Cal. Therm. 15046	$\Delta\theta$ Pt. 4725	Mean $\Delta\theta$	Heating cell used	Mean current	Mean P. D.	Duration of heating current	Energy	Heat capacity of calorimeter
1916		Ohms	$^{\circ}\text{C}$	Ohms	Ohms	Ohms		Amps.	Volts	Secs.	Joules	J/deg.
Feb. 11	1	29.3499	+44.96	10.124		10.124	10-ohm..	1.3059	13.143	600.00	10298	1017.2
Feb. 12	1	25.7379	+ 8.14	10.479		10.479	...do....	1.3066	13.128	600.09	10293	982.3
	2	26.7643	+18.58	10.364		10.364	...do....	1.3058	13.129	600.00	10286	992.5
Feb. 15	1	20.4315	-45.24	11.173		11.173	...do....	1.3073	13.105	600.00	10279	920.0
Feb. 16	2	22.2694	-26.82		9.860	9.860	Lewer 5.	1.7566	8.840	599.95	9317	944.9
Feb. 17	1	20.8346	-41.21	9.995	9.993	9.994	...do....	1.7510	8.804	599.95	9249	925.2
	2	21.8240	-31.31	9.807	9.806	9.807	...do....	1.7462	8.784	600.09	9205	938.6
	3	22.7947	-21.56	9.680	9.679	9.680	...do....	1.7452	8.784	599.97	9197	950.2
	4	23.7508	-11.82	9.565	9.563	9.564	...do....	1.7442	8.784	600.04	9193	961.0
	5	24.7212	- 2.15	10.025	10.026	10.025	...do....	1.7954	9.047	600.05	9747	972.2
Feb. 18	1	26.0946	+11.77	9.978	9.978	9.978	...do....	1.8034	9.090	600.03	9836	985.8
	2	27.0704	+21.68	9.858	9.854	9.856	...do....	1.8006	9.082	600.04	9813	995.6
	3	28.0569	+31.78	10.252	10.252	10.252	...do....	1.8450	9.312	600.00	10309	1005.5
	4	29.0552	+41.94	10.160	10.161	10.160	...do....	1.8445	9.316	600.08	10311	1014.9
	5	30.0415	+52.16	10.070	10.072	10.071	...do....	1.8438	9.316	599.98	10306	1023.3
Feb. 19	1	20.3308	-46.24	11.231		11.231	10-ohm..	1.3099	13.129	599.96	10318	918.7
	2	21.4152	-35.40	10.441		10.441	...do....	1.2726	12.762	599.96	9744	933.2
	3	22.4747	-24.78	10.805		10.805	...do....	1.3036	13.079	599.97	10230	946.7
	4	23.5417	-14.05	10.658		10.658	...do....	1.3028	13.077	600.01	10222	959.1
Feb. 21	1	24.4684	- 4.71	10.711		10.711	...do....	1.3125	13.180	599.96	10379	969.0
	2	25.5193	+ 5.93	10.550		10.550	...do....	1.3098	13.159	600.08	10343	980.4
	3	26.5520	+16.41	10.408		10.408	...do....	1.3076	13.142	600.00	10310	990.6
	4	27.5687	+26.75	10.287		10.287	...do....	1.3062	13.133	600.03	10293	1000.6
	5	28.5706	+36.99	10.170		10.170	...do....	1.3045	13.124	600.04	10273	1010.1
Mar. 8	1	27.4352	+25.39	8.890	8.885	8.888	Upper 5.	1.7141	8.630	600.12	8878	990.8
	2	28.3052	+34.27		8.849	8.849	...do....	1.7138	8.632	602.16	8908	1006.7
After emptying first filling												
Mar. 17	1	27.1856	+22.85		9.827	9.827	10-ohm..	1.2740	12.808	600.00	9790	996.2
After emptying second filling												
Mar. 27	1	20.4692	-44.86		10.668	10.668	10-ohm..	1.2792	12.820	599.83	9837	922.1
	2	21.5276	-34.28		10.492	10.492	...do....	1.2772	12.807	600.04	9815	935.5
	3	22.5658	-23.86		10.332	10.332	...do....	1.2754	12.796	600.24	9796	948.1
Mar. 28	1	23.2541	-16.95	10.178	10.175	10.177	...do....	1.2709	12.756	600.05	9728	955.9
	2	24.2573	- 6.83	10.045	10.046	10.045	...do....	1.2699	12.752	599.95	9716	967.2
	3	25.2453	+ 3.15	9.927	9.923	9.925	...do....	1.2686	12.752	599.93	9705	977.8
	4	26.2189	+13.02	9.814	9.816	9.815	...do....	1.2678	12.740	599.97	9691	987.3
	5	27.1789	+22.78	9.708	9.711	9.709	...do....	1.2671	12.738	600.01	9684	997.4

TABLE 2—Continued

Heat Capacity of Calorimeter—Continued

AFTER REPAIRS

Date	Exp. No.	Mean R Cal. Therm.	Mean θ	$\Delta\theta$ Cal. Therm. 15046	$\Delta\theta$ Pt. 4725	Mean $\Delta\theta$	Heating coil used	Mean current	Mean P. D.	Duration of heating current	Energy	Heat capacity of calorimeter
1916		Ohms	$^{\circ}\text{C}$	Ohms	Ohms	Ohms		Amps.	Volts	Secs.	Joules	J/deg.
Apr. 19	1	20.4324	-45.22	11.378	11.377	11.378	10-ohm...	1.3196	13.197	599.95	10449	918.4
	2	21.5619	-33.93	11.209	11.207	11.208	...do....	1.3196	13.205	600.27	10460	933.2
Apr. 24	1	22.1336	-28.20	11.301	11.292	11.296	...do....	1.3301	13.312	600.05	10624	940.5
	2	23.2514	-16.96	11.161	11.159	11.160	...do....	1.3310	13.333	600.14	10630	954.3
Apr. 25	1	23.8154	-11.27	11.000	11.001	11.000	...do....	1.3258	13.285	599.95	10567	960.6
	2	24.8980	- .35	10.862	10.861	10.862	...do....	1.3248	13.282	599.95	10557	971.9
	3	25.9636	+10.44	10.722	10.717	10.720	...do....	1.3233	13.274	600.00	10539	983.1
	4	27.0126	+21.10	10.596	10.600	10.598	...do....	1.3222	13.270	600.01	10528	993.4
Apr. 26	1	27.5239	+26.31	10.542	10.542	10.542	...do....	1.3218	13.270	599.99	10524	998.3
	2	28.5499	+36.79	10.421	10.423	10.421	...do....	1.3202	13.260	600.06	10505	1008.1
	3	29.5618	+47.14	10.301	10.300	10.301	...do....	1.3185	13.249	600.00	10481	1017.5

After emptying

July 5	1	20.7806	-41.80	11.115	11.112	11.114	10-ohm...	1.3076	13.081	600.00	10263	923.4
	2	21.8821	-30.72	10.933	10.930	10.932	...do....	1.3065	13.079	599.95	10252	937.8
	3	22.9629	-19.86	10.768	10.764	10.766	...do....	1.3053	13.075	599.86	10238	950.9
	4	24.0244	- 9.17	10.615	10.616	10.615	...do....	1.3042	13.071	600.03	10229	963.6
July 6	1	24.5432	- 3.94	10.392	10.389	10.391	...do....	1.2930	12.963	600.02	10057	967.9
	2	25.5644	+ 6.39	10.259	10.256	10.258	...do....	1.2927	12.957	600.01	10050	979.7
	3	26.5699	+16.60	10.149	10.147	10.148	...do....	1.2911	12.956	600.03	10037	989.1

The reductions are made according to the formula

$$N = \frac{IEt' + Bt_2h}{KAR}$$

The notation used:

N = heat capacity of the calorimeter at temperature θ , in joules per degree.

I = current, in amperes ¹¹ (mean value).

E = potential drop, in volts ¹² (mean value).

¹¹ The ampere, which is here used only as an intermediary unit, is determined by the relation $I = \frac{E}{R}$ where I = number of amperes.

¹² Where numerical values are given the joule used in this paper is determined by the relation $\frac{Q}{t} = \frac{E^2}{R}$ where Q is the number of joules transformed into heat in a given electric circuit in t seconds, E the number of volts potential drop, and R the number of ohms resistance; taking 1 volt = $\frac{1}{1.07830} \times \text{emf}$ of mean Weston normal cell at 20°C , and 1 ohm = resistance at 0°C of 106,300 cm of uniform mercury column 14.452 g in mass. The difference between the international joule, realized thus, and the absolute joule is, according to present evidence, perhaps 1 part in 3000. (Bureau of Standards Circular No. 60, 1st ed., p. 56; 1916.)

t = time.

t_1 = time at instant of starting the heating current when calorimeter is at θ_1 , t_1 is usually taken as zero.

t' = time of stopping the heating current, or duration of heating current when $t_1 = 0$.

t_2 = time when final equilibrium temperature θ_2 is observed, or duration of entire experiment when $t_1 = 0$.

h = mean thermocouple indication of the galvanometer scale during entire experiment, in millimeters.

B = coefficient of thermal leakage in joules per minute per millimeter galvanometer deflection.

R_1 = initial resistance of thermometer, in ohms.

R_2 = final resistance of thermometer, in ohms.

ΔR = change in thermometer resistance during experiment, in ohms.

K = difference factor for resistance thermometer, i. e., $\frac{\Delta\theta}{\Delta R}$

In the determinations recorded in Table 2 the mean thermocouple indication, h , during the entire experiment was in each case made zero and is therefore omitted. The duration of each experiment was about 60 minutes.

TABLE 3

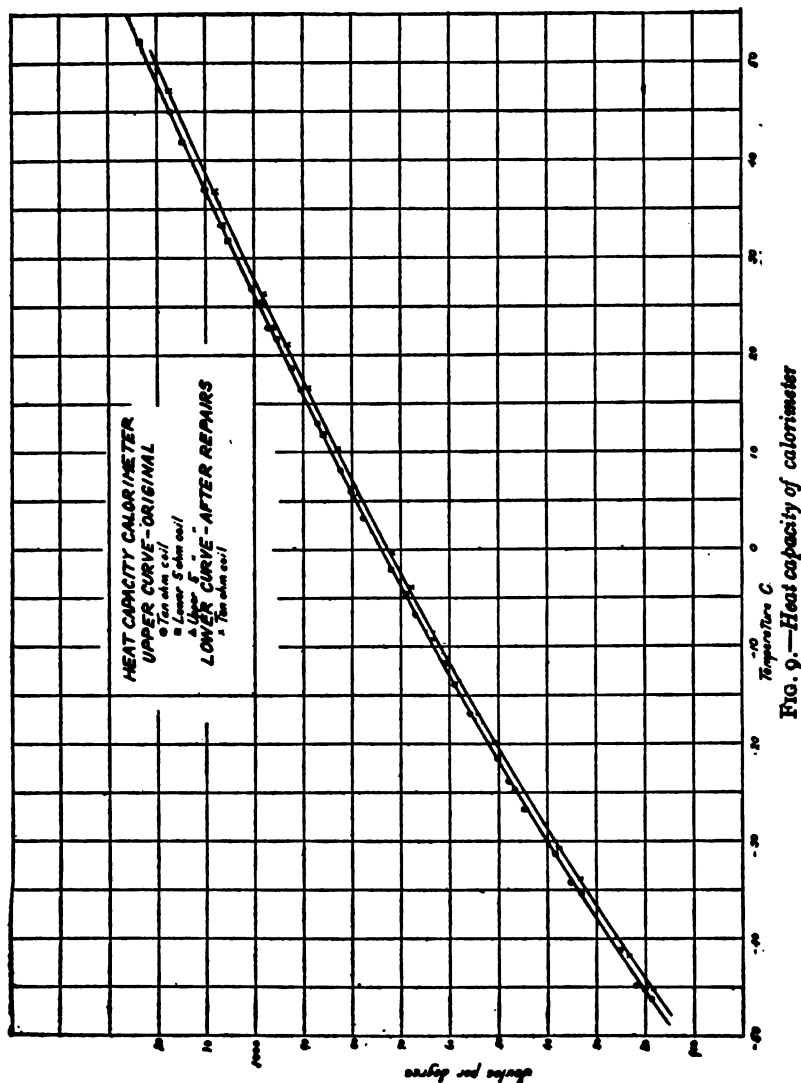
Journal of Experiments with Calorimeter, Showing the Distribution of Heat-Capacity Determinations

Heat capacity of empty calorimeter	Specific heat of liquid ammonia	Latent heat of vaporization of ammonia
Feb. 11 to Mar. 8, 1916..... Mar. 17 to Mar. 26, 1916..... Mar. 31, 1916.....	Mar. 13 to Mar. 16, 1916..... Mar. 27 to Mar. 28, 1916.....	
Heating coil failed; repairs made		
Apr. 19 to Apr. 26, 1916..... July 5 to July 6, 1916.....		Apr. 26 to July 5, 1916.

By noting the dates of the several experiments it is seen that several groups have been made. These groups bear a relation to the various fillings of the calorimeter for determinations of specific heat and latent heat of vaporization.

In the heat-capacity determinations experiments were made using each 5-ohm section of the heating coil alone, and also using both together.

In regard to the precision of the measurements of the heat capacity of the calorimeter, except for the actual change which



resulted from the repairs, the maximum deviation from the mean of any single measurement was not more than 1 part per 1000, and the average deviation from the mean was not more than 3 parts in 10 000. The results of all the determinations are shown graphically in Fig. 9.

VII. APPLICATIONS

The calorimeter has been used to determine the specific heat of liquid ammonia at saturation pressure and at constant pressure, the latent heat of vaporization of liquid ammonia, and the latent heat of compression of liquid ammonia.

The results of these determinations will be presented in separate papers in this Bulletin. It may be stated here that the precision of the measurements of specific heats and heat of vaporization is of the order of one-tenth per cent, while the heat of compression, which is of a much smaller magnitude, is determined with a precision of 3 to 5 per cent. The determinations cover the range of temperature from -40 to $+40^{\circ}\text{C}$.

The author is especially indebted for assistance in the design and construction to Dr. H. C. Dickinson; for assistance in the experimental work to M. S. Van Dusen, and for the excellence of the mechanical execution to F. Knoop, mechanician.

VIII. SUMMARY

The principle of the unstirred or "aneroid" type of calorimeter has been embodied in an instrument especially designed for determinations of the specific heat and latent heat of several substances in general use as refrigerating media.

Heat developed electrically in a coil located in the central axis of the cylindrical shell comprising the calorimeter is distributed by conduction to the calorimeter and contents whose initial and final temperatures when in thermal equilibrium are measured by a platinum resistance thermometer.

Heat from other sources is excluded by enveloping the calorimeter with a metal jacket separated from it by an air space and keeping this jacket during measurements at the same temperature as the calorimeter surface, using multiple thermocouples to indicate this equality.

The calorimeter is adapted for use between -50 and $+50^{\circ}\text{C}$ and for pressures up to 70 atmospheres in experiments where the measured heat added is used either to change the temperature of the contents or to evaporate a portion of the contents withdrawn as superheated vapor, in the first case the specific heat and in the

second the latent heat of vaporization being obtained when proper corrections are made. Such experiments are described in separate papers, this paper being devoted to the description of the instrument and its calibration.

Several features to which special attention has been given are:

1. Refinements to reduce errors in the evaluation of thermal leakage. These refinements include the following details:

(a) Location of the heater in the central axis of the calorimeter so that abrupt thermal irregularities produced by the heat developed therein may be subdued before affecting the surface.

(b) Distribution of metal connections between calorimeter and jacket in such a way that the lead conduction is unaffected by inequalities in the surface temperature of the calorimeter.

(c) By means of superficial copper sheaths protection of the calorimeter surface from abrupt variations in temperature such as may occur within during calorimetric measurements.

(d) Method of indicating relative mean surface temperatures of calorimeter and jacket by means of multiple integrating thermocouples, which permits the evaluation of and usually the annulment of the thermal leakage.

2. Provision by means of a system of radial metal vanes for the distribution of heat throughout the contents.

3. Provision for insuring the dryness of vapor withdrawn by means of baffle plate system.

4. Installation in the calorimeter of a strain-free type of platinum resistance thermometer selected to give requisite accuracy.

5. Device for rapid cooling of the calorimeter, consisting of a copper ring which can be moved within the jacket so as to thermally short-circuit the insulating air space and permit the escape of heat to the cooled jacket.

6. Thermoregulated bath for keeping the temperature of the isothermal envelope or jacket always under control of the observer in order to avoid any unmeasured heat increments by thermal leakage.

Many other details are described which have a bearing on the convenience or accuracy in the use of the instrument.

The method of manipulation in making measurements of heat capacity is described, and the results given of an extended series of observations in the temperature range from -50 to $+50^{\circ}\text{C}$ to determine the heat capacity of the empty calorimeter.

WASHINGTON, February 10, 1917.

WAVE LENGTHS OF THE STRONGER LINES IN THE HELIUM SPECTRUM

By Paul W. Merrill, Assistant Physicist

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I. INTRODUCTION

For both practical and theoretical reasons the spectrum of helium is of considerable importance in spectroscopy and related branches of science. It consists of a comparatively small number of lines well distributed from the ultra-violet to the red, and is conveniently produced in great intensity in an ordinary vacuum tube. Hence it is widely used as a reference spectrum in spectroscopic and optical work of many kinds, but the wave lengths have not been known with sufficient accuracy for the most precise wave-length measurements. It is thought that the present determinations will supply this deficiency and make possible the use of helium lines as standards of wave length. This investigation is a part of the standard wave-length program of the Bureau of Standards, which has been under way for two years and which is at present chiefly concerned with the spectra of iron, argon, and neon.

Helium is of great interest in astronomical observations as it is one of the most important elements from a cosmic viewpoint, its spectral lines being prominently observed in such radically different objects as nebulae, early—that is, blue or white—stars and the sun (flash spectrum).

As an aid to progress in the comprehension of the structure of matter nothing is of greater promise than an accumulation of accurate data in regard to spectral-line series. Thanks to the early investigations of Runge and Paschen, every line observed in the ordinary spectrum of helium may be placed in one of six series. In order to establish the mathematical form of these series and to throw light on their relations to one another as well as to the series of other elements, exact values of the wave lengths of numerous lines will be required and will grow increasingly valuable as more data become available.

Precise measurements of the stronger helium lines were therefore undertaken in order that they may serve as standards for the determination of other wave lengths, and to furnish a basis for computations of theoretical interest.

II. MEASUREMENT OF WAVE LENGTHS

1. DIRECT DETERMINATIONS FROM THE RED CADMIUM LINE

Wave lengths of 21 of the stronger helium lines were measured with the Fabry and Perot type of interferometer, which consisted of two partially reflecting plane surfaces held a few millimeters apart and exactly parallel by an invar separator, using the practice previously adopted by the Bureau.¹ The interference rings formed by the action of the parallel planes of the interferometer upon the incident light were projected upon the wide slit of a spectrograph having a single prism of rock salt, portions of them accordingly being observed as short bars across the spectral lines. The specially designed achromatic lenses allowed a great range of wave lengths to be photographed with one exposure. Thus, 2945Å and 5875Å (D_2) were obtained on one plate as were also 3888Å and 7281Å.

The helium tube employed is similar in every way to the neon tube described by Meggers.² It was filled at the Bureau with helium obtained from a London firm. The electrical excitation was furnished by stepping up from commercial alternating cur-

¹ This Bulletin, 12, p. 179, 1915; 13, p. 245, 1916 (Scientific Papers Nos. 251 and 274).

² This Bulletin, 12, p. 202 (Scientific Paper No. 251); 1913.

rent of 110 volts, 60 cycles, to about 10 000 volts (on open circuit). Primary currents from 0.5 to 1.75 a were used, causing from 3 to 11 ma through the tube. The dark space was from 1 to 2 mm in width.

Several of the helium wave lengths were compared directly with the fundamental standard by photographing the cadmium and helium spectra simultaneously upon the same plate. The cadmium lamp, electrically heated to about 300° C, was in the optical axis of the spectrograph, while the beam of helium light was brought into the axis by reflection from a partially transparent surface through which the cadmium light passed. This surface was formed by a thin film of nickel which had been cathodically deposited upon a quartz disk. Two series of photographs were made in this way, in one case the interferometer films being of copper deposited on glass and in the other of nickel on quartz. The copper interferometers employed were of 5, 10, and 20 mm separation, the nickel of 5, 10, 15, and 20 mm. The plates (Seed 27) were sensitized to the less refrangible rays by treating them with a solution containing dicyanin, pinaverdol, and ammonia. Exposures were made ranging from 4 to 15 minutes and with varying amounts of current through the tube. The results are tabulated in Table 1. The agreement of the two series is considered satisfactory except in the case of 7065A. No other reason, however, than accidental error of observation can be assigned for this difference.

The very small corrections to reduce the wave lengths to standard conditions (760 mm, 15° C) were applied.

TABLE 1

Helium Wave Lengths by Direct Comparison With Cd 6438.4696

Copper films	Nickel films	Adopted	Copper films	Nickel films	Adopted
3888.646	0.646	0.646	(5047.734)		
	(4026.190)		5875.616	0.618	0.617
4471.476	.478	.477	6678.148	.149	.149
4713.143	.143	.143	7065.186	.190	.188
4921.928	.929	.929	7281.350	.348	.349
5015.675	.675	.675			

2. RELATIVE MEASUREMENTS

The values in Table 1 served for the determination of other helium lines on photographs of the helium spectrum alone. In this series the nickel films and separations of 3, 5, 7.5, and 15 mm

were used with exposures from 20 seconds to 1 hour. With the exception of the three red lines which were not observed in this series, the ratios of wave lengths in Table 1 were satisfactorily confirmed, and are apparently accurate to 1 part in 4 000 000. The combined results of all the measurements made in this investigation appear in the first column of Table 2.

TABLE 2
Collected Helium Wave Lengths

Bureau of Standards	Rayleigh		Evers- heim	Runge and Paschen
	(a)	(b)		
2945.104				106
3187.743				701
3613.641				641
3705.003				007
3819.606				605
3888.646				638
3964.727				727
4026.189				192
4120.812				821
4143.759				766
4387.928				934
4437.549				549
4471.477	(478)	480	493	475
4713.143	(171)	142	154	074
4921.929	925	928	922	919
5015.675	680	678	683	556
5047.736				641
5875.618	616	623	639	650
6678.149	144	147	151	14
7065.188	189	197	207	22
7281.349				53

From the number and internal agreement of the individual determinations it seems that an error larger than 0.003A is scarcely to be expected and that probably most of the errors are smaller than that amount. In the case of the double lines the value, of course, refers to the stronger component. The weaker component seems to have been practically without effect upon the measurements, since accordant values were obtained from a considerable range both of effective exposures and of orders of interference.

This may be illustrated by the values obtained for D_3 . It was customary to put several exposures of varying length upon each plate. Determinations of the difference between overexposed and normal or weak images are hence available for several plates. In

units of a thousandth of an angstrom they run 0, 0, +1, +1, -3, +4, -5, the last three depending on a poor measurement in each case. The systematic difference is not larger than the accidental error. The photographs of helium alone were taken upon Seed 27 emulsion (not stained), but the longest exposure on each plate (about an hour) shows D_3 faintly. The wave lengths of this line from four of these underexposed images are practically in agreement with the other determinations, there being possibly a tendency toward a slightly higher value. For 5015A, a single line, the difference, strong minus weak exposure, is 0.000 or $\pm 0.001\text{A}$ in 11 cases out of 13. The agreement of different interferometers is shown for two lines by Table 3, which refers entirely to the direct comparisons with cadmium. The small figures in parentheses give the number of exposures upon which the value depends.

TABLE 3
Wave Lengths from Different Interferometers

[Comparisons with cadmium]

	5015A		5875A	
	Copper	Nickel	Copper	Nickel
5 mm	0.674 (2)	0.677 (2)	0.617 (3)	0.619 (3)
10 mm	.673 (3)	.674 (2)	.616 (3)	.617 (2)
15 mm		.675 (5)		.618 (6)
20 mm	.677 (3)	.676 (3)		.618 (3)

3. ELIMINATION OF DIFFERENCE OF PHASE CHANGE

The dispersion of phase change at reflection from the interferometer mirrors has been referred to as "one of the less agreeable features" of interference measurements. It is customary to find the amount of the differential effect for different wave lengths by observations with large and small path differences, and to compute the small corrections to be applied to the measured wave lengths. This procedure is, however, by no means necessary, as the whole effect can be eliminated by using differences, as was done by Priest² for visual methods. Let us find by the use of the standard line, say Cd6438A, the double thickness of a large and of a small interferometer. If the difference of these numbers be divided by the difference of the measured orders of interference for another line, it is obvious that the quotient will be the

² This Bulletin, 6, p. 573 (Reprint No. 142); 1911.

correct wave length freed from any effect of difference of phase change without having found that quantity at all. The difference of the orders will be of about the same accuracy as the larger one, for while the percentage error of the smaller may be greater the actual numerical uncertainty is less.

The same final values should, of course, be arrived at whether the difference of phase change is eliminated as suggested above or determined by the usual methods and the proper corrections applied. This is the case in the present series of measurements.

III. COMPARISON WITH PREVIOUS VALUES

The present measurements and those by Lord Rayleigh⁴ are in quite good agreement as shown by Table 2. Columns 2 (a) and 2 (b) are separate series of which the second is to be given greater weight. Eversheim's values⁵ are greater, but the differences are not uniform. Except for 3187A the grating values of the shorter wave lengths by Runge and Paschen⁶ are in good accord with the interference results. The discrepancies among the longer wave lengths are possibly due in some way to the fact that their measurements depend upon comparison lines in another order (second). This is the case for 4713, 4921, 5015, 5047, 5875, 5678, and 7065A. The mean arithmetic residual Runge and Paschen minus Bureau of Standards for these lines is 0.052A, as compared with 0.0065A for the remaining lines, for which the comparisons were in the first order. Omitting 3187A the mean residual for 12 of the shorter lines is only 0.003A.

Using an interference method which depends upon observing the disappearance of the central ring⁷ Priest found the apparent wave lengths of certain helium lines to vary with the amount of current through the tube. Variations of this kind were not observed by him for 5015A.⁸ Mr. Priest has kindly given me the exact value obtained by him for this line as 5015.679A. This is the mean of several accordant measurements.

IV. SERIES RELATIONS

When our understanding of spectral series is complete, the magnitudes of certain physical quantities can probably be computed from a single series, and from relations between different series, and it may be in this connection that accurate measurements of wave length will finally be of the most value. In the

⁴ Phil. Mag. (6), 15, p. 548; 1908.

⁵ Ze. wiss. Photo., 8, p. 148; 1909.

⁶ Astroph. Jour., 8, p. 4; 1896.

⁷ This Bulletin, 6, p. 573 (Reprint No. 142); 1911.

⁸ This Bulletin, 8, p. 1 (Reprint No. 179); 1911.

meantime they can be used to determine the applicability of the various types of formula which have been suggested, to test the so-called "combination principle," etc. For obvious reasons the present investigation includes only a small number of lines at or near the beginning of each series. These lines, although favorably situated for the purpose, will not by themselves give the best values of the series constants, particularly of the convergence frequency. Hence no extensive recomputations have been undertaken at this time, but it was thought of interest to see how closely a three-constant formula based upon three consecutive well-determined lines would represent the remainder of the series.

None of the series can be represented exactly by a formula of the type

$$n = A - B/m^2$$

where n is the reciprocal of the wave length, m represents successive integers, A and B are constants. This equation, however, gives a fairly close representation of the two first subordinate series (5875A, 4471A, etc.; 6678A, 4921A, etc.). The constant B is about the same for both and approximately 0.2 per cent larger than in the Balmer series of hydrogen.

It is well known that the Kayser and Runger formula

$$n = A - B/m^2 - C/m^3$$

will give a fair representation of the helium series. The lines for which $m = 3, 4, 5$, in each of the six series have been measured in the present investigation. The constants A, B, C , as computed from these three lines, appear in Table 4, which also contains the residuals of all the well-observed lines in each series.

In every instance the error of the representation even for $m = 6$ is very much larger than the uncertainty of observation, while the residuals show a fairly smooth and converging increase toward the terms of higher order. In some cases there is no improvement as compared with the series computed by Kayser and Runge from less accurate values. This is in agreement with the prevailing opinion that the formula contains the first three terms of a rapidly converging mathematical series which may be regarded as the expansion of a closed expression as yet unknown.

TABLE 4

Series Constants

$$n(\text{vac}) = A - \frac{B}{m^2} - \frac{C}{m^3}$$

CONSTANTS COMPUTED FROM $n=3, 4, 5$, IN EACH SERIES

	First group			Second group		
	Principal	First sub- ordinate	Second subordi- nate	Principal	First sub- ordinate	Second subordi- nate
A.....	38 469.22	29 225.53	29 147.13	32 030.82	27 176.66	27 154.51
B.....	110 501.7	109 845.2	102 963.4	109 518.3	109 783.2	107 878.3
C.....	13 027.4	152.7	96 030.4	-1 887.8	226.2	36 826.8

OBSERVED MINUS COMPUTED (ANGSTROMS)

M.....						
2.....	-18.2			+124.3		
3.....	0	0	0	0	0	0
4.....	0	0	0	0	0	0
5.....	0	0	0	0	0	0
6.....	+ 0.18	+0.021	-1.06	- 0.030	+0.013	-0.34
7.....	+ .34	+ .047	-2.2	- .091	+ .036	-0.71
8.....	+ .45	+ .078	-3.2	- .103	+ .022	-1.07
9.....	+ .56	+ .096	-4.0	- .19	+ .068	-1.30
10.....		+ .134		- .21	+ .069	-1.54
11.....				- .23	+ .072	
12.....					+ .090	

V. SUMMARY

Wave lengths of 21 of the stronger helium lines have been accurately measured by interference methods. Nine of them were compared directly with the standard cadmium line.

The possibility of eliminating the effect of apparent variation of interferometer thickness with wave length is noted.

The Kayser and Runge formula for spectral series, based upon three consecutive lines, will not reproduce accurately even the next member in any one of the six helium series.

WASHINGTON, March 14, 1917.

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S. W. STRATTON, DIRECTOR

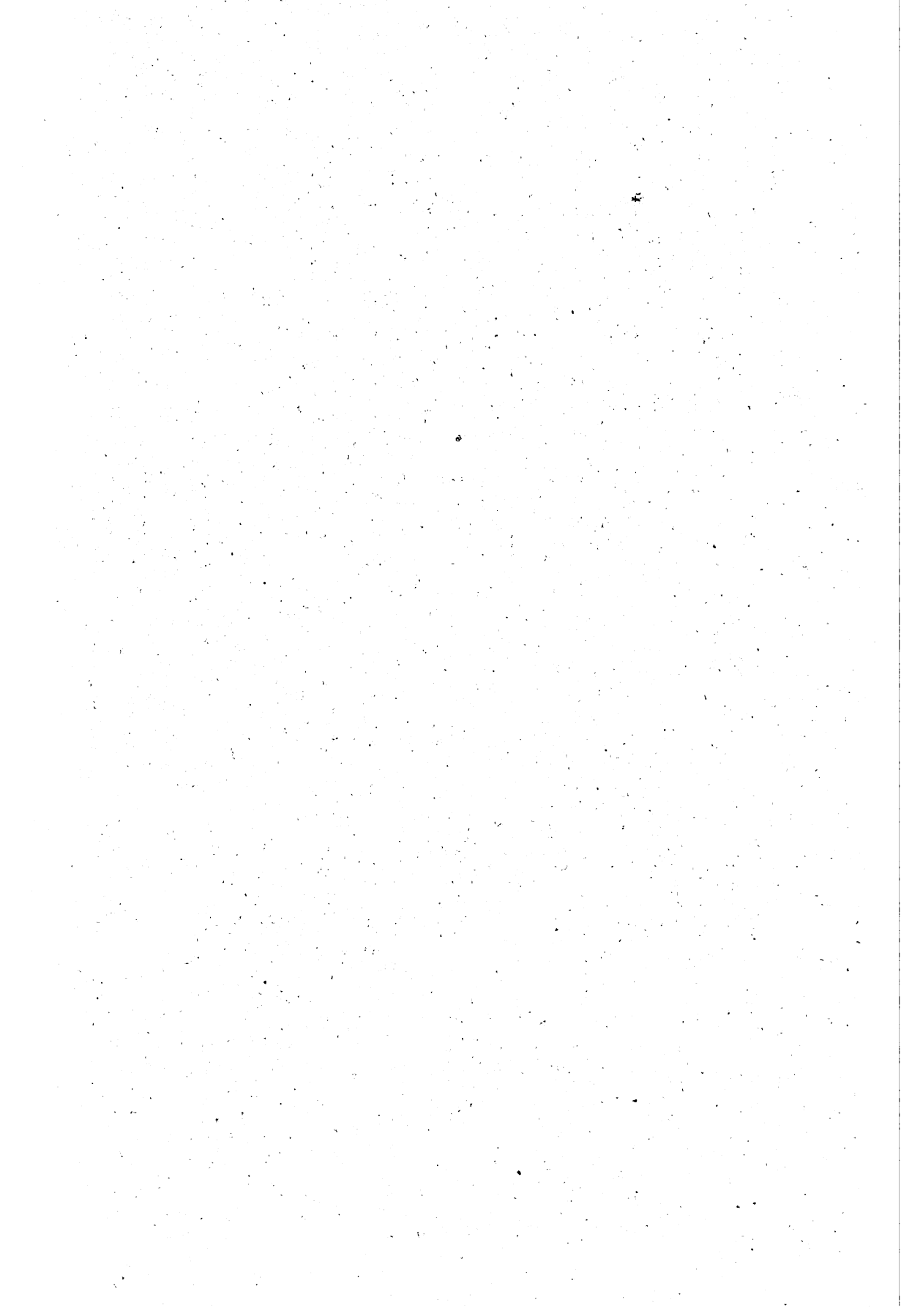
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RELATIVE SENSIBILITY OF THE AVERAGE EYE TO LIGHT OF DIFFERENT COLORS AND SOME PRACTI- CAL APPLICATIONS TO RADIATION PROBLEMS

By W. W. Coblentz and W. B. Emerson

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I. INTRODUCTION

Numerous investigations^a have been made to determine the relation between luminous sensation, "light," and radiant energy; that is to say, the relative visibility of radiation in different parts of the spectrum.

The difficulties encountered in such an investigation are due to the fact that the relative brightness varies not only for different colors but also for changes in size and brightness of the illuminated field. It is therefore necessary to work under specified conditions. Having decided upon these conditions, a further difficulty encountered is the radiometric evaluation of the light stimulus.

Some of the disagreements in previous investigations seem to be due, in part, to uncertainties in the radiometric measurements. The present investigation, therefore, was undertaken in view of this Bureau's complete equipment for evaluating light stimuli radiometrically and in view of the large staff of trained investigators available for cooperation in the work. This investigation is somewhat statistical, the aim being to obtain extensive data on the relative sensibility of the eye to light of different colors, but a complete study of color perception is not attempted.

The method of comparing the light stimuli is photometric. The application of the results obtained is photometric and radiometric; as, for example, in applying corrections to spectrophotometric measurements made on narrow spectral bands, in designing lenses, and in the determination of the factor for converting visual sensation into luminous power, i. e., the mechanical equivalent of light.

II. METHODS FOR DETERMINING THE VISIBILITY OF RADIATION

Various methods are available for determining the visibility of light in different parts of the spectrum; that is, the relative sensibility of the eye to light of different wave lengths.

1. THRESHOLD OF VISION

This method has been used by Ebert,¹ by König and Dieterici,² and by Pflüger,⁴ who determined the minimum amounts of energy required to produce a luminous sensation in different parts of the spectrum. Pflüger used a Nernst glower as a source of light and

^a Throughout this paper the reference numbers occurring in text refer to the bibliography at the end of the paper. It is therefore necessary to use letters for footnote references, the notes to which appear in the text proper.

measured the energy at the slit by means of a thermopile. He examined a dozen subjects, but the data are very irregular and, owing to the low intensities required, they are of but little use in practice.

This method of determining the relative sensibility of the eye for spectral colors shows that at very low intensities the maximum sensibility lies at 0.49 to 0.53 μ .

2. VISUAL ACUTY

By this method of comparison the visibility is defined by the amount of light, independent of color, which is necessary to enable the eye to distinguish objects, such as, for example, fine print. The most noteworthy investigation by this method was made by Langley,⁶ who determined with a bolometer the energy required to distinguish fine print which was illuminated by light from various parts of the spectrum. Recently Bender⁷ has used the method in comparison with the flicker method, to be mentioned presently.

3. CRITICAL FREQUENCY

This method has also been termed "the persistence of vision." It has been used by Allen¹² and others in studying the response of the eye of subjects having normal vision and also of subjects who are color blind. In this method a sectorized disk is rotated in the spectrum at such a speed that no flicker is perceived, the velocity of rotation giving a measure of the duration of the images of the light upon the retina. The method is based upon the principle that for the disappearance of flicker different speeds are required for different colors. The results obtained by Allen will be discussed in their proper place in this paper. It will suffice to add that the writers of the present paper tried the method and found their results in agreement with those obtained with the methods used in their work. This method and the following ones have been thoroughly investigated by Ives¹⁶ in his search for the best methods available for the photometry of lights differing in color.

4. EQUALITY OF BRIGHTNESS

In this method two differently colored lights illuminate the two parts of a photometric field, and they are said to produce the same illumination when they give a sensation of equal brightness. The difficulty with the method lies in the inability of most observers to form a judgment concerning equality of brightness which is

not influenced by differences in color. Among others the method has been used by König,⁸ Ives,¹⁶ and Houstoun.¹⁷ The latter had 52 observers. The field illumination was low. For an illumination of one-half meter candle the wave length of maximum visibility was at $\lambda_m = 0.502\mu$ and for one-six-hundredths meter candle the maximum was at $\lambda_m = 0.466\mu$. Of the nine women examined none showed systematic differences from the men.

One difficulty encountered in this method is that some observers find it impossible to make consistent comparisons when large color differences are involved. In the case of the observers examined by the present writers the majority considered it guesswork that had no meaning. In fact, their opinions came close to Helmholtz's appraisal of the method. Helmholtz¹⁵ had no confidence in his judgment concerning the equality of brightness of differently colored surfaces, although he was willing to admit that of two differently colored fields one can be so much darkened that there remains no doubt that the other is the brighter. No doubt most observers make their settings subconsciously in this manner—that is, the observer sets one part of the field too bright, then the other part of the field too bright, and finally a setting is made which is called "equality," in spite of a subconscious feeling that the two halves of the photometric field are not of equal brightness. Only a few of the subjects examined could distinguish an inequality of brightness due to slight change in intensity in the two parts of the photometric field, and even these few (including the writers, who made a prolonged test of this phenomenon) would at unexpected moments make very erratic settings in the midst of a series of closely agreeing measurements.

A few observers were found who, although they had not had previous experience in making photometric comparisons, obtained an exact coincidence in their visibility curves by this and the following method.

In view of the fact that the equality-of-brightness method is considered by some¹⁸ to be the only one that will give true results, in the present investigation it was decided to test the equality-of-brightness method at a number of points on the spectrum to determine whether, among the erratic settings made by various observers, there is a systematic deviation from the visibility curve as obtained by the flicker photometer.

From all the work that has been done by Ives,¹⁶ Crittenden, and Richtmyer,¹⁹ Middlekauff and Skogland,⁴⁰ and others the equality-

of-brightness method of photometry appears to be insensitive and often lacking in consistency. The erratic variations in the equality-of-brightness settings are so large that no trustworthy data seem to be at hand to prove definitely that there is a systematic difference in the results obtained by the equality-of-brightness and the flicker photometers.

5. THE FLICKER PHOTOMETER

In the flicker photometer the two illuminations are viewed alternately. "At a certain speed of alternations," says Cobb,²⁸ "the two colors disappear, giving place to a uniform color with a flicker superimposed upon it. Then, when the relative intensities of the two illuminations are so altered that this flicker disappears the two illuminations are by this method equal. Color flicker has a lower critical frequency than brightness flicker, and on reaching this frequency we get rid of color difference and can make a comparison of brightness."

In view of the fact that the visual sensations aroused by lights differing in color rise to their maximum brightness at different rates, the accuracy of this method of photometry has been challenged.¹⁸ The present writers, however, do not concern themselves with this phase of the question. The flicker method is quick, precise, and gives reproducible results when operated under specified conditions, as has been shown by the extensive investigations of Whitman,²¹ Tufts,²² Thürmel,²⁵ Ives,²³ Bender,²⁶ Nutting,²⁷ and others. The flicker method has merits not possessed by the other methods just discussed, and the results are definite and consistent for all observers. It is, therefore, a practical method for intercomparison of various observers and hence for the establishment of the sensibility curve of the average normal eye. If it can be shown, after a prolonged investigation of a small group of observers having normal color vision, that the curve is slightly too low in the red (as the result of underestimation due to the fact that the stimulus has not had sufficient time to arouse the maximum sensation) and too high in the blue, then all the data given herewith can be easily corrected. The time of the persons who were willing to have their eyes examined was very valuable, and it would not have been fair, at this stage of the knowledge of the subject, to ask them to devote more time to this question.

The question of photometric methods should be given further study. However, from the data presented herewith on 130 sub-

jects, whose sensibility in any part of the spectrum is rarely in agreement as close as 1 to 2 per cent (which is the error of observation of a single observer), it seems futile to devote much time to obtain a method with which any one observer can make measurements with an accuracy of a small fraction of 1 per cent.

III. APPARATUS AND METHODS

The arrangement of the apparatus is illustrated in Fig. 1.

1. THE SPECTROMETER LENSES

The spectrometer was a very substantial instrument, with an automatic attachment for maintaining the prism at minimum deviation, constructed in this Bureau's instrument shop. A photographic illustration of it is given in a previous paper.⁴¹ The collimating lens is a triple achromat of 18 cm focal length, by Zeiss. For the present investigation the exactly similar objective lens was replaced by an achromatic doublet, *L*, of 33 cm focal length, especially made in the Bureau's shop for this work. The special preparation of this lens consisted in giving it a very fine polish in order to eliminate stray light. The surfaces were quite free from pit marks remaining from grinding with coarse-grained ~~emery~~^{silica}, and although the polishing was not as perfect as is possible the field of view was remarkably free from diffuse light as compared with other spectrometer lenses which were tested for this defect. Using a hand spectroscope, an examination of the light emerging from the slit, *I*, as used in this investigation gave no marked indications of stray light (white light) from adjacent parts of the spectrum. The test was applied in various parts of the spectrum, and it was concluded that no corrections were to be made for diffuse light^a in view of the fact that if it were present in a measurable amount it would be included in the direct radiometric measurement. It is to be understood, of course, that this apparatus, like all others, is not entirely free from diffuse light. Using intense light from a Nernst glower placed close to the slit, diffuse light was detected with a hand spectroscope.

2. THE PRISM

The prism was of light flint glass. With the 33 cm focal length lens it produced a spectrum which was 14 mm in length between the red and the blue-violet helium lines ($\lambda = 0.667\mu$ and $\lambda = 0.447\mu$).

^a See Appendix 1 at the end of this paper.

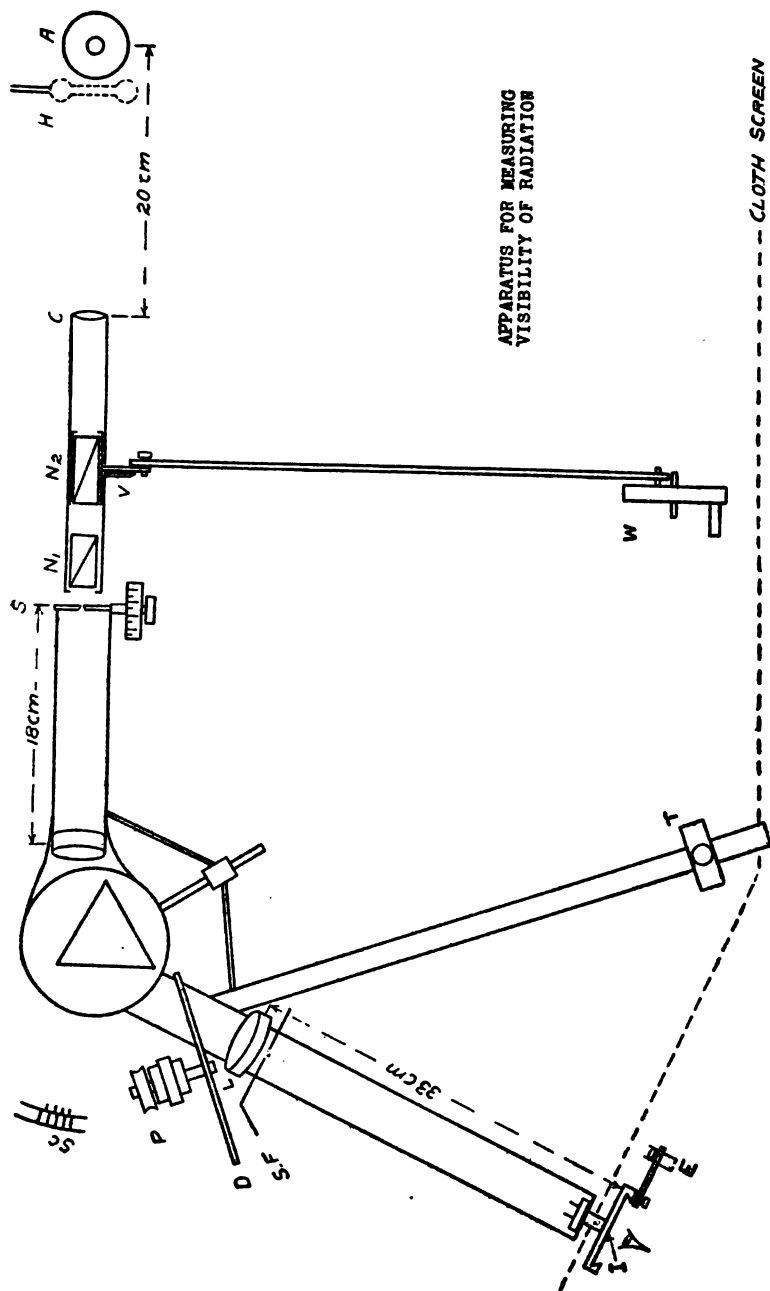


FIG. 1.—Arrangement of apparatus used in obtaining visibility data

The surfaces were given a special polish to reduce diffuse light. The prism and the collimating telescope were entirely inclosed to eliminate extraneous light. The spectrum (the spectrometer circle *Sc*) was calibrated by means of the helium lines, the adjustments being maintained by means of the yellow helium line from a tube inserted temporarily at *H*. For making this adjustment the eyepiece *E* was focused upon the slit *I*.

3. THE LIGHT SOURCES

For a standard source of white light, *T*, Fig. 1, a duplicate set 8, 16, and 38 candlepower vacuum tungsten lamps was provided. These lamps were carefully seasoned and standardized by Mr. Mulligan, of the photometric division. They were operated at 1.2 watts per candle, which specifies their color. For most of the work the 8-candlepower lamps were used, and the constancy of their light was frequently tested by intercomparing them in this apparatus, using some part of the spectrum of the acetylene flame as a standard of comparison. It is understood, of course, that it is important to operate the lamp at a constant current while obtaining a visibility curve, but a change in candlepower by aging of the lamp is of minor importance. This lamp was operated from a storage battery, the current being easily maintained constant by means of a milliammeter (the scale of which was read by means of a low-power lens). A change of current of 0.001 ampere would affect the readings by 1 to 2 per cent, which is better than the individual settings of the average observer. By sliding the lamp *T* along its support, the illumination could be varied from 15 to 800 meter candles.

As a standard source of spectral light a cylindrical acetylene flame, *A*, was used. The intensity of the light emitted is quite constant. On several occasions it was found that the flame burned at a somewhat lower (but constant) intensity although the gas pressure remained the same, but the cause of this could not be determined. This, however, did not affect the results.^a

The acetylene flame was used as a standard of spectral radiation because of its high intensity and because its spectral-energy distribution⁴⁰ is well known and easily maintained constant within the limits of errors in the visual measurements. A tungsten lamp

^a The acetylene was used as it came from the generator, without further purification. Rands (Phys. Rev., XIV) has shown that ordinary acetylene is 99 per cent pure. Acetylene might contain a small amount of H₂S, but spectroradiometric investigations (Coblentz, Investigation Infra-red Spectra, Publication No. 35, p. 44, Carnegie Institute, Washington, 1905), do not show impurities.

might have been used, but its energy distribution would be difficult to determine and the lamp would require frequent standardization.

4. THE ENERGY MEASUREMENTS

The visual measurements are not sufficiently refined to warrant undertaking the long and difficult task of determining directly the energy value of the light stimulus in different parts of the spectrum. In fact, in the yellow-green, where but little energy is required to excite a luminous sensation, it would be quite impossible to measure this energy accurately with a radiometer. To avoid this difficulty, the usual procedure is to measure radiometrically the energy in an intense source and reduce it (by a known amount) by absorption screens, Nicol prisms, etc., to the intensity utilized in the visual observations.⁶⁵

In the present investigation the distribution of energy in the spectrum of the acetylene flame ⁴⁰ was determined in the usual way by placing the flame at *S*, Fig. 1, and exploring the complete visible spectrum with a thermopile placed in the illuminator attachment *I*, as described elsewhere.⁴² This gives the prismatic-energy distribution, and hence the energy of the stimulus as applied to the eye. Proper corrections (which are small) of course must be applied for losses by reflection from the Nicol prisms and the lens when the acetylene flame is placed at *A*, as used in the visual observations. When the flame is placed at *A* and the Nicol prisms *N* and condensing lens *C* are in place, the light is reduced by one-half, so that accurate energy measurements could be made only in the longer wave lengths beyond 0.6μ . The results obtained, however, were in agreement with those obtained with the flame ^a at *S*.

The energy distribution was redetermined after a month of visual observations and found in exact agreement with the earlier measurements.^b The candlepower of the flame changes with pressure, which is easily kept constant; but, as shown in a more complete paper ⁴⁰ on this subject, this does not affect by an appreciable amount the relative energy distribution in the spectrum.

In any part of the spectrum the energy arriving at *I* is determined by the light transmitted through the Nicol prisms.

The flame was entirely inclosed, the light emanating through an opening 8 mm high and about 2 mm wide.⁴⁰ The image of

^a See Appendix 1, "On diffuse light and spectral energy measurements."

^b See Appendix 1 for further verification.

this opening as projected upon the spectrometer slit S was about 1 by 5 mm.

When making the energy measurements, the shield $S F$ was removed and the space between L and I was inclosed with a cardboard tube to exclude extraneous light and air disturbances from the thermopile.

As already mentioned, the amount of diffuse light present in the stimulus is very small and immeasurable. (See Appendix 1.) Numerous attempts, during the past 10 years, to measure such small amounts of stray light have always been a failure. Hence radiometrically the diffuse (white) light is negligible.

5. THE NICOL PRISMS

The light from the acetylene flame A is focused upon the spectrometer slit by means of the achromatic lens C . The function of the two Nicol prisms (N_1 is stationary) is to vary the intensity of the light falling upon the slit, and hence to vary the intensity in any part of the spectrum. The light transmitted is proportional to $\sin^2 \theta$, where θ is the angle through which the Nicol N_2 is rotated from the position in which there is complete extinction. A circular scale divided into even degrees and a vernier, V , enable one to read the amount of rotation of the Nicol. When the angular opening (θ) is 90° , the transmission is taken 100 per cent. The accuracy of the transmission for different openings (90° , 60° , 45° , 30° , 0°) was determined radiometrically by placing a thermopile at the exit opening, between S and N_2 , and after correcting for a slight amount of infra-red transmitted when the Nicols were crossed, $\theta = 0^\circ$, it was found that the observed and the computed transmissions were in agreement within the errors of observation. For example, the ratio of the galvanometer deflections for 30° and 90° openings was $(4.01 \div 16.18 =) 0.248$, while the computed transmission for $\sin^2 30^\circ$ is 0.2499.

The incomplete extinction of the infra-red energy from the flame A when the Nicols are crossed, $\theta = 0^\circ$, was not found in another pair of Nicols, hence it is not a common property, and tests made with a solution of cupric chloride show that the infra-red is easily eliminated. The device therefore is useful in making an apparatus (crossed Nicols and quartz plate) for transmitting light proportional to the visibility curve of the eye (as proposed by Mr. Priest), in place of the visibility solution mentioned in this paper.

The size of the Nicol was 8 by 10 by 25 mm. A diaphragm 7 mm in diameter, placed in front of N , confined the incident light to the central part of the optical system.

A further provision for varying the light entering the spectrometer is the bilateral slit S , which is provided with a graduated drum.

The observer at I rotated the Nicol by turning a wheel, W , which was connected with the Nicol mounting by means of a long rod.

6. THE FLICKER PHOTOMETER

The arrangement of the flicker photometer is practically the same as recommended by Ives.²³ The photometer field is 10 mm or about 2° in diameter. A "surrounding field," $S F$, was also provided, although this is not absolutely necessary for this work. To provide a convenient surrounding field, $S F$, illuminated from the standard lamp T , the objective lens L was mounted securely in a brass frame attached to a heavy iron bar, which was used in place of the usual telescope tube. The surrounding field was placed close in front of this lens. It consisted of a piece of stiff cardboard covered with white filter paper, which was covered with magnesium oxide. The opening in this cardboard was adjusted so that the lower edge of the hole (10 mm in diameter) in it came just above the optical axis of the lenses. In this manner the numerous small spectra (visible in any spectroscopy), caused by reflections from the surfaces of the lenses and prism, were entirely eliminated from the field of view. This is a very important desideratum when viewing the extreme ends of the visible spectrum.

The sector disk D , 15 cm in diameter, was made of brass, which was covered with a thick, even layer of magnesium oxide, obtained by burning magnesium ribbon. The edges of the sectors were beveled, and after depositing the magnesium oxide a sharp, smooth edge was produced by removing the overhanging oxide by means of a sharp knife.

The illumination incident upon the disk D was approximately 50 meter candles. It was adopted for use with the various observers after a preliminary investigation of various incidental phases of the subject. In this preliminary work visibility curves were obtained for intensities varying from 25 to 780 meter candles upon the sector disk. It was found that for the lower (25-meter candles) intensity the eye seemed to be more variable from day

to day in its response to stimuli in the green, and it seemed more difficult to make the settings at these low intensities. On the other hand, the 100 meter-candle intensity did not permit determining the visibility curve over a great range of the spectrum without changing the slit opening, which meant delays in completing the most important part of the sensibility curve. Hence, the 50-meter-candle intensity was adopted in view of the fact that our results are in agreement with those of other experimenters,^{52, 23, 37} showing that the shape and position of the maximum of the visibility curve of the central retina is the same, within experimental errors, for all these high intensities, as shown in Fig. 2.

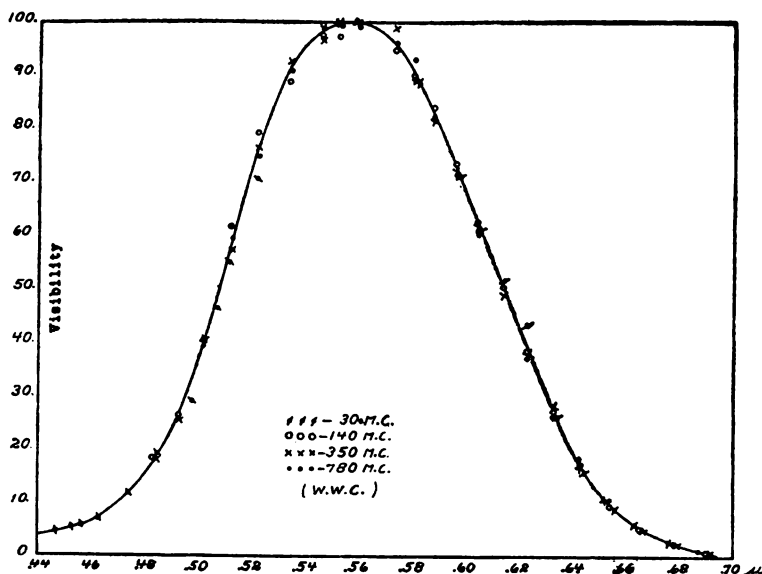


FIG. 2.—Relative spectral sensibility of the eye when subjected to stimuli of various intensities

In making these measurements the observer, at *I*, viewed a part of the spectrum coming through an opening 0.52 mm wide and 2.63 mm high.

The disk was connected by a flexible belt (rubber or coil of spring steel) with a small direct-current series-wound motor, the speed of which could be varied by means of a variable resistance in series with it. No attempt was made to record the speed, in view of the fact that it is different for different intensities, different colors, and different observers, and in view of the fact that previous investigations¹⁶ show that it has no direct bearing upon the problem under investigation.

The motor was fixed to the base of the spectrometer. The disk D and the support for the standard lamp T were securely attached to the spectrometer arm IL , which was movable, in order to view different parts of the spectrum. By this arrangement the light from the lamp T was always normal upon the disk D , and conditions were thus the same throughout the series of measurements.

This apparatus was used in a dark room illuminated only by the stray light from the acetylene flame and the lamps which were used to read the scale Sc and vernier V . A cloth screen prevented light from reaching the observer at I .

7. METHODS OF OBSERVATION

In making the photometric measurements the observer, at I , rotated the Nicol prism by means of the wheel W and adjusted the speed of the motor to produce no flicker. In a few cases the observer preferred to use minimum flicker. The assistant made the spectrometer settings Sc , read the photometer scale V (the rotation of the Nicol N_2), and recorded the data.

In the preliminary work it was found that the eye becomes adapted to the conditions of this work in less than five minutes. The observer, therefore, made a series of preliminary measurements to learn the method and to adapt the eye. Measurements were then made in the yellow and extended every 10 seconds of arc on the spectrometer circle to the end of visibility in the red. The measurements in the yellow were then repeated (and always found the same as in the beginning) and extended, for every predetermined spectrometer setting (5 and 10 seconds of arc), to the limit of visibility in the violet. The settings in the yellow-green were then repeated, when, in some cases, the sensibility was a little higher.⁴⁴ This, however, disappeared after a few settings of the Nicol prism. New settings were then made at intermediate points (the 5-seconds intervals on the spectrometer circle) extending to the end of visibility in the red, also the alternate 10-seconds settings were repeated, especially if the sensitivity of the eye had changed, which was of rare occurrence. In this manner the complete visibility curve was obtained for every 0.01μ extending from 0.49 to 0.69μ . In this series the slit S was 0.1 mm in width, the Nicol opening being about 12° in the yellow for the majority of the observers.

For those observers who could give the time the visibility curve was then extended to 0.75μ in the red and about 0.435μ in the

violet. This was accomplished by opening the slit *S* to 0.6 mm and by placing the lamp at a greater distance (73.5 cm), which reduced the illumination to about 15 meter candles on the disk. That this procedure is permissible was determined in the preliminary work, in which the visibility curve of one of the writers was determined (by using a still lower illumination) to 0.77μ in the red and 0.40μ in the violet. It is, of course, understood that on both sides of the maximum of the visibility curve, which was obtained for a 50 meter-candle illumination, measurements were made also for the 15 meter-candle illumination in order to obtain the factor for superposing the two curves. This factor was somewhat differ-

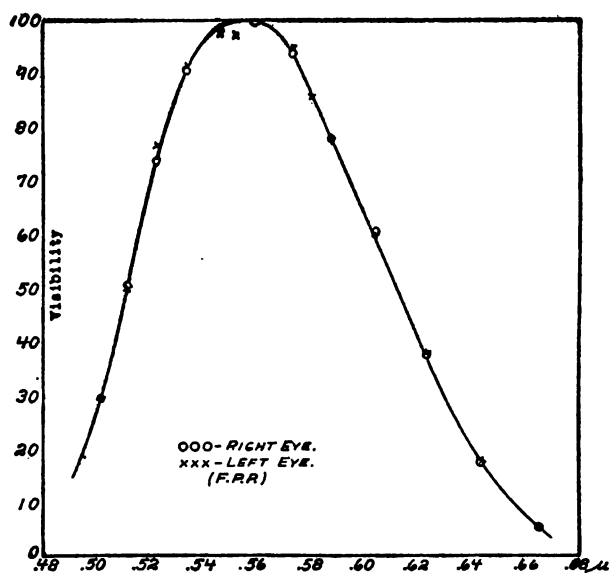


FIG. 3.—Sensibility of right and left eye

ent for the two sides of the curve, due, no doubt, to diffraction caused by the narrow slit. This, however, is of minor importance.

In the preliminary investigation it was found that the variation in the observations for the right and the left eye (W. W. C.) were from 1 to 3 per cent, which was the accuracy attained on different days using the right eye. One observer (F. P. P.) kindly volunteered to make the test of both eyes on the same day. The observations are in exact agreement throughout the curve (see Fig. 3), showing that the visibility curve, characteristic of a given observer, can be obtained by examining either the right or the left eye, as had been found by previous measurements.^{7, 12, 22} The various

observers, therefore, were permitted to use the eye which they were accustomed to using and which they considered the "best" physically.

In order to make the equality-of-brightness measurements, the disk D was placed so as to bisect (horizontally) the photometric field. In this manner one-half of the field was illuminated by the standard lamp T and the other half of the field was illuminated with spectral light of a given color. In view of the great difficulty experienced by most observers in making such settings, it was deemed of greater importance to spend the time in making numerous settings in a few parts of the spectrum instead of attempting to obtain the complete visibility curve. The results of this test will be discussed on a subsequent page.

For the information of the reader not entirely familiar with the subject, it may be added that the visibility of radiation is the reciprocal of the measurements recorded. For example, in the yellow-green the Nicol opening (rotation) may be 13° and at 0.68μ the opening may be 60° . The energy transmitted therefore would be 5 and 75 per cent, respectively, of the value observed radiometrically at these two wave lengths. In the curves given herewith the reciprocals of these energies are plotted; for by definition the sensibility of the eye, or the visibility of radiation, V_λ , at any wave length is the ratio of the luminous intensity measured in light units to the intensity of the light measured in energy units, viz, $V_\lambda = I_\lambda \div E_\lambda$. The curve obtained is the visibility for an equal-energy spectrum.

For most of the observers the visibility was determined only once, in view of the fact that but little variation of the visibility curve was found on different days (as was previously found by Bender²⁰), especially when the observations were made after a night of sound sleep. Observers suffering from lack of rest, "cold," or "grippe" gave variable results, especially in the green. Part of this difficulty seemed to be due more to the inability to fix the attention upon the flickering field than to an actual change in sensitivity of the eye.

IV. DISCUSSION OF DATA

Heretofore it has been the practice to publish the visibility curves of the individual observers. For convenience in illustration, the present data have been arbitrarily grouped according to characteristics which occur in common in certain visibility curves.

In order to classify these data, each visibility curve was set to the scale 100 at its maximum. The appropriate factors are given in column 5 of Table 1. These factors differ, of course, due to differences in retinal sensibility and, to some extent, to small changes in brightness of the acetylene flame^a and to recoating the disk with magnesium oxide. This has no effect upon the relative visibility, in view of the fact that the scale of ordinates in the spectral-energy curve of the acetylene flame might have been chosen, for example, so that the average factor would have been 1.0 instead of 1.2. (See Table 1.) Because of the extensive data it did not seem necessary to weight the visibility curves by reducing them to the same area before taking their mean value, as has been done heretofore.

These visibility curves, plotted in terms of equal maxima, were then compared with the average by viewing them over an illuminated ground glass and classified in the groups illustrated on the following pages. It is to be understood, of course, that this characteristic grouping is not always as sharply defined as illustrated, although in some instances there is a tendency toward distinct grouping.^b

This classification shows (1) wide curves with the maximum shifted toward the red, "red sensitive," (2) narrow curves with a sharp maximum in the green, and (3) curves with the maximum shifted toward the violet. A fourth group of observers has very wide curves embracing much of the three preceding groups. A fifth group has the average sensibility in the red and yellow, but has a low sensibility in the blue, while a sixth group has the average sensibility in the blue, but has a low sensibility in either the yellow or red or throughout this entire region of the spectrum.

The visibility of radiation data obtained by various observers (all but two having normal color vision) are given in Table 1. In this table column 2 gives the type of visibility curve. The classification *Sub. R.* includes several subjects whose visibility is below the average in the yellow and orange part of the spectrum.

It is to be noticed in this table that several subjects having the same type of visibility curve came in succession in the order of making observations. From this it might appear as though the

^a Caused no doubt by variation in humidity. Thus far it has not been possible to establish a change in the shape of the spectral-energy curve of acetylene (in the visible spectrum) for small changes in brightness caused by variation in humidity or gas pressure.

^b Such groups are recognized by writers on physiological optics (c. f., Nagel, von Kries, Goldhammer), who use the terms "monochromats," "dichromats," "trichromats," "red anomalous," "green anomalous," etc.

shift from the average curve might be due to lack of adjustment of the apparatus.^a

This occurrence of so many subjects in succession having the same characteristics is due to the fact that the observers were often selected from a classified list previously obtained from the photometric tests made by Crittenden and Richtmyer.²⁹

In Table 1 the fourth column gives the color of the eyes (*Bl* = blue, *Gr* = gray, *Br* = brown, *L. Bl.* = light blue), which data were obtained for the reasons explained on another page.

The last column of Table 1 gives the position of maximum visibility as read from the individual curves. The mean value of the 125 observers is $\lambda_m = 0.5576 \mu$.

Before discussing these various groups of visibility curves, it is of interest to give some references to subjects who have very marked abnormalities of color vision.

^a At the time of making the test the abnormal observations were always checked by the attendant (W. B. E.), whose visibility is very close to the average.

TABLE 1
 Visibility of Normal Subjects—Blue End of Spectrum

[NOTE.—Footnotes at end of table. See p. 193.]

Subject	Visi- bility ^a	Age.	Eye color ^b	Fac- tor	λ_m 0.427	λ_m 0.435	λ_m 0.444	λ_m 0.450	λ_m 0.456	λ_m 0.463	λ_m 0.474	λ_m 0.485	λ_m 0.493	λ_m 0.502	λ_m 0.5125	λ_m 0.523	λ_m 0.534	λ_m 0.546	λ_m 0.552	λ_m 0.559	λ_m 0.573
✓ W. W. C. ^c	W.	42	Br.		421.6	33.7	38.1	45.5	57.5	78.0	118.0	197.0	265	396	589	757	887	973	991	1000	973
E. D. T.	N.	35	Bl.	1.18									221	347	562	754	890	969	1000	969	963
✓ W. B. E.	W.	25	Bl.	1.17									253	389	575	756	910	966	958	1000	940
✓ F. P. P. ^c	G. S.	26			18.8	20.1		27.4		34.4		86.4	151	292	492	752	905	973	998	1000	938
R. F. J. ^c	N.	35				27.7		34.1		57.3		151.0	192	325	528	738	920	991	1000	999	956
C. F. S.	B. S.	25		1.05		28.6		42.4		70.0		165.0	249	382	601	818	949	1025	1000	992	943
P. D. F.	Sub. B.	29	Gr.	.965			13.0	17.6		26.4		93.1	124	242	435	720	868	969	1000	984	938
I. G. P. J.	Sub. R.	30		1.04		25.7	35.0	38.7	49.7	61.7	104.0	160.0	220	389	571	779	932	992	1000	1000	930
H. J. M. J.	N.	23			21.1	27.0	34.3	40.4	50.9	70.8	107.0	163.0	227	356	550	755	866	974	962	1000	942
C. G. P. ^c	N.	29		0.23.3	29.7	36.5	46.9	59.8	77.4	102.0	175.0		257	399	567	747	909	984	1000	998	941
W. W. B.	R. S.	30	Bl.	1.10									211	340	531	763	924	975	991	1000	1010
C. O. F.	R. S.	26	Bl.	1.24																	
J. W. S.	Sub. B.			1.12				11.6	14.3	21.2	36.0	60.0	100	210	409	681	857	965	1010	991	935
W. F. M. J.	R. S.	27					19.8	23.0	26.0	35.1	69.0	93.0	152	242	452	672	852	963	1000	995	1027
L. N. E.	R. S.	22	Bl.	1.28			24.9	27.0	35.4	48.0	74.3	109.0	173	283	478	687	850	936	971	1000	981
H. E.	B. S.	34	Bl.										291	418	552	792	906	982	1012	985	940
B. M.	R. S.	34	Bl.	1.30					20.1	30.6	51.7	73.9	124	229	437	662	803	922	964	977	1000
E. B.	Sub. R.	35	Br.	1.09									186	318	532	776	942	1017	985	978	910
P. V. W.	R. S.	26	Bl.										209	350	544	740	835	946	944	1000	987
G. E. S.	N.	27	Br.	.992									210	367	571	745	890	967	991	967	955
✓ A. N. I.	G. S.	20	Bl.	1.04				19.0	23.6	34.0	57.0	84.2	148	280	513	744	881	990		991	940
✓ E. C. C.	W.	35										198.0	295	397	608	798	906	949	1000	980	970
C. L. C.	Sub. R.	30	Br.	1.24									194	336	544	762	926	1025	1000	995	943
✓ R. G. W.	G. S.	26	L. Bl.	1.29										251	448	717	882	960	961	1010	960
C. S. C.	Sub. R.	23	Bl.	1.09									202	336	554	826	890	963	995	1000	872
M. S. V.	N.	24	Br.	1.26									230	355	553	778	903		685	1600	954

Visibility of Normal Subjects—Red End of Spectrum ✓✓

Subject	Viability	Age	Eye color	Factor	$\lambda = 0.580$	$\lambda = 0.5876$	$\lambda = 0.5964$	$\lambda = 0.604$	$\lambda = 0.613$	$\lambda = 0.623$	$\lambda = 0.633$	$\lambda = 0.643$	$\lambda = 0.654$	$\lambda = 0.665$	$\lambda = 0.678$	$\lambda = 0.690$	$\lambda = 0.703$	$\lambda = 0.717$	$\lambda = 0.730$	$\lambda = 0.746$	$\lambda = \lambda_{\max}$
$\sqrt{W. W. C.}$	W.	42	Br.		925	855	758	667	525	406	285	183.0	98.8	50.4	25.0	10.4	4.41	1.61	0.656	0.264	0.559
E. D. T.	N.	35	Bl.	1.18	926	858	774	650	536	398	273	167.0	92.8	45.2	20.3						0.559
$\sqrt{W. B. E.}$	W.	25	Bl.	1.17	898	790	712	621	502	375	271	166.0	102.0	47.0	21.7	8.91	3.88	1.45			0.558
$\sqrt{P. P. P.}$	G. S.	26			853	780	691	599	488	376	246	174.0	93.6	49.3	20.6	9.15	3.44	1.26	0.572	1.237	0.555
R. F. J.	N.	35			890	790	694	600	470	346	246	151.0	84.5	41.9	18.6	8.44	3.64	1.46	0.514	1.243	0.558
C. F. S.	B. S.	25		1.05	870	780	689	623	468	357	235	143.0	89.0	38.3	18.3	8.18	3.83	1.33	0.490	0.227	0.553
P. D. F.	Sub. B.	29		.965	910	810	730	627	466	365	250	153.0	78.1	38.9	17.7	7.27	2.84	1.05	0.395		0.557
$\sqrt{L. G. P.}$	Sub. R.	30		1.04	857	761	693	597	462	348	229	144.0	84.6	41.5	18.3	7.90	3.29	1.36	0.485	1.199	0.554
E. J. M. J.	N.	23			942	829	732	642	527	393	277	167.0	95.1	45.7	21.0	8.75	4.07	1.54	0.562	0.242	0.559
C. G. P. C.	N. S.	30			886	811	720	620	498	378	270	164.0	89.2	42.0	19.0	8.10	3.40	1.33	0.443	0.220	0.556
$\sqrt{W. B.}$	R. S.	29	Bl.	1.10	929	886	789	701	581	435	309	193.0	118.0	57.2	27.9	11.3					0.559
C. O. F.	R. S.	26	Bl.	1.24	928	898	779	705	562	427	316	194.0	113.0	56.2	25.4	10.1					0.560
J. W. S.	Sub. B.			1.12	894	817		642	387			184.0		52.4							0.560
$\sqrt{W. F. M.}$	R. S.	27			999	944	865	802	674	541	370	245.0	137.0	70.4	31.3	13.0	5.46	1.85	0.666	0.257	0.570
L. N. K.	R. S.	22	Bl.	1.28	979	948	872	788	674	533	363	244.0	129.0	69.0	30.4	13.7	5.68	1.99	0.761	0.282	0.568
H. R.	B. S.	34	Bl.		850	764	699	610	506	374	254	173.0	105.0	44.4	20.7						0.553
B. M.	R. S.	34	Bl.	1.30	962	897	839	745	626	508	359	240.0	132.0	63.4	30.2	13.0	5.28	1.88	0.757	0.319	0.568
$\sqrt{B.}$	Sub. R.	35	Br.	1.09	896	795	628	548	417	313	214	133.0	72.0	35.7	15.7						0.561
P. V. W.	R. S.	26	Bl.	1.06	962	869	812	704	592	462	330	213.0	118.0	59.8	26.9	10.8					0.562
G. K. S.	N.	27	Br.	.992	873	841	751	631	524	364	234	153.0	81.6	39.0	17.5	7.06					0.561
$\sqrt{A. N. L.}$	G. S.	20	Bl.	1.04	865	778	693	597	455	350	229	142.0	72.2	34.8	16.0	6.60	2.73	1.02	0.392	0.173	0.555
$\sqrt{E. C. C.}$	W.	35			925	820	734	646	530	412	282	188.0	103.0	51.6	23.7						0.558
C. L. C.	Sub. R.	20	Br.	1.24	852	777	669	568	438	330	216	137.0	72.1	38.5	16.8						0.554
$\sqrt{E. G. W.}$	G. S.	26	L. Bl.	1.29	883	814	715	586	476	360	239	156.0	82.7	41.0	18.3						0.558
C. S. C.	Sub. R.	23	Bl.	1.09	849	760	673	556	435	312	221	134.0	76.4	36.2	16.2						0.551
M. S. V.	N.	24	Br.	1.26	886	809	711	636	486	389		177.0		48.8							0.558

Visibility of Normal Subjects—Red End of Spectrum

Subject	Visi- bility	Age	Eye color	Refr. ind.	λ_m 0.580	λ_m 0.5876	λ_m 0.604	λ_m 0.613	λ_m 0.623	λ_m 0.633	λ_m 0.643	λ_m 0.654	λ_m 0.665	λ_m 0.678	λ_m 0.690	λ_m 0.703	λ_m 0.717	λ_m 0.730	λ_m 0.746	λ max
C. F. H.	N.	33	L. Bl.	1.27	888	706	626	484	376	246	197.0	89.4	41.8	19.0						0.556
F. A. W.	R. S.	26	Br.	1.32	942	820	737	619	468	347	228.0	131.0	59.4	28.9	12.3					.562
A. H. T.	R. S.	30		1.22	934	852	784	655	433	307	196.0	110.0	56.1	24.1						.560
H. G. B.	N.	35	Br.	1.23	913	834	745	627	524	396	167.0	93.1	44.2	20.4						.554
✓ E. D. W.	G. S.	24	L. Bl.	1.23	824	766	685	591	474	368	182.0	100.0	51.0	24.2	9.70	4.32	1.46	0.589	0.316	.556
✓ E. B. S.	R. S.	30		1.04	798	701	638	491	344	253	161	98.3	53.7	25.2	11.2	4.74	1.96			.551
C. G. C.	R. S.	23	Bl.	1.20	971	848	756	659	554	423	305	193.0	116.0	54.3	24.3	9.68				.563
J. S. P.	N.	21	Bl.	1.08	885	803	716	605	500	373	264	173.0	94.2	49.0	23.7					.558
A. H. O.	N.	29	Bl.	1.10	910	824	735	653	500	377	269	167.0	87.3	44.0	19.4					.560
✓ A. M. P.	G. S.	27	Bl.		875	793	669	594	491	351	263	170.0	89.0	46.8	20.3	8.12				.557
W. J. K.	N.	24	Br.	1.12	890	806	692	627	522	423	299	187.0	109.0	56.5	25.2	10.06				.555
R. E. L.	Sub. R.	37	Bl.	1.00	869	772	669	571	446	315	203	125.0	63.7	31.0	14.1					.553
E. D. O.	B. S.	34		1.25	874	747	658	522	400		167.0	91.0	46.6	21.8						.557
E. W. B.	B. S.	28	Bl.	1.15	877	744	635	558	432	334	221	128.0	73.5	37.4	16.4					.553
L. R. H.	Sub. R.	27	Bl.	1.37	949	860	804	702	603	453	324	209.0	116.0	59.5	25.4					.564
R. M. W.	R. S.	19	Br.	1.32	921	858	786	687	588	438	300	203.0	108.0	55.5	25.4					.560
W. S. J.	R. S.	24	Bl.	1.24	975	901	820	742	617	485	352	224.0	126.0	66.4	29.5	12.8	5.10	1.95	.762	.561
✓ E. W. W. J.	G. S.	22	Gr.		873	783	711	601	467	340	231	146.0	78.6	38.2	17.2	10.2	3.22	1.21	.528	.551
H. B. H.	N.	22	Br.	1.23	902	831	710	630	509	407	277	182.0	98.9	52.0	25.3	10.4				.558
E. F. M.	B. S.	33	Br.	1.25	885	793	709	624	519	402	270	168.0	89.7	46.6	21.2	9.74	4.38	1.59	.543	.556
H. A. B.	Sub. R.	25	Bl.	1.13	918	842	729	648	510	385	265	170.0	90.9	44.1	21.4	8.92	3.75	1.69	.694	.560
D. H. S.	Sub. R.	26	Br.	1.08	873	769	692	560	454	312	217	130.0	76.8	35.7	16.3					.552
✓ W. C. B.	W.	33	Bl.	1.23	894	816	737	657	551	411	296	186.0	110.0	51.5	23.6	10.2				.556
R. C. S.	B. S.	29	Br.	1.10	899	797	691	554	489	353	222	140.0	78.2	39.4	21.3					.555
M. H. S.	B. S.	30	Gr.	1.08	890	800	706	592	490	381	271	167.0	94.8	46.6	21.3	9.06				.564
G. W. M.	Sub. B.	46	Gr.	1.20	899	840	759	663	542	411	302	205.0	109.0	59.0	25.7	10.7				.598
✓ A. L. T.	G. S.	26	Bl.	1.03	848	799	660	534	449	328	236	140.0	82.2	40.8	19.6					.553
A. F. P.	Sub. B.	43		1.12	900	815	736	598	509	366	255	159.0	86.7	41.5	20.0	8.06	3.48	1.39	.529	.559

TABLE 1—Continued
 Visibility of Normal Subjects—Blue End of Spectrum

Subject	Vis- ibility	Age	Eye color	λ_{max} 0.427	λ_{max} 0.435	λ_{max} 0.444	λ_{max} 0.450	λ_{max} 0.456	λ_{max} 0.463	λ_{max} 0.474	λ_{max} 0.485	λ_{max} 0.493	λ_{max} 0.502	λ_{max} 0.5125	λ_{max} 0.523	λ_{max} 0.534	λ_{max} 0.546	λ_{max} 0.552	λ_{max} 0.559	λ_{max} 0.573
J. B. S.	Sub. B.	26	Br.			23.3	26.6	32.2	42.9	71.6	115.0	163	290	485	736	867	1005	990	996	921
V. J. L.	G. S.	37	Gr.			22.1	26.6	29.8	40.5	65.7	104.0	150	296	432	681	876	963	968	1000	939
M. P. S.	N.	36	Gr.								164.0	243	352	520	736	890	990	1011	951	942
J. A. S.	Sub. B.	45	Gr.			23.7	28.4	30.9	41.3	66.1	98.6	160	298	512	759	917	1000	1000	982	947
H. K. G.	N.	30	r.									200	348	564	771	905	990	1000	981	920
W. S. S.	R. S.	29										181	304	510	758	928	984	962	1000	924
V. P. G. A.	W.	35				63.2	76.4	90.5	127.0	177.0	257.0	325	464	626	786	899	945	992	1000	971
H. I. S.	N.	25			31.6	40.4	45.7	56.8	76.3	120.0	165.0	231	361	516	750	853	944	944	1000	953
E. L. P.	R. S.	33			34.2	43.0	52.6	68.6	73.1	98.2	168.0	244	389	544	696	873	952	966	1000	979
L. W. S.	Sub. B.	26	Br.			32.2	33.4	37.5	49.5	76.0	116.0	180	358	530	760	924	982	1020	968	939
J. H. D.	Sub. R.	29	Bl.			29.0	35.3	41.0	52.2	65.2	114.0	215	347	548	781	909	934	985	1015	920
R. L. S.	R. S.	31				37.8	38.6	45.3	75.5	121.0	160.0	250	362	580	705	886	971	991	1000	948
C. W. B.	B. S.	42									198.0	285	431	612	815	940	999	1000	967	924
D. E. M.	R. S.	33										265	369	576	772	899	970	980	1000	936
G. W. V.	B. S.	33	Br.									251	406	596	801	926	994	987	994	943
V. G. E. P.	G. S.	36										150	323	486	746	928	985	982	1016	953
V. O. L. S.	G. S.	32	Gr.			18.4	24.1	28.5	37.5	66.4	101.0	166	270	498	720	874	950	1000	989	955
M. J.	R. S.	24	Br.									182	307	521	706	870	911	970	969	950
V. J. F. M.	G. S.	40	Br.									193	319	546	774	915	973	1000	992	904
P. D. L.	B. S.	23	Br.			34.6	58.8	80.2	115.0	153.0	240.0	317	452	602	770	902	988	1024	966	918
R. Y. F.	N.	40	Bl.			25.2	34.7	50.6	54.4	97.3	152.0	219	359	584	784	895	981	1000	974	923
W. A. C.	R. S.	21	Br.									276	400	562	744	861	939	958	981	1000
F. J. B.	N.	39											308	514	765	914	994	1000	993	960
P. J. H.	B. S.	22	Gr.										455			844	953	1000	995	871
T. E. H.	R. S.	25	Br.									181	316	500	735	868	954	978	1020	968
H. A. E.	N.	24	Br.									180	329	501	734	913	984	1014	976	915
E. C. P.	Sub. B.	22	Br.									153	264	439	759	881	968	1000	991	958
C. H. M.	B. S.	23										270	420	615	779	939	974	1007	974	861

Visibility of Normal Subjects—Red End of Spectrum ✓

Subject	Age	Eye color	Factor	$\lambda = 0.580$	$\lambda = 0.5976$	$\lambda = 0.5964$	$\lambda = 0.604$	$\lambda = 0.613$	$\lambda = 0.623$	$\lambda = 0.633$	$\lambda = 0.643$	$\lambda = 0.654$	$\lambda = 0.665$	$\lambda = 0.678$	$\lambda = 0.690$	$\lambda = 0.703$	$\lambda = 0.717$	$\lambda = 0.750$	$\lambda = 0.746$	$\lambda = \text{max}$
F. B. S.	26	Br.	1.09	905	807	752	667	540	432	308	195.0	103.0	51.4	23.2	9.84	4.18	1.53	0.596	0.280	0.557
✓ J. L.	37	Gr.	1.06	875	792	707	560	462	337	224	150.0	82.4	42.6	19.5	9.32	3.66	1.41	.617	.325	.558
M. P. S.	36	Gr.	1.13	868	825	731	624	506	375	247	158.0	88.2	43.0	19.1						.557
J. A. S.	45	Gr.	1.17	913	826	737	609	521	394	270	175.0	95.3	48.0	22.3	10.1	4.17	1.48	.690	.331	.558
H. K. G.	30	Gr.	1.17	879	818	723	630	522	394	285	178.0	102.0	47.7	22.6						.557
W. B. S.	29		1.17	882	848	764	678	559	415	304	186.0	106.0	54.3	24.4						.560
✓ P. G. A.	35		1.23	946	872	779	705	572	416	290	185.0	102.0	50.0	22.8	9.26	4.42	1.62	.675	.364	.561
H. I. S.	25		1.10	875	818	737	629	519	400	279	178.0	96.9	51.8	22.8	9.57	4.26	1.66	.668	.365	.559
E. L. P.	33		1.24	971	904	816	738	576	446	327	210.0	110.0	56.9	25.7	10.1	4.82	1.67	.675	.320	.565
L. W. S.	26	Br.	1.19	879	835	740	620	503	385	264	162.0	92.1	44.5	20.6	7.98	3.69	1.46	.574	.303	.557
J. H. D.	29	Bl.	1.05	872	774	664	558	396	305	193	121.0	65.4	34.0	15.0	5.73	2.68	.970	.429	.224	.556
R. L. S.	31		1.25	937	891	803	682	578	452	316	202.0	109.0	54.7	27.0	10.6	5.30	1.95	.766	.383	.563
C. W. B.	42		1.17	857	769	675	592	448	345	228	142.0	83.2	38.6	17.3						.553
D. R. M.	33		1.25	921	850	751	681	563	438	310	200.0	116.0	56.5	26.5	9.85					.559
G. W. V.	33	Br.	1.24	860	776	688	599	449	355	236	142.0	85.9	42.8	19.3						.554
✓ G. E. P.	36		1.15	867	803	723	596		366		153.0		41.9	20.7						.556
✓ O. L. S.	32	Gr.	1.26	878	788	698	598	475	364	261	154.0	89.3	44.3	19.3	9.14	3.85	1.37	.518	.249	.554
M. J.	24	Br.	1.41	967	867	796	707	618	477	371	217.0	136.0	65.8	29.8	12.6					.568
✓ J. F. M.	40	Br.	1.15	867	773	660	589	492	362	257	161.0	96.5	48.6	21.5						.554
P. D. L.	23	Br.	1.20	884	808	689	600	476	351	237	147.0	84.4	41.5	17.7	7.85	3.21	1.16	.470		.557
R. Y. J.	40	Bl.	1.14	858	790	720	621	513	375	271	171.0	93.9	45.1	21.1	9.11	4.07	1.62	.526	.251	.555
W. A. C.	21	Br.	1.44	975	920	860	754	656	499	373	214.0	121.0	64.2	29.1	11.8					.559
F. J. B.	39		1.35	894	832	750	612	515	371	251	157.0	86.8	44.2	18.6						.559
P. J. H.	22	Gr.	1.30	832	757	613	538	430	327	232	147.0	80.1	40.0	20.6						.555
T. R. H.	25	Br.	1.30	959	928	854	743	625	470	337	224.0	125.0	62.0	27.7	12.6					.566
H. A. E.	24	Br.	1.16	898	814	748	605	498	394	250	162.0	89.1	42.7	17.2						.566
E. C. P.	22	Br.	1.24	884	813	726	632	515	401	274	172.0	102.0	51.2	24.6	10.4					.557
C. H. M.	23		1.36	809	727	647	535	444	321	213	132.0	74.5	35.7	16.5						.551

TABLE 1—Continued
 Visibility of Normal Subjects—Blue End of Spectrum

Subject	Visi- bilitys	Age.	Eye color ^b	Fac- tor	$\lambda =$ 0.427	$\lambda =$ 0.435	$\lambda =$ 0.444	$\lambda =$ 0.450	$\lambda =$ 0.456	$\lambda =$ 0.463	$\lambda =$ 0.474	$\lambda =$ 0.485	$\lambda =$ 0.493	$\lambda =$ 0.502	$\lambda =$ 0.5125	$\lambda =$ 0.523	$\lambda =$ 0.534	$\lambda =$ 0.546	$\lambda =$ 0.552	$\lambda =$ 0.559	$\lambda =$ 0.573
C. E. B.	N.	22	Br.	1.44									262	390	564	770	904	960	1015	994	955
C. A. B.	E. S.	33	Bl.	1.50									326	472	634	849	914	991	1000	994	943
W. S. L.	R. S.	32	Bl.	1.39									210	381	531	744	893	960	992	1000	982
T. B. F.	R. S.	34	Br.	1.42										268	462	681	822	953	945	1000	986
D. R. H.	R. S.	31	Bl.	1.39		35.2	43.4	48.9	62.2	73.6	173.0		251	362	567	752	905	965	980	1022	951
P. D. S.	R. S.	24	Gr.	1.43									267	383	583	732	844	920	957	1000	942
A. N. F.	R. S.	34	Br.	1.30										318	539	708	876	966	968	972	986
G. T. M.	R. S. ^p	26	Bl.	1.43										273	447	672	822	914	941	989	984
H. S.	N.	22	Bl.	1.34									242	318	502	782	906	971	1020	1018	950
H. B. K.	E. S.	28	Bl.	1.17									248	390	599	771	926	992	1000	983	914
S. L.	N.	26	Bl.	1.21		38.9	42.5	53.5	78.6	108.0	169.0		256	370	574	801	942	1000	984	956	946
J. L. F.	Sub. R.	25	Bl.	1.29		35.4	41.6	45.6	61.4	76.0	115.0	173.0	258	386	568	760	882	959	981	1012	968
A. B. L.	N.	26	Bl.	1.34									194	338	528	778	902	941	986	1000	898
J. O. S. P.	W.	32	Gr.	1.47				38.0	46.5	61.2	93.3	136.0	283	414	609	770	903	968	976	1000	986
W. H. Sm.	E. S.	29	Br.	1.35									264	404	630	863	950	986	1000	976	886
L. A. G.	Sub. R.	29	Br.	1.35									207	354	562	746	892	982	957	994	880
A. B.	Sub. R.	23	Br.	1.33									213	351	531	736	873	976	1009	948	892
G. C. H.	Sub. R.	26	Bl.	1.46									189	340	533	782	911	967	988	1000	918
T. R. E. ^g	Sub. R.	23	Bl.	1.45									226	339	553	789	900	941	994	1000	928
A. J. H. ^r	Sub. R.	23	Bl.	1.44										370	578	796	930	1010	985	970	905
H. M. R. ^f	E. S.	25											309	447	660	825	921	991	1000	969	895
E. C. C.	N.	23	Br.	1.28									229	358	581	784	885	962	986	1000	952
E. K. B.	Sub. B.	42	Bl.	1.28									249	463	718	881	964	986	1000	953	953
P. H.	Sub. R.	24	Br.	1.24									240	405	587	800	896	993	976	1008	994
H. S. P.	E. S.	33	Br.	1.33									254	416	618	840	980	1000	973	950	871
H. H. B.	Sub. R.	30	Br.	1.16									198	341	592	773	911	986	1013	973	835
✓ H. A. B.	G. S.	26	Bl. Gr.	1.47									189	333	536	743	906	994	991	991	924
R. D.	E. S.	28	Bl.	1.67										270	503	724	921	961	978	1000	983

Visibility of Normal Subjects—Red End of Spectrum ✓

Subject	Visi- bility	Age	Eye color	Fac- tor	$\lambda =$ 0.580	$\lambda =$ 0.5876	$\lambda =$ 0.5964	$\lambda =$ 0.604	$\lambda =$ 0.613	$\lambda =$ 0.623	$\lambda =$ 0.633	$\lambda =$ 0.643	$\lambda =$ 0.654	$\lambda =$ 0.665	$\lambda =$ 0.678	$\lambda =$ 0.690	$\lambda =$ 0.703	$\lambda =$ 0.717	$\lambda =$ 0.730	$\lambda =$ 0.746	λ max
C. E. B.	N.	23	Br.	1.44	888	806	755	649	532	401	292	186.0	110.0	54.8	25.6						0.556
C. A. B.	B. S.	33	Bl.	1.50	871	779	701	581	482	368	252	153.0	85.1	44.8	18.3						.555
W. S. L.	B. S.	32	Bl.	1.39	956	870	785	697				204.0		60.7	26.1						.560
T. B. F.	R. S.	34	Br.	1.42	959	908	804	724	635	493	350	214.0	125.0	61.4	26.4	9.26					.565
D. R. H.	R. S.	31	Bl.	1.39	956	856	763	687	589	446	311	195.0	120.0	61.2	30.5	12.8	5.10	2.03	0.804		.559
P. D. S.	R. S.	34	Gr.	1.43	966	897	802	719	599	461	320	195.0	116.0	55.4	24.5	10.4					.565
A. N. F.	R. S.	24	Br.	1.30	957	946	869	770	642	466	332	194.0	107.0	53.0	24.4						.566
G. T. M.	R. S. P.	26	Bl.	1.43	978	927	884	787	661	520	381	250.0	151.0	70.0	35.0	14.3					.570
H. S.	N.	22	Bl.	1.34	920	840	744	673	568	407	281	185.0	104.0	53.0	23.9						.558
H. B. K.	B. S.	28	Bl.	1.17	880	805	711	593	463	340	226	143.0	79.0	38.4	16.6						.556
S. L.	N.	26	Bl.	1.21	897	818	698	620	531	362	254	159.0	89.1	45.3	18.6	8.04	3.90	1.54	.594		.554
J. L. F.	Sub. R.	25	Bl.	1.29	884	792	689	590	488	341	249	141.0	82.7	42.1	18.3	7.19	3.23	1.19	.485		.559
A. B. L.	N.	26	Bl.	1.34	880	806	731	593	508	384	269	170.0	90.9	48.4	22.5	8.90	4.20	1.72	.658	0.355	.559
O. S. P.	W.	32	Gr.	1.47	954	872	799	722	596	474	333	201.0	119.0	58.8	27.6						.563
W. H. Sm.	B. S.	29	Br.	1.35	896	756	664	545	404	314	198	129.0	76.8	36.0	18.2						.549
L. A. G.	Sub. R.	29	Br.	1.35	865	760	674	582	484	346	253	164.0	96.2	46.1	19.9						.558
A. B.	Sub. R.	23	Br.	1.33	888	786	697	563	454	359	246	158.0	86.6	43.2	18.1	9.35					.558
G. C. H.	Sub. R.	26	Bl.	1.46	851	768	688	604	485	373	264	168.0	91.2	48.6	22.8						.557
T. R. E.	Sub. R.	23	Bl.	1.45	945	788	719	604	496	372	240	160.0	88.6	45.6	19.5						.557
A. J. H.	Sub. R.	23	Bl.	1.44	838	780	689	582	479	366	262	165.0	100.0	48.5	22.1						.550
H. M. R.	B. S.	23			832	774	694	567		355		161.0		46.1							.551
B. C. C.	N.	25	Br.	1.28	891	815	748	652	521	388	275	173.0	97.2	46.8	22.0						.559
G. K. B.	Sub. B.	42	Bl.	1.28	879	804	731	631		385		178.0		52.7							.557
P. H.	Sub. R.	24	Br.	1.24	852	764	653	527	399	281	187	112.0	59.9	30.3	13.9						.556
H. S. P.	B. S.	33	Br.	1.33	820	754	638	546	411	311	204	120.0	69.2	32.8	15.6						.549
H. H. B.	Sub. R.	30	Br.	1.16	734	658	558	436		237		94.6	53.3	26.4	12.9						.550
H. A. B.	G. S.	26	Bl. Gr.	1.47	874	771	689	567	451	345	236	144.0	79.3	41.5	18.6						.555
R. D.	R. S.	28	Bl.	1.67	953	876	799	716		486		226.0		65.2	31.1						.562

TABLE 1—Continued
 Visibility of Normal Subjects—Blue End of Spectrum

Subject	Visi- bility	Age	Eye color	Fac- tor	$\lambda =$ 0.427	$\lambda =$ 0.435	$\lambda =$ 0.444	$\lambda =$ 0.450	$\lambda =$ 0.456	$\lambda =$ 0.463	$\lambda =$ 0.474	$\lambda =$ 0.485	$\lambda =$ 0.493	$\lambda =$ 0.502	$\lambda =$ 0.5125	$\lambda =$ 0.523	$\lambda =$ 0.534	$\lambda =$ 0.546	$\lambda =$ 0.552	$\lambda =$ 0.559	$\lambda =$ 0.573
E. S. P.	R. S. P.	24	Br.	1.65										306	502	662	813	920	949	975	1003
W. H. St.	N.	26	Gr.	1.52										380	553	726	910	1003	986	990	974
M. D. S.	N.	21	Br.	1.37										280	520	780	911	992	1010	963	969
G. S.	G. S.	49	Bl. Gr.	1.49										259	490	716	890	972	984	1010	938
✓ J. L. B.	W.	26	Br.	1.54									373	475	614	730	878	976	990	964	966
J. B.	B. S.	20	Br.	1.38									302	446	607	769	900	1004	1000	973	910
E. E. W.	B. S.	27	Br.	1.42	37.4	49.5	66.0	76.7	89.5	119.0	182.0	244.0	339	482	648	845	970	1000	976	990	922
C. P. E.	Sub. B.	58	Br.	1.48										259	549	734	894	949		996	974
M. S.	N.	28	Br.	1.45									233	375	577	766	920	956	961	1000	940
J. C. P.	Sub. R.	37		1.37									212	332	581	782	900	981	1010	985	878
P. D. M.	R. S.	27	Bl.	1.71										305	522	729	896	969	986	1012	948
H. D. H.	N.	45	Bl.	1.72									227	349	551	762	871	972	988	1000	928
P. W. M.	B. S.	29	Bl.	1.30									242	387	604	804	956	977	1000	984	883
N. S. O.	N.	41	Gr.	1.48										353	517	723	883	935	974	1000	938
A. D. C.	N.	29	Gr.	1.57										339	526	735	918	1000	998	987	943
Mean uncorrected value.....					28.0	33.6	36.6	39.8	48.9	63.1	98.7	157.0	225	351	548	758	897	973	969	992	944
Mean value ($\lambda_{\max} = 1000$).....					28.2	33.9	36.9	40.1	49.3	63.6	99.5	158.0	227	354	552	764	904	981	997	1000	952
Corrected for slit width.....					28.0	33.7	36.7	39.7	48.6	62.8	98.1	155.0	221	346	548	765	904	981	997	1000	954
Final value (corrected for 0° Nicol and for scattered light).....					28.4	34.1	37.2	40.1	49.1	63.4	98.8	155.0	223	350	553	771	908	983	998	1000	952

Visibility of Normal Subjects—Red End of Spectrum ✓

Subject	Vis- ibility	Age	Eye color	Fac- tor	$\lambda =$ 0.580	$\lambda =$ 0.5876	$\lambda =$ 0.5964	$\lambda =$ 0.604	$\lambda =$ 0.613	$\lambda =$ 0.623	$\lambda =$ 0.633	$\lambda =$ 0.643	$\lambda =$ 0.654	$\lambda =$ 0.665	$\lambda =$ 0.678	$\lambda =$ 0.690	$\lambda =$ 0.703	$\lambda =$ 0.717	$\lambda =$ 0.730	$\lambda =$ 0.746	λ max
E. S. P.	R. S. p	24	Br.	1.65	956	939	898	797	709	546	379	265.0	153.0	82.2	36.7					0.570	
W. H. St.	N.	26	Gr.	1.52	903	827	727	644	514	406	272	166.0	95.9	46.2	22.5					.557	
M. D. S.	N.	21	Br.	1.37	899	831	757	684	556	393	279	171.0	89.0	47.3	23.4					.555	
V. E. B.	G. S.	49	Bl. Gr.	1.49	873	810	706	616	508	362	251	149.0	86.2	41.3	26.6					.558	
✓ J. L. B.	W.	26	Br.	1.54	902	840	771	663	581	447	318	197.0	109.0	56.5	28.2	13.0				.560	
J. B.	B. S.	20	Br.	1.38	866	776				381											
E. E. W.	B. S.	27	Br.	1.42	862	796	706	597	477	357	251	153.0	89.0	43.3	20.6	8.96	3.94	1.48	0.614	.551	
C. P. K.	Sub. B.	58	Br.	1.48	910	817	747	628	512	420	281	185.0	89.0	52.5	22.6	11.0				.559	
M. S.	N.	28	Br.	1.45	880	790	703	606	493	381	280	165.0	97.6	50.0	23.6					.557	
J. C. P.	Sub. R.	37		1.37	796	748	658	575	473	371	269	166.0	97.2	47.3	23.2					.552	
P. D. M.	R. S.	27	Bl.	1.71	895	878	801	725	550	430	299	192.0	108.0	57.1	24.7					.560	
H. D. H.	N.	45	Bl.	1.72	874	835	746	637		416		195.0		56.0	26.0					.558	
P. W. M.	B. S.	29	Bl.	1.30	811	728	648	527	366	296	200	128.0	63.8	32.9	14.2					.552	
N. S. O.	N.	41	Gr.	1.48	861	808	738	637	557	425	327	197.0	117.0	55.7	25.6					.559	
A. D. C.	N.	29	Gr.	1.57	913	833	738	632		381		180.0		61.9						.554	
Mean uncorrected value					893.	818	731	632	516	391	272	172.0	96.1	48.3	22.1	9.77	3.97	1.49	.591	.288	.5576
Mean value ($\lambda_{\max} = 1000$)					900	825	737	637	520	394	274	173.0	96.9	48.7	22.3	9.85	4.00	1.50	.596	.290	
Corrected for all width					901	825	736	637	518	390	266	164.0	95.6	47.7	21.7	9.55	3.86	1.43	.589	.286	
Final value (corrected for 0° Nicol and for scattered light)					899	823	734	636	517	390	267	165.0	94.5	45.9	20.2	8.56	3.36	1.21	.485	.280	.5576

a Classification of curves: N., average; W., wide; R. S., red sensitive; B. S., blue sensitive; G. S., green sensitive; Sub. R., subnormal red; Sub. B., subnormal blue
 b Bl., blue; Gr., gray; L., Bl., light blue; Br., brown; Bl. Gr., blue-gray. c Mean of 3 sets. d $\lambda = 0.422$, $V = 18.7$; $\lambda = 0.417$, $V = 16.7$; $\lambda = 0.412$, $V = 13.6$; $\lambda = 0.406$, $V = 9.18$;
 $\lambda = 0.401$, $V = 8.34$. e $\lambda = 0.566$, $V = 997$. f Mean of 2 sets. g $\lambda = 0.422$, $V = 142$. h $\lambda = 0.498$, $V = 166$. i $\lambda = 0.566$, $V = 978$. j $\lambda = 0.566$, $V = 106$. k $\lambda = 0.728$, $V = 1.08$.
 l $\lambda = 0.765$, $V = 0.120$. m $\lambda = 0.765$, $V = 0.110$. n $\lambda = 0.538$, $V = 761$. o $\lambda = 0.477$, $V = 141$. p Also color blind. q Close to normal. r Subnormal in yellow.

1. TOTAL COLOR BLINDNESS

The investigation of subjects who are entirely lacking in color sensation is of interest in connection with the various theories of color vision. At low illumination only the rods are considered to respond to light and no color is perceived. The maximum sensibility of the fovea of the normal eye lies between $0.49\ \mu$ and $0.53\ \mu$ for low illuminations.

In the case of total color blindness the maximum visibility is in the region of $0.52\ \mu$, as shown in Fig. 4 (data from Bender²⁶).

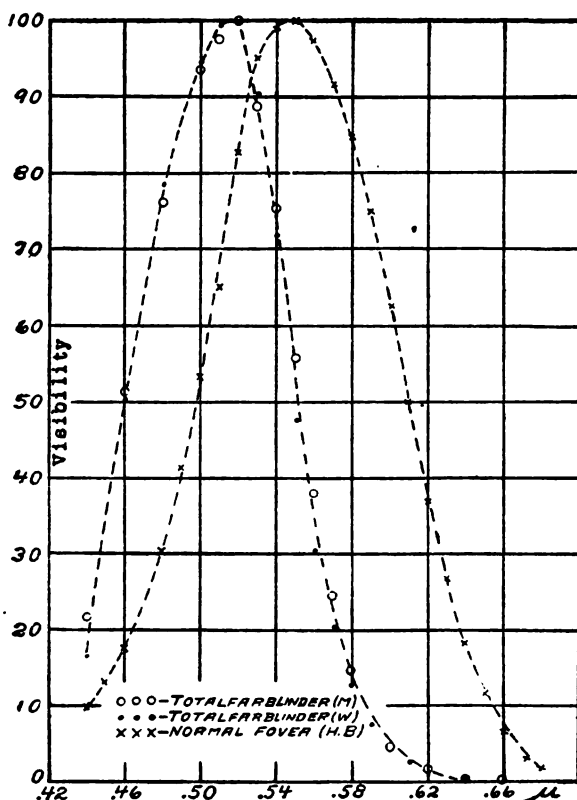


FIG. 4.—Relative sensibility of the eye to different wave lengths in the case of total color blindness (data from Bender)

From this it would appear that the color perception, in the case of so-called total color blindness, is similar to that of normal foveal color vision under very low illumination, and to that of rod vision (i. e., peripheral vision) under high illumination. As shown by Bender,²⁶ the curve for the peripheral retina (normal rod vision) coincides with that of the foveal visibility curve of totally color-blind subjects. This phenomena is of rare occurrence, and hence it is difficult to find subjects for investigation.

2. PARTIAL COLOR BLINDNESS ^a

This phenomenon is of frequent occurrence. Ferry ¹¹ found eight cases in a group of 200 students, which is close to the average (3.95 per cent among males) found by Dr. Jeffries ⁴⁸ in a group of 175,000. Among women color blindness is only about one-tenth as frequent as among men. It is well known that a partially color-blind person is not only greatly lacking in one of the fundamental color sensations, but also that he perceives other colors quite differently from the "normal."

Ferry ¹¹ gives the results of an examination of eight color-blind persons. One was a case of inherited red blindness. The visibility of this red-blind subject was very much depressed in the red as compared with that of normal color vision. The other seven persons were green blind. In all cases their persistence of vision (visibility) curves were normal except in the green, where there was a marked depression. These cases are somewhat different from the observations of de Lepinay and Nicati,⁵ whose results for dichroic vision indicate that in red blindness the visibility curve is abnormally high in the green and that in green blindness the sensibility is abnormally high in the red (but normal in the blue and, of course, below normal in the green), which agrees with the observations recorded in the present paper.

Allen ¹² has described the curve of a green-blind subject in which the visibility was normal for all parts of the spectrum except in the green, also a red-blind subject in which the visibility curve was depressed below the normal only in the red. Another red-blind subject showed depressions in both the red and the green.

Of the persons showing very marked red-green color blindness by the Holmgren yarn test, examined by Tufts,²² three had the point of maximum visibility displaced toward the green and the other three had the maximum shifted toward the red. As in the present investigation, he had another subject whose visibility curve was similarly shifted toward the red, although the observer showed no trace of color blindness by the ordinary Holmgren test.

In persons exhibiting color blindness the most common phase is a low visibility in the green, producing quite an indentation, as shown in Fig. 5. However, persons having normal vision may

^a The writers adhere to the older nomenclature, in which the term "color blindness" was applied to cases of color confusion by the Holmgren tests. It is beyond the scope of the present paper to attempt a discussion of the newer ideas concerning color vision, in view of the fact that it deals with monochromatic brightness sensation in different parts of the spectrum and does not attempt to analyze the results obtained when applying a heterochromatic stimulus.

have a temporary depression of the visibility in this region of the spectrum, as was found (W. W. C.) in the preliminary part of the present investigation.

Of the subjects examined who exhibit green color blindness, two (A. F. and J. F. S.) confused reds and greens in making the Nagel color test. Another subject (A. A. L.), an instrument maker, can not distinguish brass from copper. An unusual case

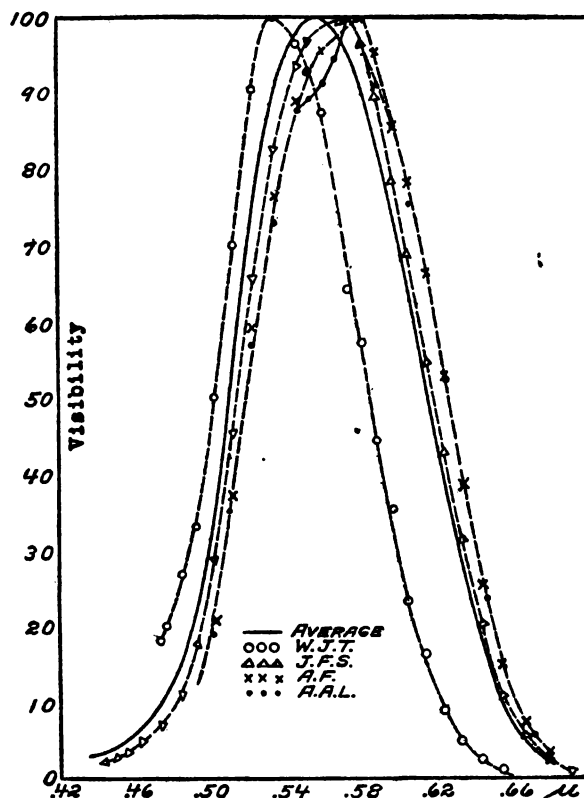


FIG. 5.—Spectral sensibility curves of eyes affected by partial color blindness

(W. J. T.) of red blindness was found. The subject is engaged in agricultural pursuits and can not distinguish red apples or red roses from the green leaves except by their shape. His visibility curve (Fig. 5) is unusually low in the red and high in the blue. A somewhat similar curve is shown in Fig. 7 (H. B. S.), but in this case the subject did not confuse colors when tested with the Nagel and the Stilling color cards.

The results of the present investigation are in agreement with those of Ferry ¹¹ and Tufts ²² and others, showing that the effect of light upon the color sense is quite independent of its effect upon the brightness sense.^a An abnormal color sense is associated with an abnormal (brightness sensibility) visibility curve; but the converse is not true, as is shown by the examples of abnormal visi-

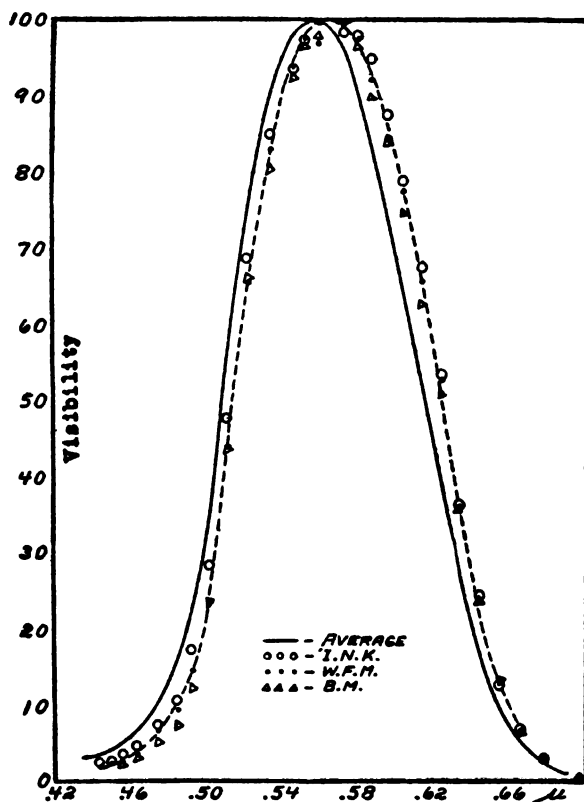


FIG. 6.—Visibility curves of observers who are red-sensitive but not color blind

bility curves (Fig. 6) of persons having normal judgments in sorting and matching colors. The data of all but two of the group of observers who were known to be color blind are given in Table 2. If all the observers had been tested, no doubt others would have been found who confused colors.

^a See Appendix 4. "On color perception versus brightness perception."

TABLE 2
 Visibility of Color-Blind Subjects

Wave length μ	J. F. S. Eye, blue factor, 1.20 $\lambda_m=0.568$	A. F. Eye, blue factor, 1.26 $\lambda_m=0.578$	C. W. K. Eye, brown factor, 1.67 $\lambda_m=0.567$	W. J. T. Eye, blue factor, 1.01 $\lambda_m=0.539$	A. A. L. Eye, brown factor, 1.75 $\lambda_m=0.573$
0.444.....	22.5		43.8		
0.450.....	29.0		50.0		
0.456.....	35.6		60.6		
0.463.....	50.4		80.1		
0.474.....	71.4		123.0	182.0	
0.485.....	108.0		180.0	α 270.0	
0.493.....	180.0		273.0	337.0	
0.502.....	300.0	210.0	403.0	502.0	208.0
0.5125.....	468.0	375.0	560.0	701.0	375.0
0.523.....	669.0	591.0	750.0	907.0	571.0
0.534.....	817.0	766.0	862.0	1003.0	730.0
0.546.....	918.0	889.0	928.0	967.0	880.0
0.552.....	947.0	928.0	945.0	926.0	892.0
0.559.....	943.0	957.0	998.0	874.0	916.0
0.566.....	1000.0		1000.0		1000.0
0.573.....	989.0	990.0	1025.0	633.0	1000.0
0.580.....	957.0	1000.0	963.0	566.0	962.0
0.5876.....	891.0	952.0	886.0	443.0	910.0
0.596 ₁	799.0	858.0	863.0	354.0	858.0
0.604.....	725.0	783.0	758.0	236.0	759.0
0.613.....	588.0	664.0	632.0	165.0	
0.623.....	483.0	530.0	483.0	90.6	526.0
0.633.....	325.0	390.0	338.0	50.6	
0.643 ₂	196.0	254.0	219.0	23.7	256.0
0.654.....	118.0	150.0	121.0	11.7	
0.665.....	56.4	73.6	61.6		73.4
0.678.....	23.6	35.0	27.3		32.2
0.690.....	8.88	15.1	11.0		15.4
0.703.....	3.83		4.71		
0.717.....	1.60		1.93		
0.730.....	.919		.830		
0.746.....	.290		.458		

$\alpha \lambda = 0.477\mu$; $V = 202$.

3. RED SENSITIVENESS

It was noticed early in the investigation that out of two dozen subjects one-fourth were abnormally sensitive in the red, as illustrated in Fig. 6. If the investigation had been terminated at that time, the shape of the visibility curve of the average normal eye and the position of its maximum would have been considerably different from the curve resulting from averaging the data of all the subjects investigated.

Of the 10 subjects examined by Bender ²⁶ the average visibility curves of three observers is separated by a wide gap from the

others in the red part of the spectrum and the maximum visibility is located at 0.55μ , while the maximum visibility of the others is located at 0.535μ . Of these three red sensitives two were women, while the remaining two women of the group of 10 subjects were abnormally sensitive in the blue-violet. From this it would appear that while color blindness may be far less frequent among women than among men their visibility curves are probably subject to as great variation as is to be found among men. Houstoun¹⁷ likewise found no systematic deviation of the observations of nine women as compared with the men.

Of the 26 red-sensitive subjects found in the present investigation (some of which may have been color blind), the visibility curve of W. F. M. is of special interest in view of the fact that two series of observations were made on a Saturday and the following Monday. The intervening day of rest and recreation had no effect upon the curve, the observations coinciding exactly throughout the whole spectrum. Even the observation of the sharp maximum at 0.573μ was repeated. Another peculiarity exhibited by this observer was the very low speed required in order to cause disappearance of flicker. The speed was only about 3 cycles per second, which shows an unusually long persistence of the visual impression. Apparently the speeds for disappearance of color flicker and brightness flicker are closely the same in this subject, who does not appear to confuse colors.^a Tufts²² had two observers who were abnormally sensitive in the red, and they showed a slight tendency to confuse oranges and yellow greens.

As shown in the composite curve of the 125 observers, Fig. 13, the distinct grouping of red sensitives, which is conspicuous in the composite curve (of only 10 observers) published by Bender,²⁴ is obliterated by a general gradation in the shift of the various red-sensitive curves from the normal to the extreme red.

4. BLUE SENSITIVENESS

Persons who are markedly sensitive in the blue are not found quite so frequently as are the red sensitives just described. A very marked example is given by Nutting,²⁷ also one by Bender,²⁴ in which the maximum visibility is greatly shifted toward the violet, and in which the whole visibility curve is shifted as compared with the average normal curve.

As was to be expected, a close parallelism was found between the observations of Crittenden and Richtmyer²⁸ and the results

^a See Appendix 4 for further tests on this subject.

of the present investigation. In their test of the transmission of yellow and blue solutions (or glasses) they found one observer (H. B. S.) whose data indicated an apparently high sensibility in the blue or low sensibility in the red as compared with the group of normals. As shown in Fig. 7, the peculiarity of the vision of this subject is in its low sensibility in the red and a high sensibility in the blue, thus causing a shift of the whole curve. A

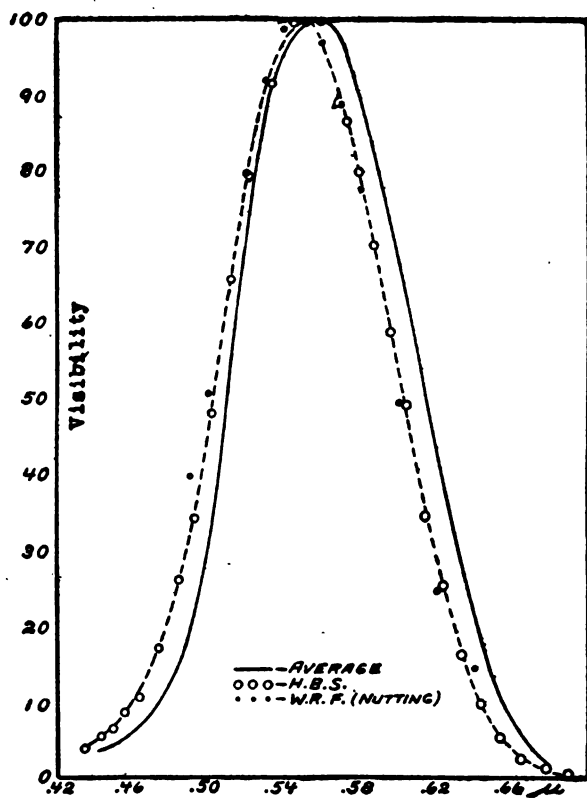


FIG. 7.—Visibility curves of blue-sensitive subjects

similar case has just been described (W. J. T.) in which this shift of sensibility was accompanied by color blindness to the red. Other observers (e. g., I. G. P. and K. B.) who were slightly more sensitive in the violet than the normals, as indicated by the observations of Crittenden and Richtmyer, were found to have visibility curves which are somewhat above the normal in the blue part of the spectrum.

5. GREEN SENSITIVENESS

A characteristic group of observers have narrow visibility curves in the sense that the curves fall below the normal in the yellow and the blue and terminate rather sharply at the maximum in the green. The group of subjects examined by Nutting²⁷ has several marked examples of this type, about one-fourth of the whole group showing this characteristic to some extent. In Fig.

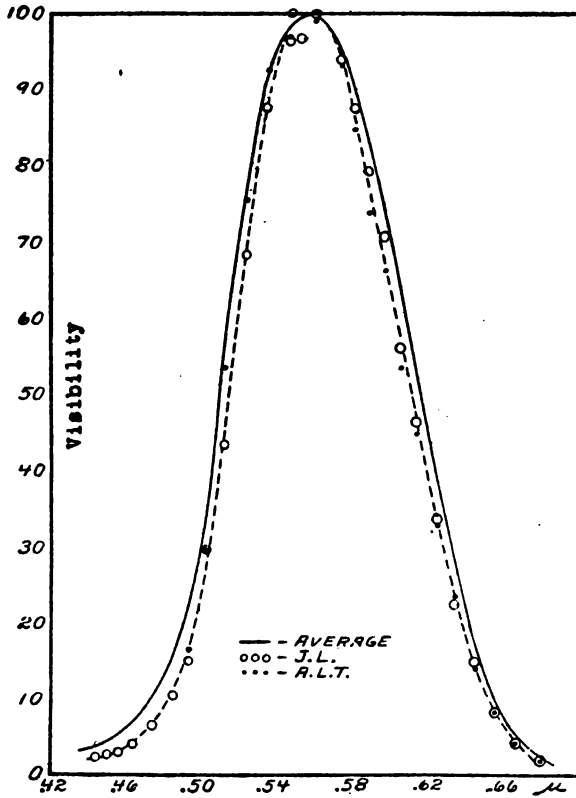


FIG. 8.—Visibility curves of green-sensitive subjects

8 the visibility curves of a number of subjects are given which exhibit this peculiarity. In the green-sensitive subject the response of the red and the blue is not up to the average eye.

6. RED-BLUE SENSITIVENESS

This class includes a characteristic of infrequent occurrence, in which the visibility curve is very wide as compared with the normal curve. Curves of this classification are given in Fig. 9. This

type is not as common as that of red sensitiveness, only 1 marked example being present among the 21 observers recorded by Nutting²⁷ and 7 among the 130 examined in the present investigation.

7. SUBNORMAL BLUE SENSITIVENESS

Among those who might be classed as yellow sensitive by some tests is a group of observers whose visibility curves coincide very

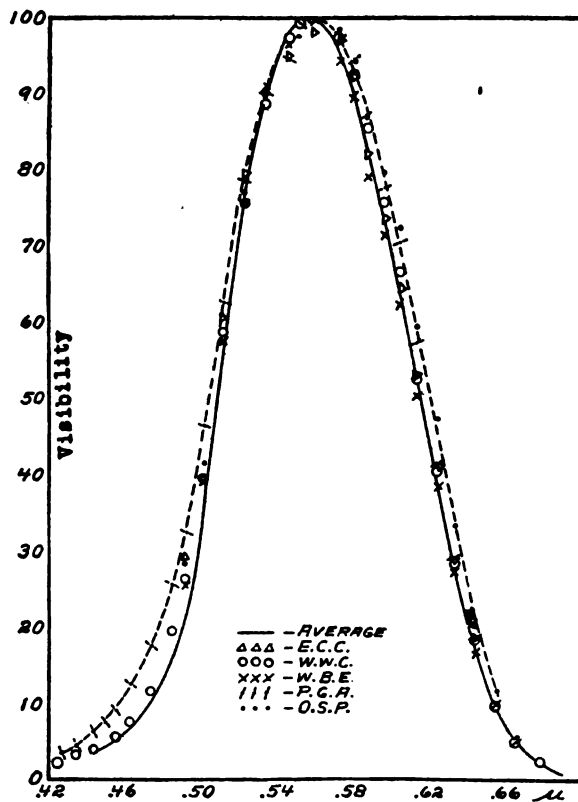


FIG. 9.—Visibility curves of subjects having a high sensibility over a wide range of wave lengths

closely, if not exactly, with the average visibility curve throughout the spectrum except in the blue violet, where the visibility falls below the average value. Before making the test several of the observers in this group reported that their eyes were not very sensitive to the blue. The curves of several observers having a low sensibility in the blue, but having the average visibility in the remainder of the spectrum are given in Fig. 10.

8. SUBNORMAL RED SENSITIVENESS

Among those who might be classed sensitive in the blue-green by some tests (e. g., the tests by Crittenden and Richtmyer²⁹) is a group of observers whose visibility curves are low in the red and yellow but coincide very closely with the average visibility curve in the green and blue parts of the spectrum. Illustrations of this type of color sensibility are given in Fig. 11. They differ

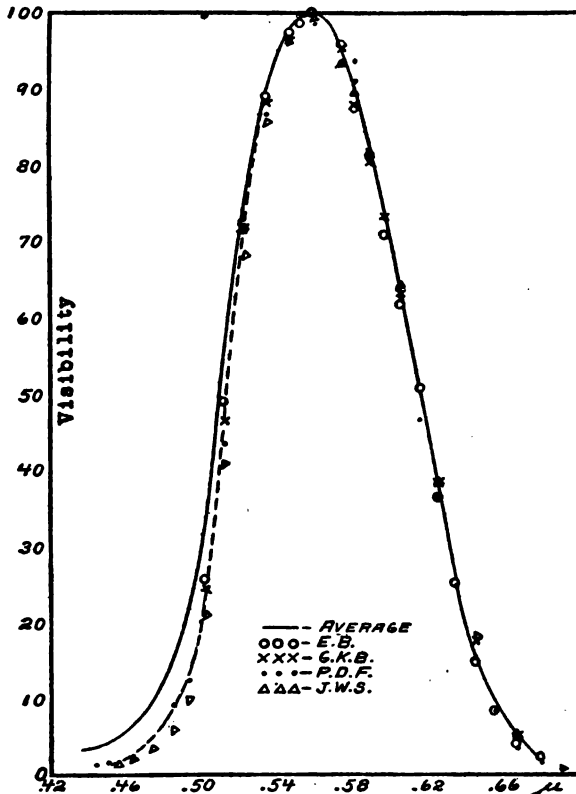


FIG. 10.—Visibility curves of sub-normal blue-sensitive subjects

from the curves (Fig. 7) of high sensibility in the blue in that in the latter the whole visibility curve is shifted to the blue. Furthermore, in the latter the shape of the curve in the blue is different from that of the average curve, so that it is not possible to superpose them nicely in the blue, as is possible with the curves in the present and in the preceding classification.

This classification includes a number of subjects whose visibility curves fall below the average in the yellow and orange

instead of the deep red, but it did not seem necessary to classify them in a separate group.

9. AVERAGE COLOR VISION

In this group are to be found about one-fourth of the total number of persons examined. The visibility curve is smooth and free from indentations, and it is quite symmetrical with the maximum

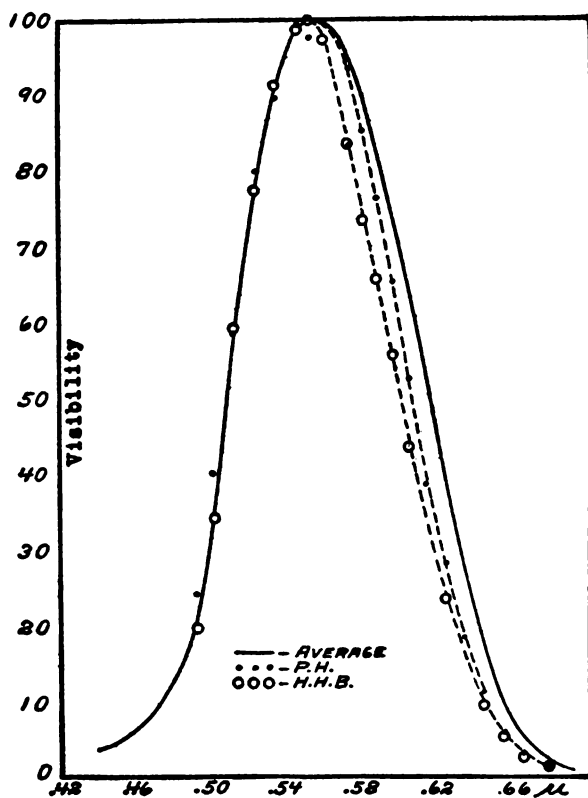


FIG. 11.—Visibility curves of sub-normal red-sensitive subjects

visibility at about 0.557μ . It is the type of curve one obtains after arbitrarily eliminating the red and the blue sensitive observers, and it may be slightly different from the visibility curve of the "average eye," which means the inclusion of all individuals who are considered to have "normal" color vision. In Fig. 12 are shown the observations of a group of observers whose visibility curves are closely the same as that of the average eye.

In Table 3 are given statistical data, by various observers, regarding the number of persons in each type of color sensibility mentioned in this paper. The characteristics of the eyes examined by Tufts are not fully described.

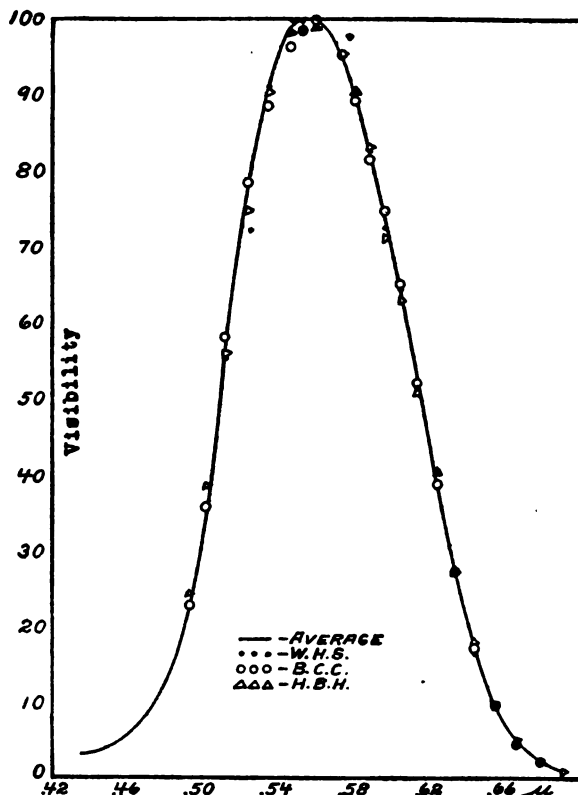


FIG. 12.—Visibility curves of observers having a sensibility close to the average

TABLE 3

Grouping of Subjects According to Dominant Color Sensibility

Subjects	Observer Tufts	Observer Ives	Observer Bender	Observer Nutting	Observers Coblentz and Emerson
Total number.....	18	18	10	21	125
Red sensitive.....	2	5	3	7	26+11 Sub. B
Blue sensitive.....	2	5	3	5	20+17 Sub. R
Green sensitive (narrow).....		5	2	4	13
Red-blue sensitive (wide).....	4	1		1	7
Average.....	a 10	2	2	4	29

a Classification by Tufts; curves not published.

The data given by the various investigators mentioned in Table 3 are sufficiently complete to show that among a group of persons having "normal" color vision (roughly estimated) 60 per cent of the cases fall into three quite evenly divided groups which are either (1) red sensitive, (2) blue sensitive, or (3) average. Similarly 30 per cent of the cases examined are quite evenly divided into three groups which fall below the average in (1) the red, (2) in the blue, or (3) in both the red and the blue, thus giving rise to an apparently high sensitivity in the green. One person in about 20 has a wide visibility curve as compared with the average.

The composite visibility curve of all these observers is illustrated in Fig. 13. It is of especial interest in showing the small range of variation in sensibility in the region of 0.51 to 0.52μ .

10. REMARKS ON OBSERVATIONAL DATA

Under this title are recorded various comments regarding the characteristics of the data obtained by various observers given in Tables 1 and 2.

W. W. C.—Equality-of-brightness settings are usually close to the flicker settings, but in the midst of these closely agreeing observations an occasional set of readings was obtained which differed by 20 per cent from the normal settings. (See Table 4.) No disproportionate increase in sensibility was observed as the result of adaptation when exposing the eye to the blue rays. The fatigued (right) eye and the unfatigued (left) eye gave the same flicker readings. For the 30-meter candle illumination there seemed to be a tendency for supernormal sensibility at 0.623μ on some days.

C. G. P.—A poor visibility curve (Mar. 9, 1916) was caused by fatigue from reading during the greater part of the preceding night. His equality-of-brightness curves (for Mar. 4 and 9) coincided better than the flicker curves. Prolonged exposure to the blue did not increase the sensibility to the yellow. For equality of brightness his Nicol readings gradually increase (sensibility decreases) to a normal—probably slow adaptation.

B. M.—His eye was more sensitive to the yellow after making readings in the blue. Parinaud's⁴⁴ data indicate a disproportionate increase in sensibility of the eye for the blue rays, due to adaptation.

W. M. S.—A rather poor curve; eyes tire very easily.

R. C. S.—Very low and irregular readings in the yellow when making equality-of-brightness settings.

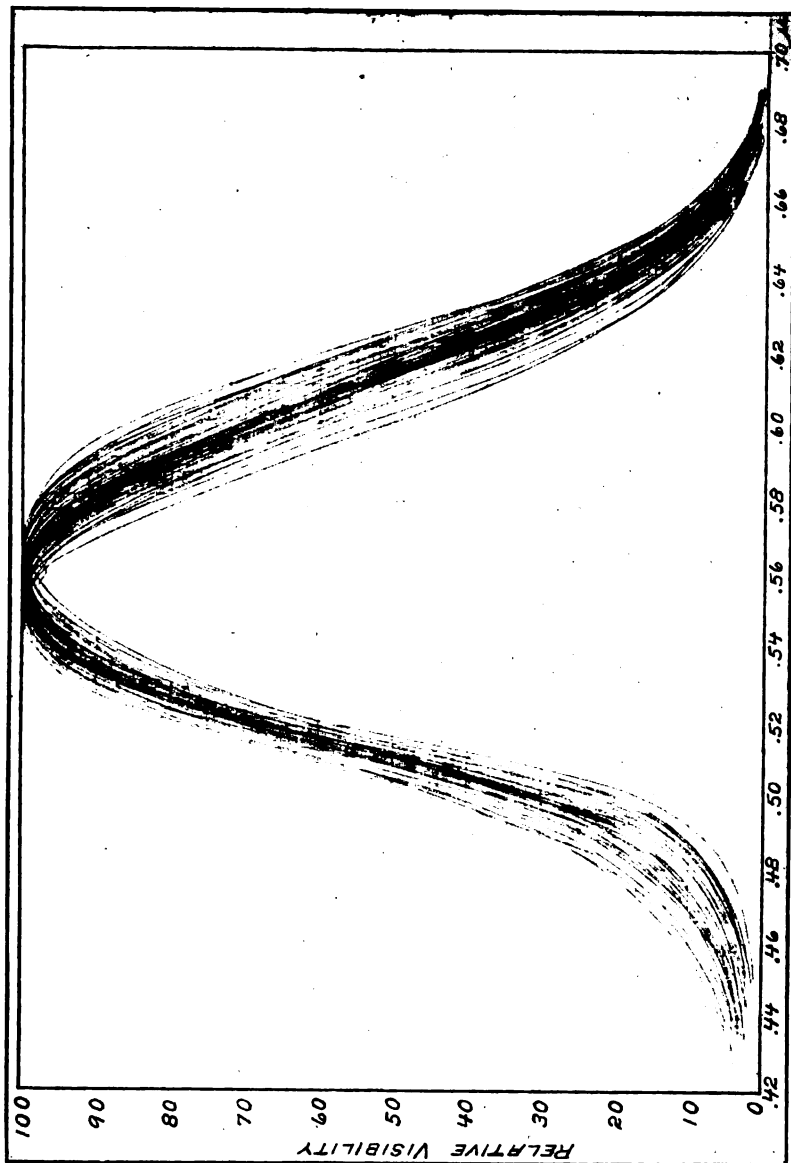


FIG. 13.—Composite visibility curve of 125 persons

J. W. S.—Unusual settings for equality of brightness. Practically the same settings (sensibility) throughout the spectrum. (See Table 4.)

J. S. P.—Equality-of-brightness settings are low. After being shown the appearance of the photometric field for a flicker setting his measurements gradually returned to the former equality-of-brightness settings.

F. P. P.—Made several sets of observations (Feb. 23, 25, 26, and Mar. 1, 1916). On June 27 he repeated the flicker observation of February 23, using the right and the left eye. The results for the two eyes are identical, and they agree with his previous observations.

P. J. H.—The visibility curve of this subject is very unusual in that it is higher than the average in the blue, 0.48μ ; lower in the blue-green, 0.53μ ; normal in the green, 0.555μ ; and lower than the average in the yellow and red, 0.57μ to 0.66μ . The equality-of-brightness settings are extremely high, especially in the blue.

11. FATIGUE AND ADAPTATION

If the fatigue is the same for white and for colored light, then it should not affect the settings. From the data at hand it would appear that there is no rule as to what one should expect.

Pfüger,⁴ Bender,²⁶ and others report a great constancy in the observations made on different days. In the present investigation it was found that closely agreeing observations were obtained on different days, provided the subject was in normal health (free from "cold," "grippe," etc.) and had sufficient sleep (rest) the preceding night.

From the data published by Allen,¹³ after fatiguing the eye with white light from the electric arc the sensibility is quite proportionately reduced throughout the spectrum and, in general, fatigue caused by exposure to light of a given wave length (say in the green) depressed the sensibility in that part of the spectrum. It is to be noted, however, that the test was made before the eye could recover from the effect. Tufts,²² using the flicker photometer, found no effect upon the visibility curve as the result of fatiguing the eye with white or colored light (red, yellow, blue), but an effect was observed after prolonged exposure to red light.

In the present investigation prolonged tests were made on the effect of fatigue upon the flicker-photometer settings. For example, making a setting with the right eye after a long series

of observations and immediately thereafter repeating this setting with the left eye (which had not been used at all) gave the same results. Similarly, making a no-flicker setting with the right eye, then staring at the uncovered standard lamp or at the sky, did not change the setting. These tests were made at 0.604μ , 0.576μ , and 0.456μ . Only at the latter position in the spectrum was the no-flicker setting changed, and this was not greater than the observational errors (3 to 4 per cent) which one would make in such a test.

All the tests show that the eye adapts itself to the conditions of the experiment in a few minutes. Only in a few cases (see "Remarks on observational data") did the sensibility change due to adaptation in changing from the blue to a less refrangible part of the spectrum.

12. FLICKER VERSUS EQUALITY-OF-BRIGHTNESS MEASUREMENTS

As already mentioned, it is an unsettled question whether the flicker method gives the same results as the equality-of-brightness method. At the conclusion of the flicker-photometer measurements the observer made measurements in selected parts of the spectrum, using the equality-of-brightness method and an illumination of 50 meter candles on the disk, as previously described.

From data obtained by other tests, when using the flicker method of photometry, one would expect to find the observations underestimated in the red and overestimated in the blue as compared with similar measurements made with the equality-of-brightness method.¹⁸ This is due to the fact that the sensations aroused by lights differing in color appear to rise to their maximum brightness at different rates when using the flicker method. A further investigation of this important question is desirable, using a large number of observers and apparatus that remains in exact adjustment to insure that some of the phenomena observed are not due to instrumental difficulties.

Of the five subjects examined by Ives,²³ (using an illumination of 250 meter candles) one observer underestimated the red and overestimated the blue (which is the expected result), two overestimated the red and underestimated the blue, one had a symmetrical curve and one was indeterminate. Using an illumination of 10 meter candles upon the disk, all five observers overestimated the red and underestimated the blue. From these data it seemed quite probable that, for the intensity (50 meter candles)

used in the present work, there would be about as frequent overestimation as underestimation in any part of the spectrum when using the flicker method. Hence, as already explained, the flicker method was adopted to obtain the complete visibility curve, and the question of overestimation or underestimation was tested by using the equality-of-brightness method in selected parts of the spectrum which are best adapted to demonstrate this alleged effect.

In the preliminary tests by the writers it was found that on some days there was an underestimation and on other days an overestimation in the same part of the spectrum, when measurements were made with the flicker method as compared with the equality-of-brightness method.

The results of the present investigation are given in Table 4. The mean value for 110 observers, using the flicker method, differs but little from that of the 125 observers.

C. G. P.:	208.0	223.0	- 6.7	559.0	714.0	- 21.7	800.0	800.0	0.0	421.0	413.0	+ 1.9	94.7	110.0	- 13.5
Mar. 4, 1916.....	206.0	261.0	+ 13.4	784.0	820.0	- 4.4	860.0	827.0	+ 4.0	426.0	363.0	+ 17.4	88.9	83.2	+ 8.1
Mar. 9, 1916.....	192.0	268.0	- 28.3	726.0	794.0	- 8.6	919.0	821.0	+ 11.9	362.0	362.0	0.0	76.0	80.7	- 5.8
Mar. 14, 1916.....	286.0	211.0	+ 35.5	892.0	763.0	+ 16.9	786.0	886.0	- 11.3	639.0	435.0	+ 47.2	187.0	118.0	+ 58.4
W. W. B.....				749.0	742.0	+ 0.9	954.0	898.0	+ 6.2	567.0	427.0	+ 32.7	135.0	113.0	+ 19.5
C. O. F.....				483.0	681.0	- 29.1	506.0	817.0	- 38.1	342.0	387.0	- 11.6			
J. W. S.....															
W. F. M.....	151.0	156.0	- 3.2	956.0	672.0	+ 42.3	960.0	944.0	+ 1.7	459.0	540.0	- 15.0	86.6	137.0	+ 36.8
L. N. K.....	176.0	173.0	+ 1.7	756.6	687.0	+ 10.0	948.0	948.0	0.0	409.0	533.0	- 23.3	103.0	129.0	+ 20.1
B. M.....				1043.0	662.0	+ 37.6	1273.0	897.0	+ 41.9	774.0	508.0	+ 52.6	242.0	132.0	+ 83.3
K. B.....	288.0	186.0	+ 57.6	1030.0	776.0	+ 32.8	491.0	735.0	- 33.2	83.6	313.0	+ 73.4	53.8	72.0	- 25.3
C. C. G.....	890.0	141.0	+ 530.0	2100.0	690.0	+ 204.0	1060.0	888.0	+ 19.4	310.0	423.0	- 26.7	146.0	116.0	+ 25.9
J. S. P.....	209.0	267.0	- 21.7	492.0	743.0	+ 33.8	822.0	803.0	+ 2.4	204.0	373.0	- 45.3	54.4	94.2	- 42.3
W. J. K.....	480.0	232.0	+ 107.0	2275.0	786.0	+ 189.0	1119.0	806.0	+ 38.8	761.0	423.0	+ 80.0	243.0	109.0	+ 123.0
P. V. W.....	271.0	209.0	+ 29.7	1140.0	740.0	+ 54.0	905.0	869.0	+ 4.1	671.0	462.0	+ 45.3	210.0	118.0	+ 78.0
G. K. S.....	200.0	210.0	- 4.8	657.0	745.0	- 11.8	708.0	841.0	- 15.8	529.0	364.0	+ 45.3	83.2	81.6	+ 2.0
A. N. I.....	155.0	148.0	+ 4.7	677.0	744.0	- 9.0	753.0	778.0	- 3.2	247.0	350.0	- 29.4	33.7	72.2	- 53.3
E. C. C.....	460.0	295.0	+ 55.9	1130.0	798.0	+ 41.6	948.0	820.0	+ 15.6	529.0	412.0	- 28.4	123.0	103.0	+ 19.4
L. C. G.....	1983.0	194.0	+ 235.0	1983.0	762.0	+ 160.0	1303.0	777.0	+ 67.7	529.0	330.0	+ 60.2	186.0	72.1	+ 88.6
B. G. W.....				686.0	717.0	- 4.3	802.0	814.0	- 1.5	181.0	360.0	- 49.7	55.0	82.7	- 33.5
C. F. H.....	214.0	205.0	+ 4.9	590.0	794.0	- 25.7	767.0	812.0	- 5.5	284.0	376.0	- 24.5	83.6	89.4	- 6.5
F. A. W.....	641.0	193.0	+ 232.0	2878.0	721.0	+ 299.0	1135.0	910.0	+ 24.7	859.0	468.0	+ 83.5	246.0	131.0	+ 87.7
A. H. T.....				748.0	762.0	- 1.8	806.0	852.0	- 5.4	377.0	433.0	- 12.9	106.0	110.0	- 3.6
H. G. B.....	351.0	215.0	+ 63.2	1115.0	826.0	+ 35.0	859.0	834.0	+ 3.0	603.0	396.0	+ 52.3	107.0	93.1	+ 14.9
E. D. W.....	282.0	200.0	+ 41.0	800.0	707.0	+ 13.2	988.0	766.0	+ 29.0	603.0	388.0	+ 55.4	131.0	100.0	+ 31.0
H. B. S.....	286.0	341.0	- 14.8	1207.0	792.0	+ 52.4	713.0	701.0	+ 10.3	274.0	253.0	+ 8.3	56.4	53.7	+ 5.0
E. R. E. L.....	230.0	170.0	+ 35.3	467.0	775.0	- 39.8	804.0	772.0	+ 4.1	259.0	315.0	- 17.8	32.5	63.7	- 49.0
R. M. W.....	460.0	253.0	+ 82.0	840.0	726.0	+ 15.7	784.0	860.0	- 8.8	516.0	453.0	+ 13.9	139.0	116.0	+ 19.8
L. R. H.....				557.0	724.0	- 23.1	766.0	858.0	- 10.7	400.0	438.0	- 8.7	93.4	108.0	- 13.5
W. S. J.....	181.0	242.0	- 25.2	532.0	771.0	- 31.0	782.0	901.0	- 13.2	597.0	485.0	+ 23.1	225.0	126.0	+ 76.5
R. W. W.....	301.0	176.0	+ 71.0	1259.0	777.0	+ 62.0	626.0	783.0	- 20.0	468.0	340.0	+ 37.7	145.0	78.6	+ 84.5
H. B. H.....	272.0	246.0	+ 10.6	857.0	748.0	+ 14.6	864.0	831.0	+ 4.0	702.0	407.0	+ 72.4	174.0	96.9	+ 75.8
E. F. M.....	458.0	335.0	+ 36.7	1151.0	786.0	+ 46.4	926.0	793.0	+ 16.8	470.0	402.0	+ 16.9	166.0	89.7	+ 85.0
A. A. B.....				899.0	715.0	+ 25.7	458.0	842.0	- 45.6	373.0	385.0	- 3.1	77.7	90.9	- 14.5

TABLE 4—Continued.
Comparison of Flicker and Equality-of-Brightness Measurements in Selected Parts of the Spectrum—Continued.

Observer	Wave length, 0.493 μ			Wave length, 0.523 μ			Wave length, 0.597 μ			Wave length, 0.623 μ			Wave length, 0.654 μ		
	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent
D. H. S.	263.0	201.0	+ 30.7	1177.0	800.0	+ 47.1	830.0	769.0	+ 7.9	410.0	312.0	+ 31.4	91.2	76.8	+ 18.7
W. C. B.	360.0	343.0	+ 5.0	756.0	831.0	- 9.0	948.0	816.0	+ 3.9	458.0	411.0	+ 11.4	95.3	110.0	- 13.4
R. C. S.	173.0	296.0	- 41.5	250.0	814.0	- 69.3	408.0	797.0	- 48.8	95.4	353.0	- 73.0	55.4	78.2	- 29.2
M. H. S.	269.0	284.0	- 5.3	673.0	819.0	- 17.8	675.0	800.0	- 15.6	198.0	381.0	- 48.0	68.9	94.8	- 27.3
G. W. M.	163.0	183.0	- 11.0	631.0	764.0	- 17.0	1012.0	840.0	+ 20.5	489.0	411.0	+ 19.0	164.0	109.0	+ 50.5
A. L. T.				1140.0	750.0	+ 51.1	705.0	799.0	- 4.6	402.0	328.0	+ 22.5			
A. P. P.				528.0	798.0	- 28.5	588.0	815.0	- 27.8	317.0	366.0	- 13.4	96.8	86.7	+ 11.7
F. B. S.	203.0	163.0	+ 24.5	598.0	786.0	- 20.1	948.0	807.0	+ 17.4	602.0	432.0	+ 39.4	138.0	103.0	+ 34.0
J. L.	292.0	150.0	+ 94.7	446.0	681.0	- 34.5	768.0	902.0	- 3.0	278.0	337.0	- 17.5	87.9	82.4	+ 6.8
J. A. S.	196.0	160.0	+ 22.5	1134.0	759.0	+ 49.4	818.0	826.0	- 1.0	563.0	394.0	+ 43.0	146.0	95.3	+ 53.2
H. K. G.	510.0	200.0	+ 155.0	1061.0	771.0	+ 37.6	837.0	818.0	+ 2.3	488.0	394.0	+ 23.9	134.0	102.0	+ 31.4
W. S. S.	203.0	181.0	+ 12.2	735.0	758.0	- 3.0	891.0	848.0	+ 5.1	615.0	415.0	+ 48.2	138.0	106.0	+ 30.2
P. G. A.	182.0	325.0	- 44.0	493.0	766.0	- 37.3	832.0	872.0	- 4.6	927.0	416.0	+ 123.0	89.0	102.0	- 12.7
H. I. S.	211.0	231.0	- 8.6	794.0	750.0	- 2.1	770.0	818.0	- 5.9	408.0	400.0	+ 2.0	101.0	96.9	+ 4.2
E. L. P.	224.0	244.0	- 8.2	778.0	696.0	+ 11.8	980.0	904.0	+ 8.4	636.0	446.0	+ 42.6	148.0	110.0	+ 34.6
L. W. S.	452.0	180.0	+ 151.0	904.0	760.0	+ 22.9	850.0	835.0	+ 1.8	353.0	385.0	- 8.3	162.0	92.1	+ 75.9
J. H. D.	172.0	215.0	- 20.0	508.0	701.0	- 24.9	492.0	774.0	- 36.5	214.0	305.0	- 29.8	55.9	63.4	- 14.5
E. L. S.	221.0	250.0	- 11.6	727.0	705.0	+ 3.1	1461.0	891.0	+ 66.3				161.0	109.0	+ 47.7
C. W. B.	462.0	285.0	+ 62.0	1236.0	815.0	+ 51.6	887.0	769.0	+ 15.3	374.0	345.0	+ 8.4	113.0	83.2	+ 35.8
D. R. M.	356.0	265.0	+ 34.3	850.0	775.0	+ 10.1	839.0	850.0	+ 1.0	275.0	438.0	- 37.2	80.2	116.0	- 30.8
O. L. S.	145.0	166.0	- 12.7	907.0	728.0	+ 26.0	573.0	788.0	- 27.3	408.0	364.0	+ 10.7	155.0	98.3	+ 73.5
M. J.	354.0	182.0	+ 94.6	1593.0	706.0	+ 126.0	1007.0	867.0	+ 16.1	551.0	477.0	+ 15.5	267.0	186.0	+ 96.4
J. F. M.	227.0	193.0	+ 17.6	560.0	774.0	- 27.6	624.0	773.0	- 19.3	118.0	362.0	- 67.4	84.0	96.5	- 12.9
P. D. L.	285.0	517.0	- 10.1	893.0	770.0	+ 16.0	944.0	808.0	+ 16.8	423.0	351.0	+ 20.5	95.9	84.4	+ 13.6
E. Y. Y.	322.0	219.0	+ 47.0	974.0	784.0	+ 24.2	1019.0	790.0	+ 28.0	588.0	375.0	+ 43.5	146.0	93.9	+ 55.5

F. J. B.	594.67	18	1	1050.0	765.0	+ 37.2	832.0	832.0	0.0	371.0	371.0	0.0	69.5	86.8	- 20.0
P. J. H.	531.0	181.0	+ 193.0	2011.0	735.0	+ 174.0	466.0	757.0	- 38.1	509.0	327.0	+ 55.6	149.0	80.1	+ 86.0
T. R. H.	357.0	180.0	+ 96.3	726.0	734.0	- 1.1	946.0	928.0	+ 1.9	846.0	470.0	+ 80.0	310.0	125.0	+ 148.0
H. A. E.	292.0	153.0	+ 90.8	1019.0	739.0	+ 34.2	509.0	813.0	- 12.3	824.0	394.0	- 17.8	112.0	89.1	+ 25.7
E. C. P.	205.0	270.0	- 24.1	812.0	773.0	+ 4.2	805.0	727.0	- 37.4	554.0	401.0	+ 38.1	143.0	102.0	+ 40.2
C. H. M.	205.0	270.0	- 24.1	812.0	773.0	+ 4.2	805.0	727.0	+ 10.7	254.0	321.0	- 20.9	56.3	74.5	- 24.4
C. E. B.	285.0	326.0	- 18.7	2710.0	770.0	+ 252.0	1007.0	806.0	+ 25.0	706.0	401.0	+ 76.1	91.1	85.1	+ 7.1
C. A. B.	546.0	681.0	- 19.9	1079.0	908.0	+ 18.8	322.0	493.0	- 10.9	368.0	368.0	- 2.9	144.0	125.0	+ 15.2
T. B. F.	717.0	752.0	- 4.6	663.0	856.0	- 22.5	259.0	446.0	+ 41.9	94.8	120.0	- 21.0	303.0	116.0	- 21.0
D. R. H.	627.0	708.0	- 11.4	844.0	946.0	- 10.8	509.0	466.0	+ 9.2	117.0	107.0	+ 9.3	117.0	107.0	+ 9.3
P. D. S.	412.0	267.0	+ 54.3	1333.0	708.0	- 11.4	844.0	946.0	- 10.8	509.0	466.0	+ 9.2	117.0	107.0	+ 9.3
A. N. F.	697.0	672.0	+ 3.7	878.0	927.0	- 5.3	570.0	520.0	+ 9.6	150.0	151.0	- 0.7	150.0	151.0	- 0.7
G. T. M.	299.0	242.0	+ 23.6	903.0	782.0	+ 15.5	668.0	840.0	- 20.5	362.0	407.0	- 11.1	91.2	104.0	- 12.3
H. S.	704.0	248.0	+ 184.0	1549.0	771.0	+ 101.0	836.0	805.0	+ 3.9	585.0	340.0	+ 72.2	178.0	79.0	+ 125.0
H. B. K.	341.0	256.0	+ 33.2	854.0	801.0	+ 6.6	618.0	813.0	- 24.5	465.0	362.0	+ 28.5	140.0	89.1	+ 57.0
S. I.	177.0	258.0	- 31.4	586.0	760.0	- 22.9	778.0	792.0	- 1.8	399.0	341.0	+ 17.0	118.0	82.7	+ 42.7
J. L. F.	154.0	194.0	- 20.6	324.0	778.0	- 58.4	741.0	806.0	- 8.1	302.0	384.0	- 21.4	85.2	90.9	- 6.3
A. B. L.	244.0	283.0	- 13.8	634.0	770.0	- 17.6	554.0	872.0	- 36.5	237.0	474.0	- 45.8	99.8	119.0	- 16.1
O. S. P.	210.0	264.0	- 20.5	661.0	863.0	- 23.4	966.0	756.0	+ 27.8	385.0	314.0	+ 22.6	99.5	76.8	+ 29.6
W. H. S.	194.0	213.0	- 8.9	608.0	736.0	- 17.4	579.0	786.0	- 26.4	448.0	359.0	+ 24.8	116.0	86.6	+ 33.9
A. B.	198.0	189.0	+ 4.7	996.0	782.0	+ 27.4	898.0	768.0	+ 16.9	582.0	373.0	+ 56.0	129.0	91.2	+ 41.4
G. C. H.	486.0	226.0	+ 115.0	1394.0	789.0	+ 76.7	876.0	788.0	+ 11.2	683.0	372.0	+ 83.6	182.0	88.6	+ 106.0
T. R. E.	384.0	229.0	+ 67.6	1267.0	784.0	+ 61.6	895.0	815.0	+ 9.8	653.0	388.0	+ 40.4	101.0	100.0	+ 1.0
A. J. H.	311.0	240.0	+ 29.5	1165.0	718.0	+ 62.3	916.0	804.0	+ 13.9	531.0	385.0	+ 38.0	123.0	97.2	+ 26.5
B. C. C.	311.0	240.0	+ 29.5	1165.0	718.0	+ 62.3	916.0	804.0	+ 13.9	531.0	385.0	+ 38.0	123.0	97.2	+ 26.5
G. K. B.	311.0	240.0	+ 29.5	1165.0	718.0	+ 62.3	916.0	804.0	+ 13.9	531.0	385.0	+ 38.0	123.0	97.2	+ 26.5
P. H.	192.0	254.0	- 24.4	250.0	840.0	- 70.2	152.0	734.0	- 79.3	93.6	311.0	- 70.0	50.6	69.2	- 26.9
H. S. P.	960.0	198.0	+ 385.0	2356.0	773.0	+ 205.0	601.0	658.0	- 8.6	632.0	237.0	+ 167.0	160.0	53.3	+ 200.0
H. H. B.	243.0	189.0	+ 28.6	548.0	743.0	- 26.2	358.0	771.0	- 53.6	299.0	345.0	- 13.3	83.6	79.3	+ 5.4
H. A. B.	702.0	662.0	+ 6.0	914.0	939.0	- 2.6	546.0	546.0	- 2.6	546.0	546.0	- 13.1	106.0	95.9	+ 10.5
E. S. P.	859.0	726.0	+ 18.3	757.0	827.0	- 8.4	353.0	827.0	- 8.4	353.0	406.0	- 13.1	106.0	95.9	+ 10.5
W. H. St.	508.0	780.0	- 34.8	795.0	831.0	- 4.3	178.0	393.0	- 54.8	41.1	89.0	- 54.8	41.1	89.0	- 54.8
M. D. S.	692.0	716.0	- 3.3	1005.0	810.0	+ 24.1	365.0	365.0	+ 24.1	365.0	365.0	+ 0.8	86.2	86.2	+ 0.8
E. B.	692.0	716.0	- 3.3	1005.0	810.0	+ 24.1	365.0	365.0	+ 24.1	365.0	365.0	+ 0.8	86.2	86.2	+ 0.8

TABLE 4—Continued
Comparison of Flicker and Equality-of-Brightness Measurements in Selected Parts of the Spectrum—Continued

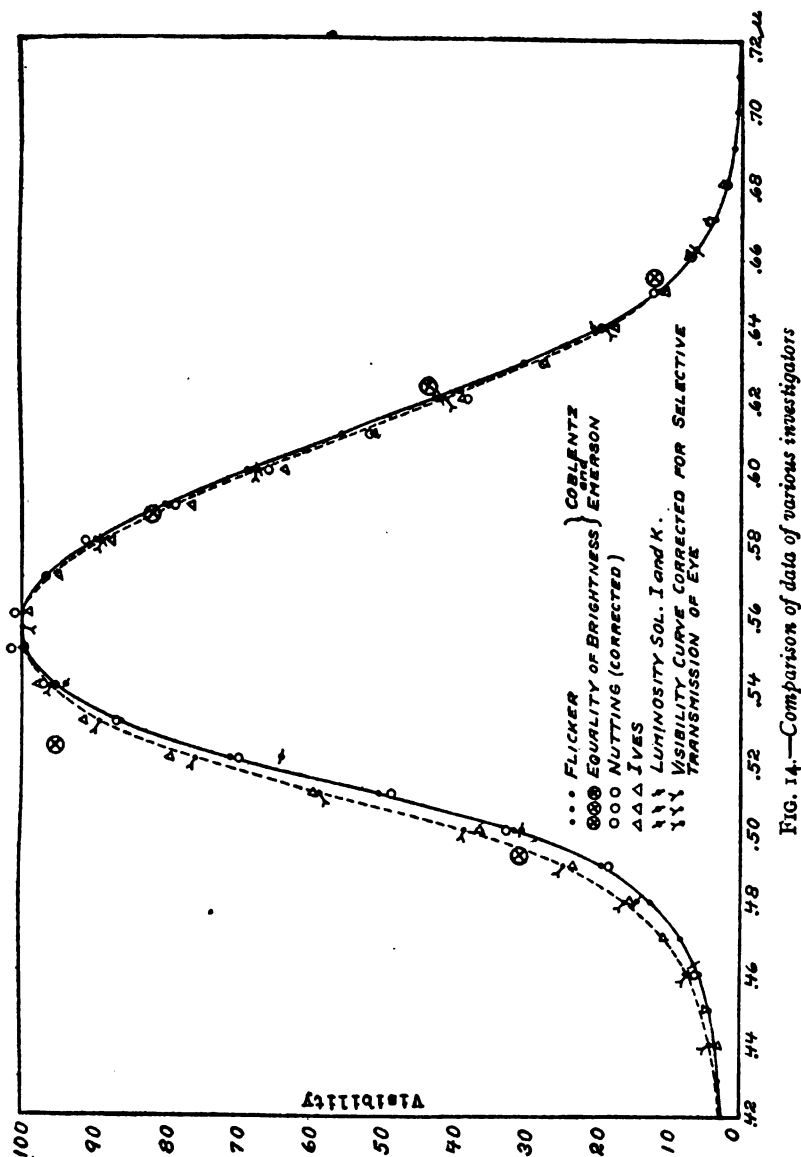
Observer	Wave length, 0.493 μ			Wave length, 0.523 μ			Wave length, 0.587 μ			Wave length, 0.623 μ			Wave length, 0.654 μ		
	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent	Equality of brightness	Flicker	Δ Per cent
J. L. B.....	324.0	373.0	- 13.1	672.0	730.0	- 8.0	948.0	840.0	+12.9	321.0	447.0	- 28.2	103.0	109.0	- 5.5
E. E. W.....	453.0	339.0	+ 33.6	1057.0	845.0	+ 25.1	785.0	796.0	- 1.4	492.0	357.0	+ 37.8	104.0	88.0	+ 16.9
M. S.....	249.0	233.0	+ 6.9	797.0	766.0	+ 4.0	976.0	790.0	+23.6	494.0	381.0	+ 29.6	110.0	97.6	+ 12.7
J. C. P.....	263.0	212.0	+ 24.0	985.0	782.0	+ 25.9	811.0	748.0	+ 8.4	326.0	371.0	- 12.1	72.8	97.2	- 25.1
P. D. M.....				1410.0	729.0	+ 93.4	1065.0	878.0	+21.3	684.0	430.0	+ 59.1	220.0	108.0	+104.0
P. W. M.....	313.0	242.0	+ 29.3	980.0	804.0	+ 21.9	708.0	728.0	- 2.7	351.0	296.0	+ 18.6	96.9	63.8	+ 51.9
N. S. O.....				985.0	723.0	+ 29.3	768.0	808.0	- 4.9	348.0	425.0	- 18.1	175.0	117.0	+ 49.6
Mean of 110 eyes	300.8	226.4		929.8	757.2		815.1	810.5		442.1	395.5		121.3	96.7	
Δ $\frac{E_B - F_1}{F_1} =$	+32.9%			+22.8%			+0.6%			+11.8%			+25.4%		
Mean of 125 eyes		225.0			758.0			818.0			391.0			96.1	

The observations using the equality-of-brightness and the flicker photometer, obtained at a given wave length, are arranged side by side in order to facilitate comparison. From this table it is to be observed that there is no regularity in the over or under estimation (of equality of brightness as compared with the flicker photometer) at the same wave length or at different wave lengths on different days. For example, the visibility reading of one observer (W. W. C.) at 0.623μ was 41.6 per cent of the maximum visibility by the equality-of-brightness method and 43.6 by the flicker method, or an underestimation of 4.6 per cent by the equality-of-brightness method. Two days later these measurements were repeated, giving an overestimation of 3.8 per cent, which would be expected from psychological data. However, the sum of all the measurements (Table 4, underestimation in per cent is minus (-) and overestimation is plus (+)), is practically zero. That is to say, the visibility curves obtained by equality-of-brightness and by flicker photometry are the same for this observer. It is to be noted, however, that in the midst of these very closely agreeing settings there are very erratic ones deviating 20 per cent or more, although they were made in a region of the spectrum (0.5876μ) where there was the least color difference in the two sources of light. Some observers (e. g., W. C. B.), without having had previous training in photometry, made very closely the same settings by these two methods of photometry. Other observers made very erratic settings (e. g., J. W. S.), which were different (c. f., F. P. P. and C. G. P.) as regards overestimation and underestimation on different days.

In the yellow ($\lambda = 0.5876\mu$) there are about as many observers who overestimated as underestimated in making the photometric settings, and the magnitude of the variations (0 to 50 per cent) are within reasonable limits. In the blue and in the red ends of the spectrum a greater number of observers overestimated than underestimated their equality-of-brightness settings. Moreover, the magnitude of this overestimation varied from 5 to 500 per cent. These variations in the photometric settings by different observers are out of all proportion as compared with similar settings made with the flicker photometer, and it is not a fair test to take the mean of the under and over estimated values.

The mean values of the equality-of-brightness measurements (Table 4, corrected for slit width) are indicated by the large crossed circles in Fig. 14. If we admit that such settings have a definite

meaning, then the visibility of radiation curve, for the central retina, obtained by the equality-of-brightness method of photometry is higher in the red and in the blue (just the opposite from the



expected¹⁸ result) than the one obtained with the flicker photometer, and its maximum lies at shorter wave lengths (0.55μ) than obtains for the flicker method. Although this tentative conclu-

sion is substantiated by the direct measurements and by the total number of persons who overestimated in making the settings, it is evident that a far greater number of persons must be examined before exact conclusions can be drawn.

As already stated, most of these observers objected to making the quality-of-brightness settings, saying that no meaning could be attached to such measurements and that it was all guesswork.

13. EFFECT OF AGE UPON VISIBILITY

It is well known that the absorption of the ocular media is highly selective in the blue and violet and that this absorption increases with age.¹¹ The age of the majority of the subjects

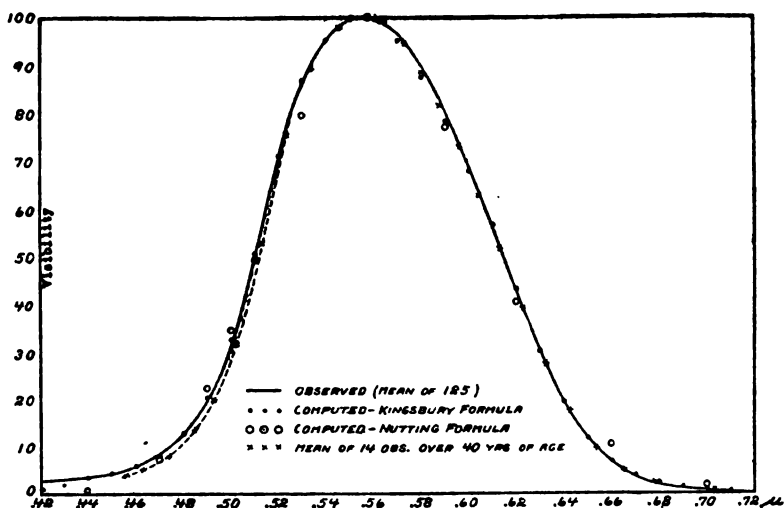


FIG. 15.—Comparison of observed and computed data; also the effect of age upon visibility

examined was between 22 and 32 years, and the question has been raised whether the average curve of those persons who were above 40 years of age is markedly different from that of the younger subjects. The present investigation contains 14 subjects who were 40 or more years of age. Their average visibility curve is given in Fig. 15.

Of this number, the grouping of the visibility curves is as follows: 1 wide, 1 blue sensitive, 2 green sensitive, 4 normal, and 6 sub-blue sensitive. This group is conspicuous for its lack of red sensitives and its large number of subnormal blue sensitives. From this large number of subjects (which is half of the total number examined, see Table 3, who have a low sensibility in the

blue) it would appear as though the effect of age should be considered in deciding upon an average visibility curve in the blue and violet part of the spectrum.

It would be of interest to study a group of observers whose occupation requires working under artificial light of a definite color (e. g., in photographic work, such as the manufacture of photographic materials) to determine whether prolonged exclusion of parts of the visible spectrum has any effect upon the visibility curve.

V. THE VISIBILITY CURVE OF THE AVERAGE EYE

The visibility data of the average eye is based upon the mean value of the observations of 125 subjects given in Table 1. Two of these subjects confused colors, but their visibility curves do not differ much from the curves of the red-sensitive observers, some of which did not show color blindness by the Nagel color-card test. The omission of these two observers would lower the average visibility curve, at most, by less than 0.3 per cent in the red.

In order to simplify the computations, three correction factors were omitted in computing the individual curves. These correction factors, therefore, are to be applied to the average curve. The most important correction to be applied is due to the selectivity of the response of the eye for energy of different wave lengths. In other words, a correction is to be made for change in slope of the visibility curve. For this purpose, for the intensities used, the response of the retina for a stimulus of a given (color) wave length and for the adjacent wave lengths comprised in the slit image may be taken proportional to the intensity of the stimulus as shown in Fig. 2. Furthermore, for any given intensity the response of the retina is proportional to the mean visibility of the energy passing through the observing slit. The visibility curve, therefore, is to be corrected for change in slope by applying a slit-width correction similar to the Runge⁴⁹ formula used in spectral radiation, but with this difference that the function involved is visibility (V_λ) instead of radiation (E_λ). In other words, the Runge formula is applied in the form

$$V(\lambda) - 1/6 V'(\lambda) + \dots \quad (1)$$

This correction is most effective in the sharp bends in the visibility curve.

A correction is applied also to make the zero of the scale reading correspond with the minimum light transmission through the crossed Nicol prisms. This correction, amounting to 0°.1, was determined at the start, and therefore might have been incorporated with other data used in reducing the various visibility curves.

A further correction⁸⁸ is applied to eliminate the effect of spectroscopic field light (scattered light) which is perceptible in the extreme blue and red ends of the visible spectrum. As discussed elsewhere in this paper, this scattered radiation has no effect upon the radiation measurements, but it has a measurable effect upon the visual measurements in the red end of the spectrum.

This correction to the visual measurements was determined by making the visibility observations with and without absorption screens of blue, Crookes's sage-green, and Corning signal-red glasses. The transmissions of these glasses were accurately determined by means of a photoelectric cell, a spectrophotometer and a spectroradiometer, in which care had been taken to screen off the scattered radiation.

In the blue (0.45 to 0.48 μ), and in the green, (0.52 μ), the visibility measurements were not affected by absorbing the scattered light. But in the red the visibility measurements required a correction which increased quite uniformly from 1 per cent at 0.65 μ to 28 per cent at 0.75 μ .

The visibility curve, Fig. 14, resulting from making these corrections differs but little from the uncorrected curve. These data are given at the bottom of Table 1. In Table 5 these data are given for the even wave lengths, read from the visibility curve.

TABLE 5
Average Visibility for Even Wave Lengths

Wave length	Average visibility	Wave length	Average visibility	Wave length	Average visibility	Wave length	Average visibility
μ		μ		μ		μ	
0.400	0.010	0.490	0.194	0.580	0.898	0.670	0.0338
.410	.017	.500	.316	.590	.800	.680	.0178
.420	.024	.510	.503	.600	.687	.690	.0085
.430	.029	.520	.710	.610	.557	.700	.0040
.440	.033	.530	.862	.620	.427	.710	.00203
.450	.041	.540	.954	.630	.302	.720	.00097
.460	.056	.550	.994	.640	.194	.730	.00048
.470	.083	.560	.998	.650	.115	.740	.00028
.480	.125	.570	.968	.660	.0645	.750	.00020

It is relevant to add a few remarks in regard to the energy measurements. In view of the difficulties involved in eliminating stray light, etc., the present writers have adopted the radiation measurements made with the apparatus used in the present investigation, because they agree with the measurements⁴⁰ made with two other kinds of spectroradiometers, one of which (the mirror spectrometer) is perfectly achromatic and is free from stray light introduced by the lenses. In view of the fact that our visibility curves differed in the red and in the blue, Dr. Nutting very kindly submitted the acetylene burner used in his visibility measurements. The energy distribution was found the same as in the burner used in the present work, instead of agreeing with the standard spectral-energy curve of acetylene previously published,^{40a} which was much higher in the red. From data submitted it appeared that the energy measurements were in coincidence at 0.51μ and 0.56μ , but from 1 to 3 per cent too high (all within experimental errors and possible stray radiation between these two points, which depressed the visibility curve below the present curve by amounts varying from 1 to 3 per cent). Making this correction brings the two visibility curves in exact coincidence throughout the curve on the short wave-length side of the maximum. Applying the spectral-energy distribution of acetylene recently determined, instead of the older determinations (which applied to a flat flame viewed edgewise), which were much higher in red, the visibility curve is raised to close coincidence with the present curve. (See Fig. 14.) The slight difference at 0.60μ to 0.62μ may be due to the large number of very green sensitive subjects found in Nutting's work.^a The present visibility curve would be still higher in the red if this investigation had been terminated after the examination of the first 24 individuals, of whom about one-fourth were red sensitive.

As indicated in Table 4, the visibility curve of about 110 subjects is in very close agreement with the average curve of the 125 subjects in the blue and the green and differs from it by less than 1 per cent in the yellow and the red.

The visibility curve obtained by Ives,²⁸ Fig. 14, is somewhat higher in the blue and lower in the red. No direct-energy measurements were made, the energy distribution having been obtained by computation on the assumption that the source (a tungsten lamp) had a certain temperature. The maximum of the curve is

^a No white surrounding field was used, as was done in the present work.

shifted to the blue just as though the temperature of the lamp had been higher than it was supposed to be, thus giving relatively more energy in the blue than in the red. In other words, the energy distribution that was applied to the measurements may have been too low in the blue and too high in the red. Aside from this possible cause of disagreement in the visibility curves, one must consider also the total number of subjects examined. The present curve, between wave lengths 0.49μ and 0.69μ , is based upon six times as many subjects as had been previously examined for establishing the average visibility curve. Between wave lengths 0.43μ and 0.49μ the present curve is based upon from 30 to 36 observers (only one observer, W. W. C., made measurements to 0.40μ) and beyond 0.69μ it is based upon measurements made by 39 observers. In the violet these data are somewhat higher than observed by Nutting,³⁷ and in the red they are higher than the values published by Hyde and Forsythe.¹⁹ (See Table 6.) In the present investigation the aim has been to obtain an accurate average visibility curve, of a large group of observers, between 0.48μ and 0.70μ . Owing to stray light, the extreme violet end of the spectrum should be further investigated; eliminating stray light by means of absorption screens or by placing two spectroscopes in series.

TABLE 6

Comparison of Different Data for the Red End of the Spectrum

Wave lengths	Coblentz and Emerson	Hyde and Forsythe	Nutting	Ives	König
0.620 μ	220.0	252.0	227.0	221.0	189.0
.630	156.0	164.0	164.0	156.0	143.0
.640	100.0	100.0	100.0	100.0	100.0
.650	59.3	58.0	62.0	59.0	61.0
.660	33.2	30.0	34.0	^a 39.0	33.0
.670	17.4	15.7	18.6	^a 25.0	15.0
.680	9.2	7.6	8.0	^a 15.0	
.690	4.4	3.8	4.7		
.700	2.06	1.87	1.2		
.710	1.04	.91			
.720	.50	.45			
.730	.25	.22			
.740	.14	.11 ₁			
.750	.10 ₂	.05 ₂			
.760		.02 ₂			
.770		.01 ₂			

^a Extrapolated values.

The question has been raised as to whether the visibility curve would be modified by using light stimuli in which the energy distribution is normal, e. g., using a grating spectrum. In the blue the length of spectrum (in wave lengths) comprised within a given slit width is only one-fifth⁴⁰ that included in the same slit opening when observations were made in the red. However, in either case (prismatic or normal spectrum) the retinal impression is the mean visibility of the light passing through the slit, and in order to eliminate this effect a correction is applied to the observed visibility curve as already described.

As a result of an extensive investigation of the transmission of certain blue and yellow solutions as determined visually and by means of a physical photometer, Ives and Kingsbury⁴⁰ concluded that the solution of salts used to produce a transmission curve which coincides with the visibility curve of the average eye would have to be modified so as to produce a higher transmission in the red than was indicated by the average visibility curves of Ives and of Nutting then at hand. The data pertaining to their luminosity-curve solution (*I* and *K*) are plotted in Fig. 14. It is interesting to note that their predicted curve coincides very closely (especially in the red) with the average visibility curve obtained in the present investigation.

The transmission of the ocular media, especially the yellow spot, is highly selective in the blue and violet. In a recent communication Troland⁴¹ applied a correction to the visibility curves published by Nutting in order to eliminate this selective absorption.

This correction produces a very symmetrical curve. Applying a similar correction to the energy measurements in order to eliminate this selective absorption, the present average visibility curve is changed into a very symmetrical form, as indicated by the dotted curve in Fig. 14. The uncorrected curve represents the visibility of radiation impinging upon the average eye, no attempt being made to localize the effect within the eye. Correcting for selective absorption traces the stimulus to the retina, and this corrected curve represents the retinal visibility curve of the average eye. As mentioned by Troland, the marked symmetry of this (cone) visibility curve leads to the inference that luminosity is due to a single photochemical process in the retina.

As already mentioned, the present visibility curve is higher in the red than previously observed, but it is in agreement with what is to be expected by other tests. Hence, while the present visibility curve, as a whole, disagrees with previous determinations,

it approaches closely (especially in the red) to what is to be expected as the result of other tests in which the spectral visibility of radiation is indirectly involved.

VI. SOME PRACTICAL APPLICATIONS TO RADIATION PROBLEMS

It is important to obtain a mathematical equation of the curve of the relative visibility of radiation of the average eye, with which equation it is possible to relate visual sensation, "light," with radiant energy, especially with the energy radiated from a black body, the constants of which are fairly well established.

1. A VISIBILITY OF RADIATION EQUATION

During the past few years various mathematical equations have been proposed by Goldhammer,³⁷ Hertzprung,³⁸ Nutting,³⁹ and Kingsbury⁴⁷ to express the visibility of radiation of the average eye as a function of an equal-energy spectrum. While such an equation would be of value (1) in defining the visibility curve, (2) in computing radiant efficiencies and the mechanical equivalent of light, etc., unfortunately, in the present state of our knowledge of the processes involved in visual consciousness, such a mathematical equation must necessarily be empirical.

The simple visibility equation

$$V_{\lambda} = V_m R^n e^{n(1-R)} \quad (2)$$

has been used by Goldhammer,³⁷ Nutting,³⁷ and others. In this equation $R = \lambda_{\max} \div \lambda$ and the parameter V_m is the ratio of the candle or lumen to the watt at $\lambda = \lambda_{\max}$. As used by Nutting $n = 181$ and $\lambda_m = 0.555\mu$. In order to fit the computed curve to the observed curve, the constants used in the present work are $\lambda_m = 0.558\mu$ and $n = 170$. This gives a curve which, as shown in Fig. 15, does not fit the observations very closely except in the yellow and the blue. In view of the fact that there were no indications that such a simple equation could fit the present observations, the three-term formula proposed by Kingsbury⁴⁷ was modified by changing some of the constants and adding a fourth term. The formula, which is a modification of equation (2), is:

$$V_{\lambda} = 0.999(R_1 e^{(1-R_1)})^{200} + 0.035(R_2 e^{(1-R_2)})^{400} \\ + 0.130(R_3 e^{(1-R_3)})^{1000} + 0.084(R_4 e^{(1-R_4)})^{2000} \quad (3)$$

In this formula $R_1 = \frac{0.556}{\lambda}$, $R_2 = \frac{0.455}{\lambda}$, $R_3 = \frac{0.610}{\lambda}$, and $R_4 = \frac{0.525}{\lambda}$.

A fifth term $R_5 = 0.58 \div \lambda$, and a large exponent, say, $n = 5000$, would bring exact coincidence between the observed and computed data at $\lambda = 0.58\mu$. As shown in Fig. 15 and Table 7, between the wave lengths $\lambda = 0.45\mu$ and $\lambda = 0.66\mu$ the observed and the computed data are in agreement within 2 to 3 per cent, which is about as accurate as the visibility observations, including the energy measurements. In Table 7 the second column gives the computed data for the first term of equation (3) and the sixth column gives the summation of all the terms.

TABLE 7

Comparison of Observed and Computed Data Using the Modified Kingsbury Formula

Wave length λ	First term of equation	Second term of equation	Third term of equation	Fourth term of equation	Sum of all Σ	Observed	Δ Per cent from observed
0.400.....	0.000	0.001			0.001	0.010	- 90.0
0.410.....	.000	.004			.004	.017	- 76.0
0.420.....	.000	.009			.009	.024	- 62.0
0.430.....	.001	.018			.019	.029	- 34.0
0.440.....	.003	.028			.031	.033	- 6.1
0.450.....	.008	.034			.042	.041	+ 2.4
0.460.....	.022	.034			.056	.056	0.0
0.470.....	.050	.028			.078	.083	- 6.0
0.480.....	.103	.020		0.000	.125	.125	0.0
0.490.....	.189	.012		.000 ₆	.201	.194	+ 3.6
0.500.....	.311	.006		.007	.324	.316	+ 2.5
0.510.....	.463	.003		.036	.502	.503	- 0.2
0.520.....	.632	.001		.077	.710	.710	0.0
0.530.....	.791	.000		.077	.868	.862	+ 0.7
0.540.....	.916		0.000	.038	.954	.954	0.0
0.550.....	.987		.000 ₆	.010	.997	.994	+ 0.3
0.560.....	.994		.003	.001	.998	.998	0.0
0.570.....	.939		.012	.000	.951	.968	- 1.8
0.580.....	.839		.036		.875	.898	- 1.4
0.590.....	.708		.074		.782	.800	- 2.2
0.600.....	.567		.113		.680	.687	- 1.0
0.610.....	.434		.130		.564	.557	+ 1.3
0.620.....	.318		.114		.432	.427	+ 1.2
0.630.....	.223		.078		.301	.302	- 0.3
0.640.....	.151		.042		.193	.194	- 0.5
0.650.....	.098		.018		.116	.115	+ 0.9
0.660.....	.0619		.0063		.0682	.0645	+ 5.7
0.670.....	.0379		.0018		.0397	.0338	+ 17.4
0.680.....	.0224		.0004		.0228	.0178	+ 28.1
0.690.....	.0130		.0001		.0131	.0085	+ 54.2
0.700.....	.0073		.0000		.0073	.0040	+ 82.5
0.710.....	.0040				.0040	.00203	+ 97.0
0.720.....	.0021				.0021	.00097	+ 116.5
0.730.....	.0011				.0011	.00048	+ 129.0
0.740.....	.0006				.0006	.00028	+ 114.0
0.750.....	.0003				.0003	.00020	+ 50.0

2. THE MINIMUM RADIATION VISUALLY PERCEPTIBLE

Calculations have been made of the minimum energy required to excite the retina to visual perception. Evidently the minimum energy required will depend upon surrounding conditions. For example, a sixth-magnitude star is considered the limit of vision in viewing stars. When the eye is adapted to complete darkness its sensibility is much greater than when viewing a star.

Assuming that a sixth-magnitude star is the limit of visibility (at sea level) of the central retina and that the pupillary opening is 3 mm in diameter, Drude⁵⁷ calculated that the eye would respond to 12×10^{-8} meter candles, i. e., a sixth-magnitude star is as bright as a Hefner flame at 11 000 meters. The total radiation⁵⁸ from a Hefner flame is 26×10^{-6} g-cal. per cm^2 per second (29×10^{-6} for a sperm candle) at 1 m. The radiant luminous efficiency⁵⁴ of the sperm candle is 0.0024, so that for the Hefner flame it is about 0.0027. This gives a value of 7×10^{-8} g-cal. (or 29×10^{-8} watt) per cm^2 per second for the luminous energy density at 1 m. For a pupillary opening of 3 mm and a candle at 11 000 meters the light intercepted by the eye would be $(29 \times 10^{-8} \times 0.07 \div 11\,000^2 =) 1.7 \times 10^{-16}$ watt = 1.7×10^{-9} erg. per cm^2 .

The sensitivity of the eye may be estimated also from direct measurements⁵⁵ of the heat from stars. The calibration of the radiometer was 1 mm deflection = 34×10^{-14} g-cal. per cm^2 per minute = 85.5×10^{-16} watt per cm^2 per second. The sixth-magnitude stars gave deflections of 0.5 mm for blue stars to 1.5 mm for red stars (say, 1 mm on an average), depending upon their color. From measurements made on the transmission of stellar radiation through a cell of water the radiant luminous efficiency may be 0.2 (0.1 for red stars to 0.4 for blue stars). Hence, the luminous energy intercepted by a pupillary opening of 0.07 cm^2 is $(85.5 \times 10^{-16} \times 0.2 \times 0.07 =) 1.2 \times 10^{-16}$ watt or 1.2×10^{-9} erg., which is in agreement with the preceding computation.

Another method of estimating the minimum radiation which is visually perceptible is to use data on the mechanical equivalent of light, viz, 1 lumen per cm^2 = 1.6×10^{-7} watt per cm^2 . A pupil opening of 3 mm will intercept $(0.07 \times 1.6 \times 10^{-7} =) 1.1 \times 10^{-8}$ watt and for 1.2×10^{-8} meter candle this amounts to 1.3×10^{-16} watt or 1.3×10^{-9} erg.

These computations show the extraordinary sensibility of the eye as compared with a radiometer, which had to be combined with a 3-foot reflecting mirror in order to attain the same sensi-

bility.^a The ratio of the area of the mirror to the radiometric receiver was 1 to 7 000 000. In other words, the eye is 7 000 000 times as sensitive as the radiometer receiver used two years ago to measure heat from stars. Recent improvements in the radiometer⁵⁸ reduce this value to about $\frac{1}{800000}$ as sensitive as the eye.

As a further illustration of the extraordinary sensibility of the eye, the above calculations show that it responds to visually perceptible energy of the order of 1.3×10^{-18} watt (3×10^{-14} g.-cal. per minute), at which rate it would take 60 million years to raise the temperature of 1 gram of water 1° .^b

Since writing this paper Dr. Burns has called our attention to a very interesting paper by H. D. Curtis⁶¹ on the limits of unaided vision. One difficulty of making the test of star visibility is in finding the position of the star, after which it is comparatively easy to recognize it. Another difficulty arises from the fact that the background of the sky is not perfectly dark. Given as artificial aids the direction in which the object lies and the screening off of the light of the sky (by looking through a long tube), Curtis could easily see stars of the 7.5 to 8.0 magnitude, but stars of the 8.3 to 8.5 magnitude were seen with difficulty. This means brightnesses one-fifth to one-sixth as great as used in the above calculations. The pupillary aperture would be 6 to 7 mm. Using these values, the above computations give about 8×10^{-10} erg., which is a reasonable estimate of the sensibility of the eye.

VIII. SUMMARY

The object of the present investigation was the determination of the spectral visibility of radiation curve of the average eye, as based upon a large group of observers. The total number of persons subjected to test was 130, of which number 7 were known to be color blind.

One of the most important measurements involved in the work was the radiometric evaluation of the light stimulus, which was a cylindrical acetylene flame. The distribution of energy in the spectrum of the acetylene flame was determined with great care.

^a To be exact, this comparison should be made in terms of the "light" instead of the total radiation from a star. Assuming a stellar luminous efficiency of 20 per cent, it would require a 7-foot mirror to attain the sensibility comparable with that of the eye.

^b Since writing the above section of this paper a discussion of this same subject has been published by Ives.⁵⁹ He used a more probable pupil opening of 6 mm and Russel's⁶⁰ value of 1 meter candle = -14.18 stellar magnitude. This gives a somewhat higher value than the above, viz. 3.8×10^{-9} erg. per cm.² Using a 6-mm pupil the present computations would be increased fourfold.

As this paper goes to the press, a paper by Russel⁶² has appeared in which the data by Ives⁵⁷ are recalculated on the basis of the observations of star visibility by Curtis⁶¹ and recent measurements of the aperture (8.5 mm) of the pupil by Stevenson.⁶³ His calculations of 7.7×10^{-10} erg., for the minimum energy perceptible, is in agreement with the present calculations. This rate of energy reception by the eye corresponds to about 200 elementary quanta of radiation per second, or 1 g.-cal. in 1700 million years.

The spectral-energy curve determined with the spectroradiometer used in the present investigation was found in agreement with similar measurements made with two other instruments of an entirely different type of construction.

The methods employed in obtaining the spectral-visibility curve were photometric. The complete visibility curve extending from 0.46μ to 0.74μ was determined with a flicker photometer by comparing the various spectral colors with a standard incandescent lamp. Similar measurements were made at five points in the spectrum using the equality-of-brightness method of photometry, the object being to determine whether there is a systematic difference in the measurements made by these two methods of photometry.

It was found that the various observers experienced little or no difficulty in making the photometric comparisons when using the flicker photometer. On the other hand, only a few observers were able to make accurate settings with the equality-of-brightness photometer, especially in the blue and in the red. This difficulty of forming a judgment of equality of brightness which is not influenced by differences in color seemed to be aggravated by the feeling that such a comparison had no meaning. In the yellow (0.5876μ) about as many observers overestimated as underestimated in making the equality-of-brightness settings as compared with the flicker photometer, and the magnitude of these variations (0 to 50 per cent) were within reasonable limits. On the other hand, when comparing less saturated colors, in the blue and in the red ends of the visible spectrum, a greater number of observers overestimated than underestimated their equality-of-brightness settings. Furthermore, the magnitude of this overestimation varied from 5 to 500 per cent. For the same observer these measurements varied greatly from day to day, whereas his flicker measurements were repeated to within several per cent. Because of these erratic settings the data do not appear to be convincing evidence that the visibility curve determined with the equality-of-brightness photometer differs from the visibility curve obtained with the flicker photometer.

From the data obtained on 14 observers above 40 years of age, it appears that age has a perceptible effect upon the spectral-visibility curve.

The same visibility curves were obtained for the right and left eye of a given observer.

The spectral-visibility curves of no two persons appear to be exactly alike, although there are some which are closely alike. When a spectral-visibility curve does not coincide with the average, there is usually a marked departure from the average visibility in a given spectral region. This gives rise to (1) wide visibility curves with the maximum shifted toward the red, i. e., "red sensitive," (2) narrow curves with a sharp maximum in the green, and (3) curves with the maximum shifted toward the violet. A fourth group of observers has the average visibility in the red and yellow, but has a low sensibility in the blue, while a fifth group has the average sensibility in the blue but has a low sensibility in either the yellow or the red or throughout this entire region of the spectrum.

The data available indicate that 60 per cent of the cases examined fall into three quite evenly divided groups (i. e., 20 per cent roughly estimated, in each group) which are either (1) red sensitive, (2) blue sensitive, or (3) average. Similarly, 30 per cent of the cases examined are quite evenly divided into three groups which fall below the average sensibility in either (1) the red, (2) in the blue, or (3) in both the red and the blue, thus giving rise to an apparently high sensibility in the green. One person in about 20 has a very wide visibility curve as compared with the average.

The point of maximum visibility is very different for different observers. The maximum visibility of the average of 125 subjects is at $\lambda = 0.5576\mu$. Correcting the visibility curve of the average eye for selective transmission of the ocular media including the yellow spot, produces a quite symmetrical curve with a maximum $\lambda = 0.556\mu$.

The present results are in agreement with previous conclusions that the effect of a light stimulus upon the color sense is quite independent of its effect upon the brightness sense. Subjects were found who were normal with respect to color sensation, but who were abnormally sensitive to brightness sensation. The abnormal color sense of so-called color-blind subjects is associated with an abnormal visibility curve.

Tabulated data are given of the visibility of radiation of the average eye as a function of an equal-energy spectrum. A mathematical equation is given of the average visibility curve.

Calculations are given showing that the eye responds to light having an intensity of less than 8×10^{-17} watt, or 8×10^{-10} erg.

WASHINGTON, October 16, 1916.

APPENDIXES

APPENDIX 1.—ON DIFFUSE LIGHT AND SPECTRAL-ENERGY MEASUREMENTS

During the past 10 years repeated attempts have been made to measure the scattered light superposed upon the spectrum under investigation. The invariable result was to find that, provided proper precautions had been taken to prevent reflection of the ends of the spectrum from the sides of the telescope tube and thence upon the radiometer, the effect of scattered radiations ("field light") upon the spectral-radiation measurements was quite uniform and negligible ^a in the visible spectrum.

When using a mirror spectrometer, one test was to attempt to measure the energy in the spectrum at 0.32μ , where the reflecting power of silver is practically zero. Hence the energy measured at this point in the spectrum is due to diffuse radiation. Another test was to attempt to measure the residual energy in the yellow and green part of the spectrum after transmission through Schott's red glass, which is completely opaque to these radiations. The results of these tests indicated that the correction for stray light in the region from the violet to the red was less than 1 part in 300.

In view of the fact that the crucial part of the present investigation has been the energy measurements, this question was recently investigated anew in order to establish beyond all reasonable doubt the accuracy of the radiometric evaluation of the light stimulus.

Before describing the present tests for diffuse light, it is of interest to describe the construction of the radiometric attachment to the spectroscope used in the present investigation.

Until very recently the spectroscopes obtained on the market were not properly constructed to prevent light from being reflected from the side of the telescope tube and from the beveled edges of the slit jaws.

In the present instrument the knife-edge jaws which form the slit are placed with the flat side toward the incident light.^b The knife edge is curved to fit the image of the slit of the collimating telescope. These slits and the inside of the tube are painted with lampblack, after which the knife edges are wiped smooth with a thin wedge of wood. The inside of the telescope tube was entirely free from springs and attachments to the slits such as are to be found in some instruments. The radiometer end of the telescope tube was about 3 cm in diameter. Hence, when used without diaphragms, one end of the spectrum falls upon the side of the tube when the other end is on the slit. When the blue end of the spectrum was on the slit, enough red light was reflected from the side of the tube into the radiometer slit to increase the true value in the blue by 5 to 50 per cent.

Suitable diaphragms gradually decreasing in opening as they approached the slit prevented reflections from the side of the telescope tube. These diaphragms were painted with lampblack (reflecting power 3 per cent), then smoked by holding them in the flame of a sperm candle. The diffuse reflecting power of soot deposited by holding the plate in the candle flame is very small—as low as 0.5 per cent.^c

^a Coblenz, this Bulletin, 10, p. 53; 1913.

^b Coblenz, Journal Franklin Institute, 175, p. 151; 1913.

^c Coblenz, this Bulletin, 9, p. 301; 1913.

This manner of construction of the receiver end of a spectroradiometer has been employed for a number of years, thus eliminating the possibility of reflection from the side of the telescope tube,^a and hence reducing the scattered light to the unavoidable "field light", which comes from the prism and lenses. In certain kind of investigations (e. g., transmission spectra) this field light can be reduced by inserting absorption screens. In very exact measurements of spectral-energy distributions it would be more practicable to use two spectroscopes in series.

The diffuse light in the spectroscope used in the foregoing investigation was determined by measuring (with a thermopile) the transmission through a sample of Schott's "monochromatic" red glass (No. 2745; thickness=3.5 mm), which is opaque to radiation of wave lengths less than 0.6μ .

In the first determination of the transmission of red glass the source of light was the acetylene flame used in the visibility measurements. At $\lambda=0.604\mu$, where this glass is not entirely opaque, the transmission was about 0.6 per cent, but throughout the remainder of the spectrum from the yellow ($\lambda=0.5876\mu$) to the blue no radiation transmitted through the red glass could be detected. The sensitivity of the thermopile and meteorological conditions were such that a deflection of 1 part in 130 could have been detected.

In the second determination of the diffuse radiation transmitted through this red glass the source of radiation was a Nernst glower. The intensity in the yellow was more than 30 times as great as in the acetylene flame. The amount of diffuse radiation varied quite uniformly from 3.8 per cent at 0.59μ to 6 per cent at 0.50μ , and from this point the transmission increased rapidly to 18 per cent at 0.42μ . This result was at variance with all previous tests, and in view of the fact that the Nernst glower has far more infra-red radiation than the acetylene flame in the region of the spectrum ^b from 0.7μ to 2μ it was evident that the diffuse radiation was chiefly infra-red radiation.

This test, therefore, was repeated, using a 1-cm cell of a 3 per cent solution of cupric chloride in addition to the red glass. The cupric-chloride solution is practically opaque to the infra-red ^c beyond 0.8μ and has a high transmission throughout the visible spectrum.

Using this combination to eliminate the infra-red radiations, the transmission was found to be about 0.5 per cent at 0.604μ , as observed with the acetylene flame. At $\lambda=0.5876\mu$ the transmission (which is due entirely to diffuse radiation) was found to be 0.3 to 0.4 per cent, while at 0.50μ the (transmission) diffuse radiation amounted to 0.1 to 0.2 per cent. At this point the total deflection was 102 mm and the readings could be made to 0.1 to 0.2 mm; i. e., the observations were read to 1 to 2 parts in 1000. At 0.45μ the diffuse radiation was of the order of 1 part in 300. The test with the Nernst glower is, therefore, in entire agreement with that on the acetylene flame, showing that the radiometric correction for diffuse radiation (which is also of low visibility) is entirely negligible in this work.

As used in the visibility measurements the diffuse light was even less than observed in these tests. This is due to the selection of the part of the optical system which showed the least diffuse light and the "ghost" spectra visible along the axis of the lenses. In view of these facts the data, uncorrected radiometrically for diffuse reflection, are published as given in the foregoing pages.

As already mentioned, the difference between the visibility curve obtained by Nutting and the one obtained in the present work seems to be due chiefly to the

^a Ives, *Phys. Rev.*, **6**, p. 339, 1915, and Souder, *Phys. Rev.*, **8**, p. 310, 1916, comment on the great amount of diffuse light which vitiated their spectral-radiation measurements in the blue. Nothing is mentioned, however, concerning the provision of diaphragms to prevent reflection of the spectrum from the side of the telescope tube.

^b Coblenz, this Bulletin, **7**, p. 265; 1911. The present investigation differed from the previous ones in that a Zeiss uncemented triple achromatic lens was used to focus an image of the flame or the glower on the spectrometer slit.

^c Coblenz, This Bulletin, **7**, p. 655; 1911; **9**, p. 110, 1912.

difference in our determinations of the distribution of energy in the spectrum of the acetylene flame, especially in the region of 0.59μ to 0.72μ . The spectral-energy measurements, therefore, were verified once more before publishing these data. In the first series of measurements the spectral-energy curve was found to coincide within 1 per cent with the previous measurements in the spectrum from 0.6μ to 0.72μ .

To make this test more crucial, the work was repeated by concentrating our efforts upon measurements at two selected points in the spectrum. At the first point, $\lambda=0.604\mu$, the energy curves used by Nutting and by the writers are in close coincidence, while at $\lambda=0.717\mu$ the energy curve used by Nutting is from 12 to 14 per cent higher than the curve used in the present work. The ratio of the spectral radiation at $\lambda=0.717\mu$ to the radiation at $\lambda=0.604\mu$ should be 3.75 to 3.8 if the former curve is correct. On the other hand, the energy curve used in the present investigation indicates this ratio to be 3.36 to 3.39. Accordingly, a day was spent in making measurements on this ratio. By so doing, the slow change in galvanometer sensitivity which occurs when determining the complete spectrum was eliminated. In all, six sets of measurements, involving 236 readings, were made. The mean value of the six sets was (3.38, 3.32, 3.41, 3.43, 3.35, 3.36) 3.37, which indicates an exact agreement of the present energy measurements with our previous determinations. The most crucial part of the present investigation has been verified after a lapse of almost a year, and hence there appears to be no reason for further withholding these data from publication.

WASHINGTON, November 21, 1916.

APPENDIX 2.—A SCREEN WHICH TRANSMITS RADIATIONS PROPORTIONAL TO THE AVERAGE VISIBILITY CURVE ^a

In certain investigations it is often desirable to use a screen which is opaque to all the infra-red and ultra-violet radiations and which transmits the visible radiations in proportion to the visibility curve of the average eye. Such a screen would be especially useful in a physical photometer, in radiant luminous-efficiency measurements, etc.

The transmission of light through a solution of certain inorganic salts in water forms a convenient and quite accurate copy of the visibility curve of the average eye. Various combinations of salts have been proposed, the most recent being by Ives and Kingsbury.^b As already mentioned, the concentration of salts recommended by these two investigators gives a transmission curve which coincides very closely with the visibility curve of the average eye except in the green. The object of the present paper is to describe a modification of the above solution which gives a transmission screen which coincides more closely with the visibility curve of the average eye. The lack of coincidence in the blue is of minor importance.

This solution has the following composition:

Cupric chloride ($\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$).....	grams..	5.7
Cobalt ammonium sulphate [$\text{Co}(\text{NH}_4)_2(\text{SO}_4)_2 \cdot 6\text{H}_2\text{O}$].....	do....	1.2
Potassium chromate (K_2CrO_4).....	do....	0.16
Nitric acid (HNO_3).....	do....	0.123
Water (H_2O).....	mols (cc)...	100

The salts were analyzed for purity. The nitric acid is expressed as 100 per cent acid (equivalent to 1.3 cc nitric acid, specific gravity 1.05 at $15^\circ/4^\circ$).

This solution is to be used in a glass cell 1 cm in thickness with uncolored glass windows. An additional cell of water 3 to 4 cm in thickness is to be used to absorb

^a These data were obtained in collaboration with Mr. A. N. Finn of the chemical division, who prepared the solutions, and Mr. H. J. McNicholas, who made the spectrophotometric settings.

^b Ives and Kingsbury, *Phys. Rev.*, (2) 6, p. 319, 1915.

the infra-red rays.^a To reduce reflection losses these two cells may be placed in contact with a film of glycerin intervening, the edges being covered with paraffin. The maximum transmission of this (double) cell was found to be 61 per cent at 20° C.

The transmission of the separate constituents was determined, and from this the proper concentrations for matching the visibility curve were estimated. In view of the fact that the "computed" and "observed" transmissions at 20° C. are in exact agreement for the mixture of the salts (see Fig. 16) they can be used in a single cell, although one would naturally expect to use the constituents in separate cells. The visibility curve is 1.2 per cent larger than the superposed (at 0.56 μ) transmission curve of the screen between $\lambda=0.44\mu$ and $\lambda=0.72\mu$.

WASHINGTON, January 9, 1917.

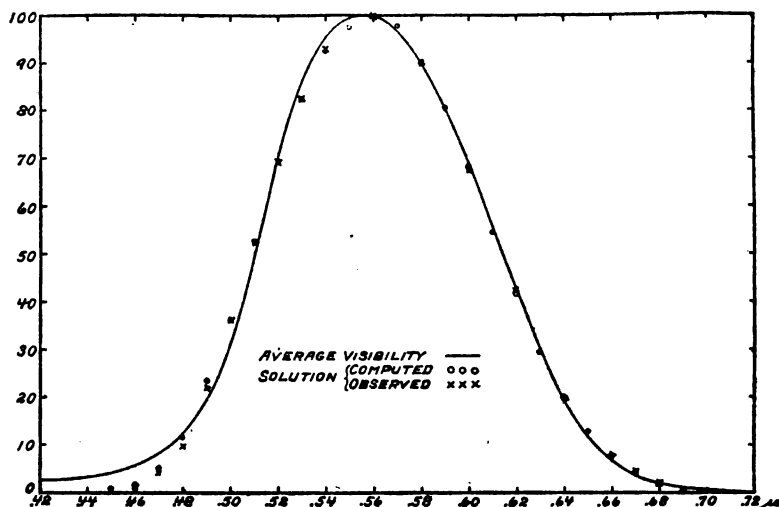


FIG. 16.—Comparison of average visibility curve with that of the proposed transmission screen

APPENDIX 3.—TEST OF A PHYSICAL PHOTOMETER

As already mentioned in the beginning of this paper, an important application of the visibility data is in physical photometry.^b In a previous investigation of a physical photometer it was found that very satisfactory results could be obtained.^c It was, therefore, of interest to repeat the test, using the luminosity screen described in Appendix 2. In view of the fact that the linear thermopile which was used in the present test was exposed to the air, conditions were not as favorable as in the previous experiments. However, the data obtained indicate that with this instrument accurate measurements can be easily and quickly made. The present test was made upon a standard vacuum tungsten lamp obtained from the photometry division of this Bureau. The lamp was operated at the currents specified to give 1.02 and 1.23 watts per candle, respectively, under which conditions the ratio of the luminous intensities (candlepowers^d) was =1.405.

In the first series of measurements of these intensities, determined radiometrically, the maximum galvanometer deflection was only 15 mm and the ratio was =1.418, or

^a This Bulletin, 7, p. 655, 1911; 9, p. 110, 1912; Journal Franklin Inst., 180, p. 355; 1915.

^b Ives, Trans. Illum. Eng. Soc., 10, p. 101, 1915; Ives and Kingsbury, Phys. Rev., 7, p. 319; 1915.

^c Coblenz, Jour. Franklin Inst., 180, p. 355; 181, p. 233; 1916.

^d In accordance with the Middlekauff-Skogland equation, this Bulletin, 11, p. 483, 1915.

a difference of 1 per cent, which is smaller than the errors of observation. In the second series of measurements, made some hours later with the lamp closer to the thermopile (maximum deflection 450 mm) and with more favorable meteorological conditions, the ratio of the radiometric intensities was =1.401. This is a difference of only 0.3 per cent between the photometric and the radiometric determinations. Using a vacuum thermopile the galvanometer reading would have been steadier and a higher precision would have been attainable.

WASHINGTON, January 24, 1917.

APPENDIX 4.—ON COLOR PERCEPTION VERSUS BRIGHTNESS PERCEPTION

As already mentioned in the main part of this paper, an abnormal color sense is associated with an abnormal brightness-sensibility curve, but the converse does not seem to be true. While, of course, many subjects must be tested in order to thoroughly

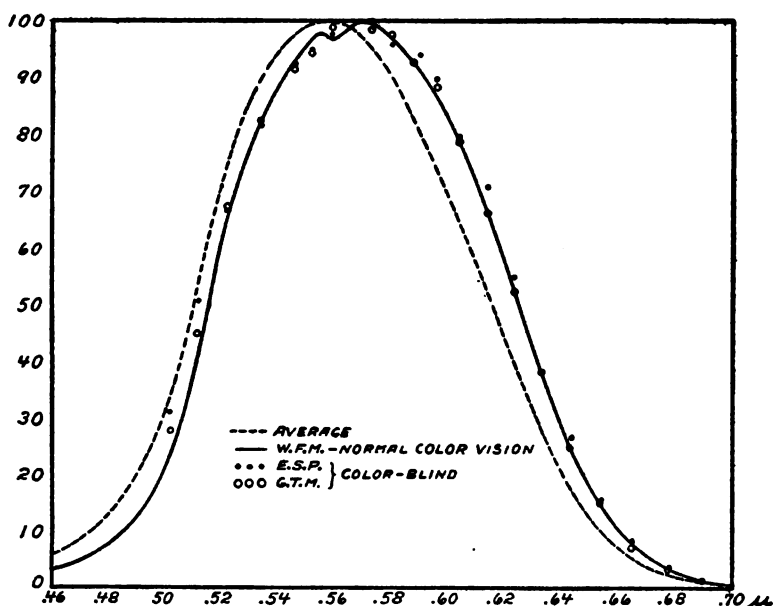


FIG. 17.—Curve illustrating abnormal color sense as compared with brightness sense

establish this fact, it is of interest to record further data on the color vision of one subject (W. F. M.), who is abnormally sensitive in the red, but who shows no signs of color blindness by the most crucial tests thus far used.^a

In the present examination for color blindness each eye was examined separately, the other being blindfolded. The first test made with the Stilling and the Nagel color cards gave no indications of color blindness.

The second test was one devised by Drs. Dunlap and Loring, who consider it better than the Holmgren yarn test. This test consists in classifying a large number of yarns with a set of 11 yarns (red, yellow, green, blue, blue-purple, gray, pink, brown, green-blue, red-purple, and yellow-green) of low color saturation. The subject showed no color confusion in making this classification.

^a These tests were made in the psychological laboratory of John Hopkins University. The writers wish to acknowledge their indebtedness to Drs. Dunlap and Loring for their courtesy in making these tests.

In the perimeter test (not completed) nothing unusual was found in the color perception of the peripheral retina.

In the spectrometer test the subject named the colors correctly throughout the spectrum.

In the tobacco-scotoma test, the fovea was explored with a colored object (red or green) 17 mm. in diameter, at a distance of 3 meters. There were no indications of a color-blind spot in the fovea.

From this brief summary of the results of the tests for color blindness it is quite evident that the color sense of this subject is quite different from that of other subjects who showed color blindness when examined by much simpler tests, e. g., the Nagel color-card test.

The sensibility curves of these two types of color perception are given in Fig. 17. This is an excellent illustration of abnormal color sense accompanied by abnormal brightness sense, and the converse example, which shows that abnormal brightness sensibility does not necessarily indicate abnormal color perception.

WASHINGTON, March 10, 1917.

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CALCULATION OF THE CONSTANTS OF PLANCK'S RADIATION EQUATION: AN EXTENSION OF THE THEORY OF LEAST SQUARES

By Harry M. Roeser

During the past few years several methods have been proposed for determining from experimental measurements the constants of Planck's equation for the distribution of energy in the spectrum of a uniformly heated inclosure or black body. The equation is as follows:

$$E = c_1 \lambda^{-5} (e^{\frac{c_2}{\lambda \theta}} - 1)^{-1}$$

In this equation E is a measure of the energy, λ the wave length, θ the absolute temperature, c_1 is a constant depending on the units in which E is measured and gives the scale of ordinates, c_2 is a constant which affects the shape of the curve and is sometimes called the "constant of spectral radiation." On account of the ease with which it may be eliminated from computations and its minor importance in indicating the actual distribution of energy, passing mention is usually deemed sufficient for c_1 . The determination of c_2 has properly been the center of most concern, and several methods of computing it from observed data on E and λ at constant temperature or E and θ at constant wave length have been offered.¹ Recent discussions have been devoted to observations at constant temperature, the curve in this case being called an "isothermal," and the methods of solution differ mainly in the manner of combining observations and in the use of correction terms to the transcendental expression for c_2 in terms of the other magnitudes involved. Some variations have appeared in the results given by these different methods. C. E. Van Orstrand has suggested as a remedy a solution by least squares. The carrying out of this suggestion is one of the purposes of this paper. A

¹ A concise summary of current methods and a detailed explanation of a new one may be found in this Bulletin, 18, p. 535, 1916 (Scientific Paper No. 287), by J. H. Dellinger; see also Buckingham and Dellinger, this Bulletin, 7, p. 393, 1911; W. W. Coblentz, this bulletin, 10, p. 2, 1914; W. W. Coblentz, this Bulletin, 18, p. 459, 1916 (Scientific Paper No. 284). Copious references to early German methods of attack are given in Coblentz's first paper. Phys. Rev. 28, p. 466, 1909; 32, p. 591, 1911; Ann. d. Phys., 309, p. 277, and p. 649, 1901; 311, p. 192, 1901.

method of attack is indicated and a least-square solution for the constants made, using some of the data that have been used in the past to determine c_2 by other methods. Theoretical reasons why the results of some of the previous investigations differ will also be given.

THE METHOD OF ATTACK

The rationale of the least-square reduction employed herein is as follows. Consider the Planck equation as stated above,

$$E = c_1 \lambda^{-5} (e^{\frac{c_2}{\lambda \theta}} - 1)^{-1}$$

A set of n observations being given on E and λ at constant temperature θ , it is required to determine the most probable values of the constants c_1 and c_2 .

The function may be written,

$$E = c_1 \lambda^{-5} e^{\frac{c_2}{\lambda \theta}} (1 - e^{-\frac{c_2}{\lambda \theta}})^{-1}$$

If $E_1, \lambda_1; E_2, \lambda_2; \dots E_n, \lambda_n$ are the n observations all of equal weight, the observation equations are

$$\begin{aligned} E_1 &= c_1 \lambda_1^{-5} e^{\frac{c_2}{\lambda_1 \theta}} (1 - e^{-\frac{c_2}{\lambda_1 \theta}})^{-1} \\ E_2 &= c_1 \lambda_2^{-5} e^{\frac{c_2}{\lambda_2 \theta}} (1 - e^{-\frac{c_2}{\lambda_2 \theta}})^{-1} \\ &\quad * \qquad * \qquad * \\ E_n &= c_1 \lambda_n^{-5} e^{\frac{c_2}{\lambda_n \theta}} (1 - e^{-\frac{c_2}{\lambda_n \theta}})^{-1} \end{aligned} \tag{1}$$

In these equations the parentheses on the right-hand side containing the exponential is relatively insensitive to a change in c_2 , as long as c_2 is a fairly close approximation to its proper value. The equations may be made linear and explicit in $\log c_1$ and c_2 , by substituting in the parentheses a reasonably approximate value c'_2 for c_2 , which may be determined by trial or by any method whatever, taking the logarithms of both sides and properly weighting the transformed observation equations.

Thus, after taking logarithms and transposing,

$$\begin{aligned} \log c_1 - \frac{c_2}{\lambda_1 \theta} &= \log E_1 + 5 \log \lambda_1 + \log (1 - e^{-\frac{c'_2}{\lambda_1 \theta}}), \text{ weight } E_1^2 \\ \log c_1 - \frac{c_2}{\lambda_2 \theta} &= \log E_2 + 5 \log \lambda_2 + \log (1 - e^{-\frac{c'_2}{\lambda_2 \theta}}), \text{ weight } E_2^2 \\ * \qquad * \qquad * \qquad * \qquad * \qquad * \\ \log c_1 - \frac{c_2}{\lambda_n \theta} &= \log E_n + 5 \log \lambda_n + \log (1 - e^{-\frac{c'_2}{\lambda_n \theta}}), \text{ weight } E_n^2 \end{aligned} \tag{2}$$

The logarithmic transformation employed here is by no means new and is quite frequently used as a medium for reducing data by least squares. However, while weighting the observation equations so transformed is evidently necessary in order to make a proper least-square solution, the principle is generally neglected in texts and papers dealing with practical means of handling observation equations to make them linear in the quantities sought, and consequently a sort of inherent inaccuracy depending on the scheme of reduction exists in many commonly accepted current methods of which the above is one.

It may be well to justify the principle here. Consider any observed value E_1 . Let its probable error be r_1 . Let $\varphi(E_1)$ be any function of E_1 , and R_1 be the probable error of $\varphi(E_1)$.

Then²

$$R_1^2 = \left(\frac{\partial \varphi(E_1)}{\partial E_1} \right)^2 r_1^2 \quad (3)$$

Now the relative weights of two quantities are inversely as the squares of their probable errors.³ That is, if P_1 be the weight of $\varphi(E_1)$ and p_1 be the weight of E_1 , then

$$P_1 : p_1 :: \frac{1}{R_1^2} : \frac{1}{r_1^2}$$

or,

$$P_1 = \left[\frac{p_1}{\frac{\partial \varphi(E_1)}{\partial E_1}} \right]^2 \quad (4)$$

In the particular case above where equations I are all of unit weight or reduced to a unit weight basis,

$$p_1 = 1, \text{ and } \varphi(E_1) = \log E_1$$

thus P_1 , which is the weight of any one of equations II becomes,

$$P_1 = \left(\frac{1}{\frac{\partial \log E_1}{\partial E_1}} \right)^2 = E_1^2$$

which was to be shown.

² Merriman, *Least Squares*, pp. 78-79; 1913.

³ Merriman, *Least Squares*, p. 60, 1913; Johnson, W. W., *Theory of Errors and Method of Least Squares* p. 53, 1892.

Equations 2 may be condensed into the following form:

$$\begin{aligned}\log c_1 - m_1 c_2 &= M_1, \text{ weight } E_1^2 \\ \log c_1 - m_2 c_2 &= M_2, \text{ weight } E_2^2 \\ * & \quad * \quad * \quad * \\ \log c_1 - m_n c_2 &= M_n, \text{ weight } E_n^2\end{aligned}\quad (5)$$

in which all quantities except $\log c_1$ and c_2 are either known or may be determined from the observations.

The normal equations whose solution gives the most probable values of $\log c_1$ and c_2 are

$$\begin{aligned}\log c_1 \sum E^2 - c_2 \sum E^2 m &= \sum E^2 M \\ -\log c_1 \sum E^2 m + c_2 \sum E^2 m^2 &= -\sum E^2 m M\end{aligned}\quad (6)$$

Whence,

$$\log c_1 = \frac{\sum E^2 M \cdot \sum E^2 m^2 - \sum E^2 m M \cdot \sum E^2 m}{\sum E^2 \cdot \sum E^2 m^2 - (\sum E^2 m)^2}, \quad (7)$$

and if common logarithms are used

$$c_2 = 2.3026 \cdot \frac{\sum E^2 \sum E^2 m M - \sum E^2 m \cdot \sum E^2 M}{\sum E^2 \cdot \sum E^2 m^2 - (\sum E^2 m)^2}. \quad (8)$$

If c_2 does not agree well with the approximate value, c'_2 , of equations 2, another solution can be made using the computed value as c'_2 .

APPLICATION OF THE METHOD

The application of the above process is straightforward and will not require an excessive amount of time or trouble of one possessing even a moderate amount of skill as a computer. The following 12 observations taken with a vacuum bolometer, fluorite prism, and mirror spectrometer were furnished by Dr. W. W. Coblentz.⁴

TABLE 1

Observations of Energy, Measured by Galvanometer Deflection, and Wave Length

[Absolute temperature 1350° C]

E (galvanom- eter de- flection)	λ (microns)	E (galvanom- eter de- flection)	λ (microns)	E (galvanom- eter de- flection)	λ (microns)
3.12	0.9338	23.95	1.526	12.10	4.446
6.75	1.054	19.20	3.569	11.02	4.638
11.80	1.197	17.45	3.760	9.10	5.001
18.03	1.357	15.05	4.031	7.70	5.329

⁴ These data have been used for other determinations of c_2 . See Coblentz, this Bulletin, 18, p. 474; 1916.

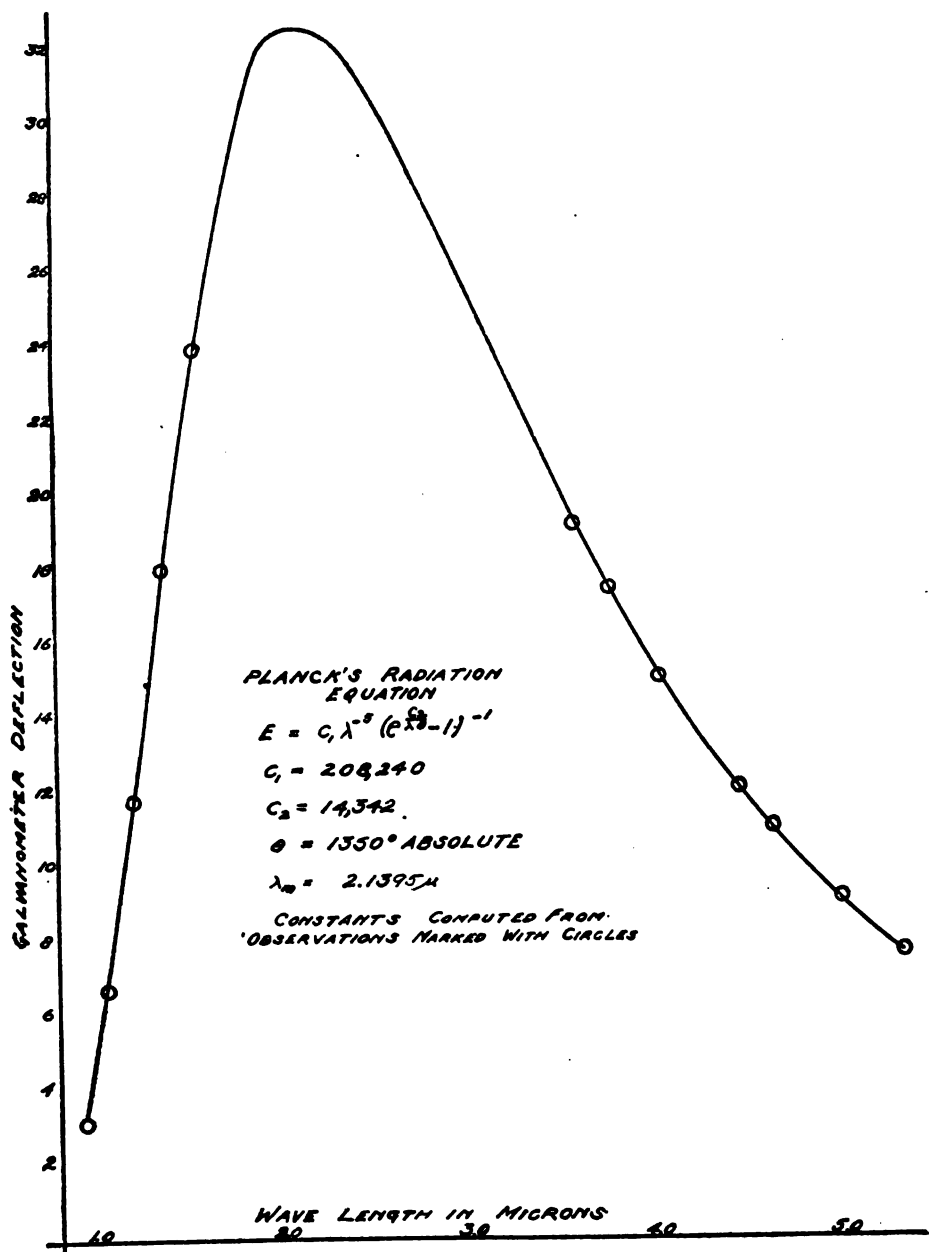


FIG. 1.—Planck's radiation curve, computed from the data of Table I.

The author carried out the computations to seven significant figures, using a seven-place Vega logarithm table and an eight-place computing machine. Five significant figures will amply suffice for practical purposes and will materially reduce the amount of labor required. The value of c' , used in equations 2 to reduce them to equations 5 was $c' = 14\,340$. A table of computed coefficients is herewith given.

TABLE 2
Coefficients for Equations 5

$m - \frac{1}{\lambda\theta}$	$M - \log E_1$ $+ 5 \log \lambda_1$ $+ \log (1 - e^{-\frac{c'_2}{\lambda\theta}})$	E^2	$m - \frac{1}{\lambda\theta}$	$M - \log E_1$ $+ 5 \log \lambda_1$ $+ \log (1 - e^{-\frac{c'_2}{\lambda\theta}})$	E^2
0.0007932542	0.3454193	9.7344	0.0001970055	4.0911939	304.5025
.0007027901	.9434885	45.5625	.0001837610	4.1722971	226.5025
.0006188310	1.4622922	139.2400	.0001666458	4.2804091	146.4100
.0005458664	1.9187218	325.0809	.0001597112	4.3274943	121.4404
.0004854133	2.2966665	573.6025	.0001481185	4.3990489	82.8100
.0002075485	4.0233168	368.6400	.0001390018	4.4561279	59.2900

Let $c_2 = 1,000\,v$.

Equations 5 become

$$\begin{aligned}
 \log c_1 - 0.7932542\,v &= 0.3454193, \text{ weight } 9.7344 \\
 \log c_1 - .7027901\,v &= 0.9434885, \text{ weight } 45.5625 \\
 \log c_1 - .6188310\,v &= 1.462292, \text{ weight } 139.2400 \\
 \log c_1 - .5458664\,v &= 1.918722, \text{ weight } 325.0809 \\
 \log c_1 - .4854133\,v &= 2.296666, \text{ weight } 573.6025 \\
 \log c_1 - .2075485\,v &= 4.023317, \text{ weight } 368.6400 \\
 \log c_1 - .1970055\,v &= 4.091194, \text{ weight } 304.5025 \\
 \log c_1 - .1837610\,v &= 4.172297, \text{ weight } 226.5025 \\
 \log c_1 - .1666458\,v &= 4.280409, \text{ weight } 146.4100 \\
 \log c_1 - .1597112\,v &= 4.327494, \text{ weight } 121.4404 \\
 \log c_1 - .1481185\,v &= 4.399049, \text{ weight } 82.8100 \\
 \log c_1 - .1390018\,v &= 4.456128, \text{ weight } 59.2900
 \end{aligned}$$

The normal equations 6 become

$$\begin{aligned}
 2402.816 \log c_1 - 824.2166\,v &= 7645.762 \\
 824.2166 \log c_1 - 359.4441\,v &= 2144.792
 \end{aligned}$$

whence

$$\log c_1 = 5.3185650, \quad c_1 = 208,240, \quad c_2 = 1000\,v = 14,342.1$$

This value of c_2 agrees very closely with the assumed value, 14 340, in equations 5. Had there been a wide difference, as much as 1 per cent, say, the computed value of c_2 could be used

for the c' , of equations 5 and the computations repeated until a satisfactory agreement was attained.

Should a second solution be deemed advisable, it may be accomplished very rapidly and with little labor. The coefficients of the unknowns in the normal equations will not be altered by a change in c' . The known terms will be altered slightly, due to slight changes in the last few of equations 5 from the right-hand side of the distribution. However, the normal equations are so insensitive to changes in c' , that after a reasonably judicious selection of a first value a second solution will rarely be necessary.

In regard to units, c_2 is expressed in micron degrees. The scale of ordinates is affected by c_1 , which depends upon the units in which E , the emissivity, is measured. In the above observations from which c_1 is calculated, E is measured by the deflections of a galvanometer whose sensitivity varies under different experimental conditions. The value of c_1 is, therefore, of an arbitrary nature and will hold only for data measured by this same galvanometer scale under similar experimental conditions.

A table of residuals follows:

TABLE 3

List of Residuals from Observations Given in Table 1. Computed from Planck's Equation, Using the Constants Obtained by the Least-Square Solution

E		Residuals, observed minus computed	(Residuals) ²
Observed	Computed		
3.12	3.360	-0.240	0.057600
6.75	6.713	.037	.001369
11.80	11.849	-.049	.002401
18.03	18.023	.007	.000049
23.95	23.863	.087	.007569
19.20	19.311	-.111	.012321
17.45	17.461	-.011	.000121
15.05	15.109	-.059	.003481
12.10	12.105	-.005	.000025
11.02	10.926	.094	.008836
9.10	9.036	.064	.004096
7.70	7.641	.059	.003481
			$\Sigma = .101349$

It may be of interest to note here the probable errors, R_{c_1} and R_{c_2} , of c_1 and c_2 , which are an indication of the precision of the values as determined. The probable errors are,

$$R_{c_1} = c_1 \times R \log c_1 = 0.6745 \sqrt{\frac{\sum v^2}{(n-2) \sum E^2 \cdot \sum E^2 m^2 - (\sum E^2 m)^2}} = 624$$

(See equation 3.)

$$R_{c_2} = 1000 R_e = 674.5 \sqrt{\frac{\sum v^2}{(n-2) \sum E^2 \sum E^2 m^2 - (\sum E^2 m)^2}} = 8.0$$

n being the number of observations. The other coefficients are those as found in equations 6.

The adjusted values of the constants written in the conventional manner are then

$$c_1 = 208,240 \pm 624, \text{ in arbitrary units.}$$

$$c_2 = 14,342 \pm 8.0, \text{ in micron degrees.}$$

DISCUSSION OF SOME CURRENT METHODS⁶

Two methods about which discussion has centered will now be briefly considered.

In the first or "equal ordinate" method, a curve is fitted by eye to the plotted observations and a line distant E_1 and parallel to the λ axis drawn. c_2 in terms of the abscissas λ_1 and λ_2 , corresponding to the points of intersection of the line with the curve has been shown⁶ to be

$$c_2 = \frac{\theta \lambda_2 \lambda_1}{\lambda_2 - \lambda_1} \left[5 \log \frac{\lambda_2}{\lambda_1} - e^{-\frac{c_2}{\lambda_2}} \right] \quad (9)$$

to a high degree of approximation. In this manner a number of values of c_2 are computed and the arithmetic mean taken as a final value.

A second method, which has come to be called the "two point" method, has been recently developed.⁷ A high approximation for c_2 in terms of any two observations, $E_1, \lambda_1; E_2, \lambda_2$, is

$$c_2 = \theta \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} \left[\log \frac{E_2}{E_1} + 5 \log \frac{\lambda_2}{\lambda_1} - e^{-\frac{c_2}{\lambda_2}} \right] \quad (10)$$

⁶ See note 1, p. 237.

⁶ Buckingham and Dellinger, this Bulletin, 7, p. 393, 1911; W. W. Coblentz, Phys. Rev., 88, p. 591, 1911.

⁷ J. H. Dellinger, this Bulletin, 18, p. 540; 1916.

and the rigorous solution is

$$c_2 = \theta \frac{\lambda_2 \lambda_1}{\lambda_2 = \lambda_1} \left[\log \frac{E_2}{E_1} + 5 \log \frac{\lambda_2}{\lambda_1} + \log \frac{1 - e^{-\frac{c_2}{\lambda_1 \theta}}}{1 - e^{-\frac{c_1}{\lambda_1 \theta}}} \right] \quad (11)$$

The former equation is accurate enough for all cases except where both observations are on the right side of the maximum.

A discussion follows to show why these two methods as they have been applied necessarily give somewhat inconsistent results.

Consider the conditions. There are given n observations all of equal weight on the quantities E and λ . These observations are operated on a pair at a time to give

$$\begin{aligned} c_2 &= \varphi_1 (E_1, E_2, \lambda_1, \lambda_2) \\ &\quad * \quad * \quad * \\ c_2 &= \varphi_1 (E_i, E_k, \lambda, \lambda_k) \end{aligned} \quad (12)$$

Herein lies the difficulty that has previously been met. It has been taken for granted that the values of c_2 obtained from the expressions 12 are of equal weight, and to get a final value of c_2 an arithmetic mean is taken. Since the probable error of a quantity, which is proportional to the square root of the reciprocal of its weight,⁸ computed from a function of observed quantities depends upon the nature of the function and the probable errors of the observed quantities, this step is obviously at fault unless the change in weight due to changes in the function is inappreciable as observations are taken from place to place in the distribution.

In what follows a system of weighting the separately computed values of c_2 is indicated that will tend to make the results by the two-point method agree with the results given by least squares.

Consider any pair of equations 5, say the i 'th and the k 'th, ($k > i$). Let them be reduced to a unit weight basis⁹ and solved for c_2 .

Thus,

$$\begin{aligned} E_i \log c_1 - E_i m_i c_2 &= E_i M_i \\ E_k \log c_1 - E_k m_k c_2 &= E_k M_k \end{aligned}$$

Whence,

$$c_2 (m_k - m_i) E_i E_k = (M_k - M_i) E_i E_k \quad (13)$$

or,

$$c_2 = \frac{M_k - M_i}{m_k - m_i} \quad (14)$$

⁸ Merriman, *Least Squares*, p. 69; 1913.

⁹ Wright and Hayford, *Adjustment of Observations*, p. 95; 1906.

which, when proper substitutions are made, is the same as equation 11.

If $E_1 = E_k$, equation 14, after certain approximations,¹⁰ becomes identical with 9.

If there are n of equations 5, there will be possible $\frac{n(n-1)}{2}$ expressions like 13, from each of which may be obtained an expression for c_2 like 14, which is a possible (but all are not equally probable) expression for the true value of c_2 . A brief elementary consideration of the error theory will show that the most probable value of c_2 is not the mean of the $\frac{n(n-1)}{2}$ equations 14. The most probable value is given by a least-square solution of the $\frac{n(n-1)}{2}$ equations 13, or,

$$c_2 = \frac{\sum E_i^2 E_k^2 (M_k - M_i) (m_k - m_i)}{\sum E_i^2 E_k^2 (m_k - m_i)^2} \quad (15)$$

In which $k > i$, k takes any value between 2 and n , inclusive, and i takes every corresponding possible value between 1 and $n-1$, inclusive. It is evident from inspection that this solution will be given by a reduction of the $\frac{n(n-1)}{2}$ equations 14, if each is weighted as

$$E_i^2 E_k^2 (m_k - m_i)^2 \quad (16)$$

It will now be shown that if equations 5 be reduced by the two-point method, giving each expression for c_2 the weight indicated by 16, and if every possible combination of the observations a pair at a time be made, the resulting value of c_2 will be identical with that given by the method of least squares.¹¹

For the sake of brevity in notation and manipulation, let four observation equations similar in type to equations 5 be considered, viz,

$$\begin{aligned} a + bx_1 &= y_1, \text{ weight } p_1 \\ a + bx_2 &= y_2, \text{ weight } p_2 \\ a + bx_3 &= y_3, \text{ weight } p_3 \\ a + bx_4 &= y_4, \text{ weight } p_4 \end{aligned} \quad (17)$$

¹⁰ See note 1, p. 237, Buckingham and Dellinger.

¹¹ Wright, Adjustment of Observations, p. 141; 1884.

In which the b of these equations corresponds to the c , of equations 5.

Equations 17 may be reduced to an equal-weight basis by multiplying each through by the square root of its weight. Thus,

$$\sqrt{p_1}a + \sqrt{p_1}bx_1 = \sqrt{p_1}y_1$$

$$\sqrt{p_2}a + \sqrt{p_2}bx_2 = \sqrt{p_2}y_2$$

$$\sqrt{p_3}a + \sqrt{p_3}bx_3 = \sqrt{p_3}y_3$$

$$\sqrt{p_4}a + \sqrt{p_4}bx_4 = \sqrt{p_4}y_4$$

Let the i 'th and the k 'th ($k > i$) equations be solved simultaneously for b . Then

$$\sqrt{p_1}\sqrt{p_k}(x_k - x_i)b = \sqrt{p_1}\sqrt{p_k}(y_k - y_i) \quad (18)$$

or,

$$b = \frac{(y_k - y_i)}{(x_k - x_i)}, \text{ weight } p_1 p_k (x_k - x_i)^2 \quad (19)$$

The number of equations of type 19 (i. e., the number of possible solutions for b from the four observation equations) will be the same as the number of combinations of four things two at a time, or $\frac{4 \times 3}{1 \times 2}$. If all these solutions be made, there will be at hand six expressions for b ,

$$\begin{aligned} b &= \frac{y_4 - y_1}{x_4 - x_1}, \text{ weight } p_1 p_4 (x_4 - x_1)^2 \\ b &= \frac{y_4 - y_2}{x_4 - x_2}, \text{ weight } p_2 p_4 (x_4 - x_2)^2 \\ b &= \frac{y_4 - y_3}{x_4 - x_3}, \text{ weight } p_3 p_4 (x_4 - x_3)^2 \\ b &= \frac{y_3 - y_1}{x_3 - x_1}, \text{ weight } p_1 p_3 (x_3 - x_1)^2 \\ b &= \frac{y_3 - y_2}{x_3 - x_2}, \text{ weight } p_2 p_3 (x_3 - x_2)^2 \\ b &= \frac{y_2 - y_1}{x_2 - x_1}, \text{ weight } p_1 p_2 (x_2 - x_1)^2 \end{aligned} \quad (20)$$

If all these values be considered as possible, and hence probable, values for the proper value of b , then the most probable value can be assumed to be the weighted mean of them all, i. e.,

$$b = \frac{\sum p_i p_k (x_k - x_i) (y_k - y_i)}{\sum p_i p_k (x_k - x_i)^2} \quad (21)$$

and it is to be shown that this is identical with the least-square solution,

$$b = \frac{\sum p \sum p x y - \sum p x \sum p y}{\sum p \sum p x^2 - (\sum p x)^2} \quad (22)$$

The denominator of 21 becomes, when expanded,

$$\sum p_i p_k (x_k - x_i)^2 = \sum p_i p_k x_k^2 + \sum p_i p_k x_i^2 - 2 \sum p_i p_k x_i x_k$$

Now $k > i$ and can take any value between 2 and 4, inclusive. i can take any value between 1 and 3, inclusive.

Suppose $k=4$.

The equation $a + bx_4 = y_4$ can be combined with the remaining three equations to obtain a separate solution for b . If $k=3$, there will be two equations that can be combined with $a + bx_3 = y_3$ to obtain a separate solution for b ; and if $k=2$, there will be left one equation that can be combined with $a + bx_2 = y_2$ to obtain a separate solution for b .

x_4 will, therefore, appear as x_k in the summations of 21 three times, x_3 will appear as x_k twice, and x_2 will appear as x_k once.

Similar reasoning will show that x_1 will appear as x_i three times, x_2 will appear as x_i twice, and x_3 will appear as x_i once.

$$\begin{aligned} & \therefore \sum p_i p_k x_k^2 + \sum p_i p_k x_i^2 - 2 \sum p_i p_k x_i x_k \\ &= \left\{ \begin{array}{l} p_1 p_4 x_4^2 \\ + p_2 p_4 x_4^2 \\ + p_3 p_4 x_4^2 \\ + p_1 p_3 x_3^2 \\ + p_2 p_3 x_3^2 \\ + p_1 p_2 x_2^2 \end{array} \right\} + \left\{ \begin{array}{l} p_1 p_4 x_1^2 \\ + p_2 p_4 x_2^2 \\ + p_3 p_4 x_3^2 \\ + p_1 p_3 x_1^2 \\ + p_2 p_3 x_2^2 \\ + p_1 p_2 x_1^2 \end{array} \right\} - 2 \left\{ \begin{array}{l} p_1 p_4 x_1 x_4 \\ + p_2 p_4 x_2 x_4 \\ + p_3 p_4 x_3 x_4 \\ + p_1 p_3 x_1 x_3 \\ + p_2 p_3 x_2 x_3 \\ + p_1 p_2 x_1 x_2 \end{array} \right\} \\ &= \left\{ \begin{array}{l} p_4 x_4^2 [p_1 + p_2 + p_3] + p_4 p_4 x_4^2 - p_4 p_4 x_4^2 \\ + p_3 x_3^2 [p_1 + p_2 + p_4] + p_3 p_3 x_3^2 - p_3 p_3 x_3^2 \\ + p_2 x_2^2 [p_1 + p_3 + p_4] + p_2 p_2 x_2^2 - p_2 p_2 x_2^2 \\ + p_1 x_1^2 [p_2 + p_3 + p_4] + p_1 p_1 x_1^2 - p_1 p_1 x_1^2 \end{array} \right\} \\ &\quad - \left\{ \begin{array}{l} p_1 x_1 [p_2 x_2 + p_3 x_3 + p_4 x_4] \\ + p_2 x_2 [p_1 x_1 + p_3 x_3 + p_4 x_4] \\ + p_3 x_3 [p_1 x_1 + p_2 x_2 + p_4 x_4] \\ + p_4 x_4 [p_1 x_1 + p_2 x_2 + p_3 x_3] \end{array} \right\} \end{aligned}$$

$$\begin{aligned}
 &= p_1 x_1^2 \Sigma p + p_2 x_2^2 \Sigma p + p_3 x_3^2 \Sigma p + p_4 x_4^2 \Sigma p \\
 &- [p_1 x_1 \Sigma p x + p_2 x_2 \Sigma p x + p_3 x_3 \Sigma p x + p_4 x_4 \Sigma p x] \\
 &= \Sigma p [p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + p_4 x_4^2] \\
 &- \Sigma p x [p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4] \\
 &= \Sigma p \Sigma p x^2 - \Sigma p x \Sigma p x \\
 &= \Sigma p \Sigma p x^2 - (\Sigma p x)^2
 \end{aligned}$$

which is identical with the denominator of 22. The numerator of 21 becomes, when expanded,

$$\begin{aligned}
 &\Sigma p_1 p_2 x_1 x_2 y_1 + \Sigma p_1 p_3 x_1 x_3 y_1 - \Sigma p_1 p_4 x_1 x_4 y_1 - \Sigma p_1 p_2 x_1 x_2 y_1 \\
 &= \left\{ \begin{array}{l} p_1 p_2 x_1 y_1 \\ + p_2 p_3 x_2 y_2 \\ + p_3 p_4 x_3 y_3 \\ + p_1 p_3 x_1 y_3 \\ + p_2 p_4 x_2 y_4 \\ + p_1 p_4 x_1 y_4 \end{array} \right\} + \left\{ \begin{array}{l} p_1 p_2 x_1 y_1 \\ + p_2 p_3 x_2 y_2 \\ + p_3 p_4 x_3 y_3 \\ + p_1 p_3 x_1 y_3 \\ + p_2 p_4 x_2 y_4 \\ + p_1 p_4 x_1 y_4 \end{array} \right\} \\
 &- \left\{ \begin{array}{l} p_1 p_2 x_1 y_1 \\ + p_2 p_3 x_2 y_2 \\ + p_3 p_4 x_3 y_3 \\ + p_1 p_3 x_1 y_3 \\ + p_2 p_4 x_2 y_4 \\ + p_1 p_4 x_1 y_4 \end{array} \right\} - \left\{ \begin{array}{l} p_1 p_2 x_1 y_1 \\ + p_2 p_3 x_2 y_2 \\ + p_3 p_4 x_3 y_3 \\ + p_1 p_3 x_1 y_3 \\ + p_2 p_4 x_2 y_4 \\ + p_1 p_4 x_1 y_4 \end{array} \right\} \\
 &= \left\{ \begin{array}{l} p_1 x_1 y_1 (p_2 + p_3 + p_4) + p_2 x_2 y_2 (p_1 + p_3 + p_4) + p_3 x_3 y_3 (p_1 + p_2 + p_4) \\ + p_4 x_4 y_4 (p_1 + p_2 + p_3) + p_1 p_2 x_1 x_2 y_1 - p_1 p_3 x_1 x_3 y_1 \\ + p_2 p_3 x_2 x_3 y_2 - p_2 p_4 x_2 x_4 y_2 \\ + p_3 p_4 x_3 x_4 y_3 + p_1 p_2 x_1 x_2 y_1 - p_1 p_3 x_1 x_3 y_1 \\ + p_2 p_3 x_2 x_3 y_2 - p_2 p_4 x_2 x_4 y_2 \\ + p_3 p_4 x_3 x_4 y_3 \end{array} \right\} \\
 &- \left\{ \begin{array}{l} p_1 x_1 (p_2 y_1 + p_3 y_2 + p_4 y_3) \\ + p_2 x_2 (p_1 y_1 + p_3 y_2 + p_4 y_3) \\ + p_3 x_3 (p_1 y_1 + p_2 y_2 + p_4 y_3) \\ + p_4 x_4 (p_1 y_1 + p_2 y_2 + p_3 y_3) \end{array} \right\} \\
 &= p_1 x_1 \Sigma p + p_2 x_2 \Sigma p + p_3 x_3 \Sigma p + p_4 x_4 \Sigma p \\
 &- [p_1 x_1 \Sigma p y + p_2 x_2 \Sigma p y + p_3 x_3 \Sigma p y + p_4 x_4 \Sigma p y] \\
 &= \Sigma p [p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4] \\
 &- \Sigma p y [p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4] \\
 &= \Sigma p \Sigma p x y - \Sigma p y \Sigma p x
 \end{aligned}$$

which is identical with the numerator of 22.

The above analysis, though limited to four observations, will apply equally well to any number of observations, odd or even. Equations 21 and 22 are, therefore, identities, and hence the two-point method may be made to give results identical with the least-square method by making all possible separate solutions for c , furnished by the data and assigning weights to them according to 16 before taking the mean.

Substitute in 16,

$$\frac{1}{\lambda_1 \theta} \text{ for } m_1, \frac{1}{\lambda_2 \theta} \text{ for } m_2,$$

there results

$$E_1^2 E_k^2 (m_k - m_1)^2 = \frac{E_1^2 E_k^2}{\theta} \left(\frac{\lambda_1 - \lambda_k}{\lambda_1 \lambda_k} \right)^2 = p$$

and since relative weights are all that are required the constant θ may be dropped,

$$\therefore p \propto E_1^2 E_k^2 \left(\frac{\lambda_1 - \lambda_k}{\lambda_1 \lambda_k} \right)^2 \quad (23)$$

The p in equation 23 approaches zero when λ_k is near to λ_1 , as may be the case when both points are near or on the same side of the maximum ordinate, and when E_1 and E_k are small.

It will be noted that in making the individual computations for c , from equations 10 and 11 that an assumed approximate value of c , is necessary in the exponential term. If, after applying the above system of weights, the mean does not agree with the assumed value, another solution will be necessary in which the computed value may be used as the approximate value in the exponential term. In no case should more than two of these processes be necessary.

A numerical illustration of the above theory is briefly represented in the accompanying Table 4. Six pairs of observations, E_1 , λ_1 , E_k , λ_k , were selected at different parts of the distribution and values of c , computed from them according to equations 10 and 11.

Table 4 is prepared to show (1) there is considerable variation in the weights of the values of c ; (2) that values of c , differing widely from the mean are likely to be of small weight; (3) that the simple arithmetic mean of a limited number of computations is likely to differ considerably from the weighted mean (the variation shown in the computations of Table 4 is probably an extreme one. Any other arrangement may show a better agreement with the

value given by least squares); (4) that for a limited number of computations the weighted mean approximates better the most probable value, i. e., the value given by a least-square solution.

TABLE 4

List of Values of c_2 Given by the "Two-Point" Method. Computed From One Arrangement of the Observations of Table 1 Using Each Observation Once

λ_1	λ_k	E_1	E_k	c_2	p	$p \cdot c_2$
0.9338	5.001	3.12	9.10	14 478	6.117	88 555.7
1.526	3.569	23.95	19.20	14 308	297.556	4 257 437.3
1.054	4.638	6.75	11.02	14 358	29.740	427 007.1
3.760	5.329	17.45	7.70	14 487	1.107	16 036.3
1.197	4.446	11.80	12.10	14 360	75.963	1 090 829.7
1.357	4.031	18.03	15.05	14 337	175.958	2 522 709.8
				$\Sigma p = 586.440$ $8 402 575.9 = \Sigma p \cdot c_2$		
				$\frac{\Sigma c_2}{6} = 14 388 \pm 21$ $\frac{\Sigma p \cdot c_2}{\Sigma p} = 14 328 \pm 10$		

The above table is computed from an arrangement of the observations using them all once. There are 11 different arrangements of the observations two at a time in each of which every observation appears only once. Thus there are 66 possible solutions for c_2 from the data of Table 1 by the two-point method. If any part of the remainder of these solutions be made and the results added to those in the table before taking the mean, the final value of c_2 will, no doubt, be altered to some extent. In case the experimenter considers the mean of a limited number of computations satisfactory for his purpose, they should be arranged so that each observation enters the computations the same number of times, and thus prevent any one point from unduly affecting the results. That is, for $2k$ observations, k , or $2k$, or $3k$, etc., to $(n-1)k$ computations will be necessary. The same number of computations will be necessary for $2k+1$ observations, and in each set of k computations one of the observations can not appear. This does not mean that as a method of approximation the two-point method is a failure in this case, for the experimenter may use his discretion and drop in order from each set that observation that can be combined with others of the group to give relatively small weights to the value of c_2 .

The above system of weighting is only theoretically correct when applied to all possible solutions for c_2 from the data. However, the system applied to a limited number tends to make the mean stable and approach the value given by least squares. By a judi-

cious selection of combinations it is possible to give a comparatively large weight to a mean value of c_2 with one or two sets of weighted computations. The pairs of observations should be selected so as to keep the difference of the λ 's large and at the same time keep the products of the E 's large. Any set of k computations in which each observation is used once need not give the same result as any other set, nor need the weights of the results of the two sets be the same. Thus it is seen that the value of a result obtained from a limited number of computations depends, first, upon the arrangement of the observations and, second, upon the number of separate computations for c_2 .

To a skilled computer the labor involved in a least-square computation is by no means prohibitive of its application, and whether it is to be preferred to the two-point method to make a solution for the most probable value of the constants depends to some extent upon the factor of personal skill. In using the two-point method for n observations there will be necessary $\frac{n(n-1)}{2}$ indi-

vidual computations for c_2 and the same number of computations for the weights. Thus the method will become impractical for a large number of observations. In fitting a curve to a few observations, say six or seven, the complete solution can perhaps be most readily accomplished with the two-point method by anyone. For a larger number of observations the advantages of the two-point method are less. The fact that the least-square computation gives both constants may or may not be of practical value. If c_1 is desired, another extended computation to determine c_1 will be necessary after determining c_2 by the two-point method.

The possibilities of the equal-ordinate method depend upon the computer's skill as a draftsman, for in fixing the curve he tacitly predetermines a value of c_2 . In a high-grade set of data one skillful with a pen may be able to fit a curve so closely by eye to the least-square position that the results of the individual calculations for c_2 will not differ appreciably among themselves, and any system of weights will therefore have no appreciable effect. An appreciable difference among individual solutions by this method signifies that the curve is not properly located. If for a given position of the curve an appreciable variation is not obtained in a reasonable number of solutions selected from place to place on the distribution, the curve can be considered properly located and the unweighted mean of the solutions should approximate well the least-square solution.

In the data used in this paper the gain in precision of the least-square method over the weighted two-point method using a limited number of computations is slight for the particular arrangement of the observations in Table 4. However, the element of chance enters to a considerable degree in the worth of the results of Table 4. It must be remembered that there are 10 other arrangements of the observations using them all once. Had any one of these happened to be selected for the illustration, the results might or might not have agreed so well both in the value of the constant and the probable error. In the same way the element of chance is a large factor in the disagreement of the simple mean with the least-square value. Individual computations for c , by the two-point method can be arranged to present a practical significance to a physicist which the least-square method does not directly possess, but consideration of these is outside the scope of this paper.¹²

SUMMARY

A least-square reduction of a set of observations of the Planck radiation-equation type has been effected by taking logarithms of both sides and assigning proper weights to the transformed equations. The method of assigning these weights has been given in a general form that can be adapted to any scheme of transformation.

How to combine and weight separate computations for c , by the two-point method so as to yield a value that agrees with the least-square value has been shown. A numerical demonstration is given that the proposed system of weights applied to an arbitrarily selected set of two-point calculations leads to a result which approximates the least-square value better than the simple mean of the set.

The author wishes to thank Dr. J. H. Dellinger for valuable assistance and suggestions in the revision of the paper.

WASHINGTON, August 19, 1916.

¹² J. H. Dellinger, this Bulletin, 18, p. 541, et seq.; 1916.

LUMINOUS RADIATION FROM A BLACK BODY AND THE MECHANICAL EQUIVALENT OF LIGHT

By W. W. Coblentz and W. B. Emerson

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I. INTRODUCTION

Numerous investigations have been made to determine the relation between visual sensation, "light," and radiant power, and, more specifically, the radiant power required to produce an intensity of 1 candle.

In order to calculate data on visual sensation from the radiation emitted by a black body, it is necessary to have a mathematical equation of the visibility curve of the average eye. Such an equation has been given in a preceding paper,¹ which includes experimental data on the average eye representing 125 observers. This equation, which fits the observations very closely, is of the form

$$V_{\lambda} = V_m R^n e^{n(1-R)} \quad (1)$$

In this equation V_{λ} represents the visibility of radiation at a given wave length relative to the visibility, V_m , at the wave length of maximum visibility, and $R = \frac{\lambda_m}{\lambda}$.

In order to make this equation fit the observed data, it has to be used in the following form:

$$V_{\lambda} = 0.999(R_1 e^{(1-R_1)})^{200} + 0.035(R_2 e^{(1-R_2)})^{400} \\ + 0.130(R_3 e^{(1-R_3)})^{2000} + 0.084(R_4 e^{(1-R_4)})^{2000} \quad (2)$$

* See bibliography at conclusion of paper for notes referred to throughout text by superior figures.

In this formula $R_1 = \frac{0.556}{\lambda}$, $R_2 = \frac{0.455}{\lambda}$, $R_3 = \frac{0.610}{\lambda}$, and $R_4 = \frac{0.525}{\lambda}$, where λ is expressed in microns.

II. LUMINOUS FLUX EMITTED BY A BLACK BODY

The visibility curve of the average eye and the mathematical equation representing it are of interest in computing the brightness (the luminous flux emitted), the luminous efficiency, etc., of a black body as a function of its temperature. Extensive computations of this type have been made by various writers, the most recent papers being by Nutting,^{3,8} Pirani,⁴ Foote,⁵ Kingsbury,⁶ etc. In many cases the numerical data were obtained by graphical multiplication and integration. For example, one could obtain the luminous efficiency by computing the spectral energy curve of a black body at a given temperature, using Planck's equation. This curve is drawn to scale, and from it a new curve is obtained in the visible spectrum by laying off ordinates proportional to the product of the ordinates of the original curve and the visibility curve. The ratio of the areas of these curves obtained by graphical integration gives the luminous efficiency. These graphical computations may be avoided by following the procedure of Kingsbury,⁶ who has reduced these data by mathematical integration. In the present paper only a few of these computations will be given in order to show the modification of Kingsbury's numerical values when using the most probable radiation constants⁷ and the new visibility curve. For this purpose it is of interest to recall that the luminosity L_λ of a black body is the product of the energy, E_λ , emitted per second and the visibility at each wave length. The integral luminosity ("luminous power," "brightness," "Helligkeit," etc., are terms commonly used when considering the light emitted) per unit area is, therefore,

$$L = \frac{1}{\pi} \int_0^\infty V_\lambda E_\lambda d\lambda \quad (3)$$

in which V_λ is obtained from equation (1) and E_λ from Planck's equation $E_\lambda = c_1 \lambda^{-5} (e^{c_2/\lambda T} - 1)^{-1}$. The value of L is in light watts per steradian per square centimeter (projected area) of the radiating surface. The factor $\frac{1}{\pi}$ is introduced in order to give the normal flux density, σ_n , of radiation per cm^2 per deg.⁴ The total hemispherical flux, on this basis, is $\sigma = \pi \sigma_n$.

In order to calculate the luminosity mathematically,⁶ it is necessary to transform equation (3) so that it can be integrated exactly. For this purpose the part of the Planck equation which is in parentheses is expanded by the binomial theorem giving:

$$E_{\lambda} = c_1 \lambda^{-\alpha} (e^{c_2/\lambda T} - 1)^{-1} = c_1 \lambda^{-\alpha} [e^{-c_2/\lambda T} + e^{-2c_2/\lambda T} + \dots + e^{-nc_2/\lambda T}] \quad (3a)$$

In view of the fact that we are interested in evaluating the energy in the visible spectrum, only the first term, $e^{-c_2/\lambda T}$, in the binomial series, equation (3a), which corresponds with the Wien equation, needs to be used in equation (3).

Writing equation (1) in the form $V_{\lambda} = \left(\frac{\lambda_m}{\lambda}\right)^n \frac{e^n}{e^{n\lambda_m/\lambda}}$ and placing $V_m = 1$, equation (3) becomes

$$L = \frac{c_1 e^n \lambda_m^n}{\pi} \int_0^{\infty} \left[\lambda^{-n-\alpha} e^{-(n\lambda_m + \frac{c_2}{T}) \frac{1}{\lambda}} \right] d\lambda \quad (3b)$$

By placing $\lambda = \frac{1}{\lambda'}$, equation (3b) becomes

$$L = \frac{c_1 e^n \lambda_m^n}{\pi} \int_0^{\infty} \left[\lambda'^{(n+\alpha-2)} e^{-(n\lambda_m + \frac{c_2}{T}) \lambda'} \right] d\lambda' \quad (3c)$$

This integral has the form of the gamma function,

$$\int_0^{\infty} x^p e^{-kx} dx = \frac{\Gamma(p+1)}{K^{p+1}} = \frac{\Gamma(n+\alpha-1)}{K^{p+1}}$$

in which

$$\begin{aligned} p &= n + \alpha - 2 \text{ and } K^{p+1} = \left(n\lambda_m + \frac{c_2}{T} \right)^{n+\alpha-1} \\ &= (n\lambda_m)^{n+\alpha-1} \left(1 + \frac{c_2}{n\lambda_m T} \right)^{n+\alpha-1} \end{aligned}$$

Hence equation (3c) becomes:

$$L = \frac{c_1 e^n \lambda_m^n \Gamma(n+\alpha-1)}{\pi (n\lambda_m)^{n+\alpha-1} \left(1 + \frac{c_2}{n\lambda_m T} \right)^{n+\alpha-1}} \quad (3d)$$

Using Stirling's formula (see English edition of Planck's "Heat radiation," p. 218) for large values of n , we have

$$\Gamma(n+\alpha-1) = (n+\alpha-2)! = \frac{\sqrt{2\pi}}{e^{n+\alpha-2}} (n+\alpha-2)^{n+\alpha-3/2}$$

The value of c_1 , which appears in equation (3d), is obtained from the relation (using $\alpha = 5$)

$$c_1 = \frac{\sigma T^4}{\int_0^\infty \lambda^{-5} (e^{\alpha \lambda T} - 1)^{-1} d\lambda} = \frac{\sigma c_2^4}{6.494}$$

This value is obtained by using the binomial expansion of the Planck equation given in equation (3a), which is integrated term by term, after which the whole is summated^b from $m = 1$ to $m = \infty$ for $\alpha = 5$.

Substituting these values, equation (3d) becomes

$$L = \frac{\sigma}{\pi} \frac{A}{\left(1 + \frac{B}{T}\right)^{n+4}} \quad (4)$$

where

$$A = 0.01922 \left(\frac{c_2}{\lambda_m}\right)^4 \frac{1}{\sqrt{n}} \left(\frac{n+3}{n}\right)^{n+1/2} \text{ and } B = \frac{c_2}{n\lambda_m}$$

Using an equation containing four terms [equation (2) instead of equation (1)] for V_λ and using the constants $\lambda_m = 0.556, 0.445, 0.610$, and 0.525μ , respectively, $c_2 = 14\,350$ and $\sigma = 5.7 \times 10^{-12}$ watt per cm^2 per deg^4 , the luminous intensity (in light-watts per steradian per cm^2) of a black body at an absolute temperature T is

$$L = \frac{5.7 \times 10^4}{\pi} \left[\frac{1.247}{\left(1 + \frac{129.05}{T}\right)^{204}} + \frac{0.0678}{\left(1 + \frac{78.85}{T}\right)^{404}} + \frac{0.0489}{\left(1 + \frac{23.52}{T}\right)^{1004}} + \frac{0.0406}{\left(1 + \frac{13.67}{T}\right)^{2004}} \right] \quad (5)$$

These data are given in Table 1. If we have the visibility constant, V_m , which is the ratio of the candle (or lumen) to the watt at the wave length of maximum visibility, these values of L can be transformed to candlepower. For this purpose in equation (3) $L = bm$, where b = brightness in candlepower per unit area and $m = \frac{1}{V_m}$ = the least mechanical equivalent of light, which is discussed on a subsequent page.

^b See Preston's *Theory of Heat* (revised by Cotter), p. 607, 1904, for the derivation of the constants α and c_2 .

TABLE 1

Data on the Luminous Intensity of a Black Body and the Mechanical Equivalent (m) of Light

T, degrees absolute	Luminous intensity L watt/cm ²	Total intensity $\sigma_0 T^4$ watt/cm ²	Radiant luminous efficiency	Brightness b, in candles per cm ² (Hyde)	m=L/b	Crown wave length μ
1200	2.34×10^{-6}	3.762	0.0000622			0.605
1600	3.45×10^{-5}	1.189	.000290			.591
1700	8.46×10^{-5}	1.515×10	.000558	5.2	0.001627	.589
1800	1.88×10^{-4}	1.905×10	.000987	11.3	.001664	.586
1900	3.85×10^{-4}	2.365×10	.00163	23.3	.001652	.584
2000	7.34×10^{-4}	2.903×10	.00253	45.0	.001631	.582
2100	1.32×10^{-3}	3.529×10	.00374	81.0	.001630	.580
2200	2.26×10^{-3}	4.250×10	.00532	138.0	.001638	.579
2300	3.69×10^{-3}	5.077×10	.00727	227.0	.001626	.577
2400	5.79×10^{-3}	6.020×10	.00962	359.0	.001613	.576
2500	8.77×10^{-3}	7.087×10	.0124	550.0	.001595	.575
2600	1.29	8.291×10	.0156	810.0	.001593	.574
3000	4.66	1.470×10^2	.0317		(c)	.570
4000	3.85×10	4.645×10^2	.0829			.563
5000	1.36×10^2	1.134×10^3	.1201			.559
6000	3.26×10^2	2.351×10^3	.1386			.556
7000	6.03×10^2	4.356×10^3	.1385			.554
8000	9.59×10^2	7.432×10^3	.1290			.553
10 000	1.84×10^3	1.814×10^4	.1014			.550

c Mean = 0.001627 = 614.6 lumens = 48.9 cpw.

III. RADIANT LUMINOUS EFFICIENCY OF A BLACK BODY

The radiant luminous efficiency of an incandescent body is the ratio of the luminous flux to the total power radiated or the ratio of the luminous intensity to the total intensity of radiation, and for a black body the *luminous efficiency* =

$$\int_0^\infty V_\lambda E_\lambda d\lambda \div \int_0^\infty E_\lambda d\lambda = L \div \sigma_0 T^4 \quad (6)$$

where E_λ is given by Planck's equation and L is given in Table 1 as computed from equation (5). It is of great importance in connection with the question of efficiency in light production. Experimental determinations of the radiant luminous efficiency of illuminants have occupied the attention of numerous investigators⁸ and for a black body numerous computations^{9,6} of this ratio have been made. The present computations (Table 1) are from the values given in the second column, which are based upon the value $\sigma = 5.7 \times 10^{-12}$ watt per cm² per deg.⁴ for the coefficient of total radiation. The results are of interest in showing that the maximum luminous efficiency of a black body is attained slightly above 6000° C and has the numerical value about 14 per cent.

IV. MECHANICAL EQUIVALENT OF LIGHT DETERMINED FROM BLACK-BODY RADIATION

One of the important applications of the equation of the visibility curve is in determining the factor for converting radiant energy into visual sensation or "light."

The unit of power is the watt. The present arbitrary practical unit of luminous flux is the lumen. The ratio of these two factors for light of maximum visibility is the stimulus coefficient or numerical factor V_m in equation (1) for evaluating the lumen (or candle) in watts of luminous flux.

The reciprocal of the coefficient V_m is commonly called the mechanical equivalent of light.^d More properly it should be termed the "least mechanical equivalent of light," in view of the fact that it is the ratio of the candle (or lumen) to the watt for monochromatic radiation having the wave length of maximum visibility.

It is possible to determine the mechanical equivalent of light^{10,11} from the radiation constants of a black body provided we know its brightness at a given temperature. Such determinations of brightness have been made by various observers,^{12,13} the most recent being by Hyde¹⁴ and his collaborators. The least mechanical equivalent, m , can be computed from the data just mentioned (Table 1) on the luminous intensity of a black body at a given temperature, at which the brightness b (in candlepower per cm²) has been measured. The mathematical formula is

$$m = \frac{L}{b} \quad (7)$$

The mechanical equivalent, therefore, depends largely upon the accuracy with which these constants are known⁷ and of course upon the accuracy of the visibility curve. The values of the least mechanical equivalent given in Table 1 are easily computed by dividing the luminosity values, L , by the brightness values, b , published by Hyde and his collaborators.¹⁴ It is to be noticed that the value of m , as obtained at various temperatures, is subject to great variations.^e Because of such great variations it is proposed to make all the measurements radiometrically,¹¹ in view

^d This term is considered a misnomer. Unlike the mechanical equivalent of heat, the actual power equivalent of a lumen is not a fixed quantity but varies with the wave length of the radiation concerned and to a slight extent with the intensity. A more precise term for V_m would be "the luminous equivalent of radiation of maximum visibility."

^e These computations are based upon an unpublished revision of the brightness data,¹⁴ kindly furnished by Dr. Hyde.

of the fact that the physical photometer^{15,16} has been found to give excellent results.

The mean value of the least mechanical equivalent, using all of the candlepower observations,* is:

1 lumen = 0.001627 watt of luminous flux.

1 watt of radiation of maximum visibility = 614.6 lumens = 48.9 candles.

The candlepower measurements should be most accurate between 1700° and 2300° Abs., within which range there is no marked color difference between the comparison lamp and the black body, also the temperature scale is not in question. Using the brightness data for this temperature range, the mean value of the least mechanical equivalent is about 0.7 per cent higher, viz:

1 lumen = 0.001638 watt of luminous flux.

1 light watt = 611 lumens = 48.6 candles.

The published note¹⁴ gives a value of 760 lumens per watt or 1 lumen = 0.00132 watt of luminous flux, the computations being based upon the older visibility curves. It is therefore of interest to recall that the value of the mechanical equivalent determined experimentally¹⁷ was 1 lumen = 0.00162 watt. A subsequent investigation by Ives and Kingsbury¹⁸ showed that the visibility curves then available gave results which were inconsistent with other experimental data. They found, experimentally, the visibility curve required in order to give consistent results, and on the basis of this experimental curve the old value of 1 lumen = 0.00162 watt was corrected to 1 lumen = 0.00159 watt, which differs by about 2 per cent from the present determination.

The new visibility curve is about 1 per cent larger than the one (corresponding to the absorbing solution) used by Ives and Kingsbury¹⁸ in obtaining the value of 0.00159. Increasing this value by 1 per cent, which is the difference in the visibility curves, gives a value of

1 lumen = 0.001606 watt of luminous flux.

In subsequent investigations by Ives and Kingsbury^{10,18} this constant has been checked in various ways, and they have concluded that the value 0.00159 is substantially correct.[†]

Probably one of the most trustworthy direct determinations yet made of the least mechanical equivalent of light is that by Ives,

[†] Some of their apparently most reliable measurements indicate a somewhat lower value, but not as low as 0.0012,¹⁹ watt, which value is inconsistent with other experimental data.

Coblentz, and Kingsbury,¹⁷ in which the radiation from the monochromatic green mercury line $\lambda = 0.5461\mu$ was evaluated. Indirectly this measurement represents the mean of 61 observers, which number is sufficient for comparing results with the present work.

The observed value¹⁷ was 1 watt = 613.6 lumens of green mercury radiation. The present visibility curve gives a value of 98.5 per cent for the visibility at $\lambda = 0.5461\mu$. Hence, correcting the observed value by 1.5 per cent we have—

1 light watt = 622.8 lumens = 49.6 candles.

1 lumen = 0.001606 watt of luminous flux of maximum visibility.

This is in exact agreement with the value obtained by using the visibility-curve solution with the physical photometer.^{17,18} This is to be expected in view of the fact that, in the determinations of the mechanical equivalent from measurements on the green mercury line, the radiation constants adopted for use in calibrating the radiometer were the same as used in the present work. Hence, the only outstanding errors would be experimental ones (also the uncertainty of the value of the constant $c_2 = 14\,350$), which should be small, owing to the large number of observers used in both investigations.⁹

It is important to remember that this value of the least mechanical equivalent of light is defined by the present visibility curve obtained under the conditions imposed as regards size and brightness of the illuminated field.

As mentioned elsewhere, an important application of the value of the mechanical equivalent of light of some spectral line is in the calibration of radiometers (in absolute measure) used in measuring light stimuli which are employed in various physiological and biological investigations involving photometric measurements. For this purpose the green mercury radiation ($\lambda = 0.546\mu$) may be used on the basis that 1 cp = 0.02 watt (1 cp = 4π lumens).

A common definition of "white light" is the light emitted by a black body at 6000° Abs. At this temperature the radiant luminous efficiency is 13.9 per cent. The radiant luminous efficiency of a black body as measured by the average eye is a maximum (14 per cent) at a temperature of about 6300° Abs. Hence, "white light" may be defined as the light emitted by a black body at 6000° C.

⁹ In a recent communication, Gerlach (Ann. der Phys., 50, p. 259; 1916) gives a new determination of $\sigma = 5.85 \times 10^{-12}$ watt per cm^2 per deg.⁴ This would indicate a value of 1 lumen = 0.0017 to 0.0018 watt. This value of $\sigma = 5.85 \times 10^{-12}$ does not harmonize with what one would expect from other data.

The mechanical equivalent of white light so defined is seven times as large as that of the light having a maximum luminous efficiency. The efficiency of such white light is only 14 per cent of that of the maximum, and may be expressed as (49.6×0.14) or 6.94 candles per watt.

V. RADIANT LUMINOUS EFFICIENCY OF A TUNGSTEN LAMP AND THE MECHANICAL EQUIVALENT OF LIGHT

In connection with the value of the least mechanical equivalent of light, as derived in the preceding part of this paper, by using the radiation constants and the brightness of a black body at various temperatures, it was of interest to obtain further data upon this important constant. Accordingly, a further determination was made, using a standardized tungsten lamp (obtained from the photometric division) as a source of light.

The lamp was operated at 1.23 watts per candle (in a specified direction), giving 26.5 cp or 26.5×10^{-4} lumens per square centimeter of surface 1 meter distant. By direct comparison with a (seasoned carbon incandescent lamp) standard of radiation²² the intensity of the total radiation was 293.5×10^{-6} watt per square centimeter at a distance of 1 meter from this lamp.

The radiant luminous efficiency of this lamp was determined by means of a thermopile, and the luminosity screen.²³ This screen has a transmission for different wave lengths approximately proportional to their visibilities.

A correction was made for shortening of the optical path by the cells containing the water and the solution. Corrections were made also for departure of the cell from perfect transmission at the maximum visibility and for difference in areas of the transmission curve of the solution and the visibility curve, using the spectral energy data of a tungsten lamp operated at closely the same color.²³ The total radiation from the lamp upon the thermopile, when the luminosity screen was not in place, was reduced by means of a sectored disk. The lamp was at a distance of 1.45 meters from the thermopile. A correction was made for radiation from the moving disk and from the surroundings when the lamp was not lighted. The first determination of the radiant luminous efficiency gave a value of 1.33 per cent. Some hours later, when the instruments were less disturbed by air currents, a more reliable series of measurements (68 galvanometer readings)

²³ Described in this Bulletin, 14, p. 167; 1917.

gave a value of 1.44 per cent. The weighted mean value (weight of second series = 4 times that of first series) is 1.42 per cent for the radiant luminous efficiency of a "40-watt" vacuum tungsten lamp operated at 1.23 watts per candle.

The least mechanical equivalent of light is obtained from a knowledge of the total radiation, the radiant luminous efficiency, and the candlepower of the tungsten lamp. The luminous flux per cm^2 on a surface at 1 meter distance was $(293.5 \times 10^{-6} \times 0.0142) = 4.16 \times 10^{-6}$ light-watt per cm^2 . Equating this to the value in lumens of the luminous flux (26.5×10^{-4} lumens = 4.16×10^{-6} watt) gives a value of 1 lumen = 0.00157 watt, which is in close agreement (about 3 per cent) with the value found in Table I.

A check upon this value is obtained from a consideration of the power applied to the tungsten lamp, etc. The lamp was rated at 1.23 watts per horizontal candle. This is equivalent to 1.567 watts per mean spherical candle or 8.02 lumens per watt. This value must be increased by 7 per cent for conduction losses at the leading-in wires²⁴, and by 2 to 3 per cent for convection losses²⁵, or a total correction of about 10 per cent. This gives an efficiency of 8.82 lumens per watt. The total luminous efficiency L_t is obtained by multiplying this value by the mechanical equivalent of light, or $8.82 \times 0.00162 = 0.0143$, which is nearly the weighted mean value obtained by direct experiments.

Our conclusions are therefore in agreement with Ives and Kingsbury¹⁸ that the value of the least mechanical equivalent of light is close to 1 lumen = 0.0016 watt of radiant energy of maximum visibility.

VI. THE CROVA WAVE LENGTH

The Crova²¹ wave length is that wave length in the visible spectrum of two sources of light at which the ratio of the luminous intensities equals the ratio of their total luminous intensities. For a black body it is that wave length at which the luminous intensity varies by the same fractional part that the total luminous intensity varies for the same change in temperature. The practical application, as introduced by Crova, was to photometer the two sources by comparing their luminous intensities at this wave length and thus avoid the question of color difference which arises in heterochromatic photometry. The mathematical derivation of the formula for determining the Crova wave length (λ_c) has been given by various writers. The formula used in the present calcu-

lation of the Crova wave lengths is a modification of Kingsbury's^a equation, using the values given in equation (2). It is

$$\lambda_0 = \frac{L}{\frac{\sigma}{\pi}} \left[\frac{2.2877}{\left(\frac{129.05}{T} + 1 \right)^{206}} + \frac{0.1505}{\left(\frac{78.85}{T} + 1 \right)^{406}} + \frac{0.0805}{\left(\frac{23.52}{T} + 1 \right)^{1006}} + \frac{0.0775}{\left(\frac{13.67}{T} + 1 \right)^{2006}} \right]^{-1} \quad (8)$$

The results are given in the last column of Table I.

VII. SUMMARY

This paper gives some applications of the curve of visibility of radiation for the average eye (125 observers) to radiation problems. A mathematical equation is given of the average visibility curve. Using this visibility equation and Planck's equation of the black body, calculations are made of the luminous flux emitted by a black body at various temperatures, also the luminous efficiency, the Crova wave length, and the mechanical equivalent of light.

The visibility curve of the average eye gives a value for the least mechanical equivalent of 1 lumen = 0.001627 watt of luminous flux of maximum luminous efficiency. The two determinations of the least mechanical equivalent of light made by Ives, Coblentz, and Kingsbury are corrected. The values obtained by their two methods of measurement, using 61 observers, are in exact agreement, giving 1 lumen = 0.001606 watt.

A further determination of the least mechanical equivalent of light was made by using a standardized vacuum tungsten lamp as a source of radiation. The measurements on this lamp gave a value of 1 lumen = 0.00157 watt of radiant energy of maximum visibility. These computations are based upon the most probable values of the radiation constants: $C_2 = 14,350$ micron deg. and $\sigma = 5.7 \times 10^{-12}$ watt per cm^2 per deg^4 . On this basis the most reliable data now available indicate that the value of the luminous equivalent of radiation of maximum luminous efficiency is of the order of:

$$1 \text{ watt} = 617 \text{ lumens} = 49.1 \text{ candles.}$$

$$1 \text{ lumen} = 0.00162 \text{ watt of luminous flux.}$$

Among other data this paper gives the determination of the radiant luminous efficiency of a vacuum tungsten lamp, the value being 1.42 per cent when operated at 1.23 watt per candle.

WASHINGTON, January 24, 1917.

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FIG. 1.—*Photograph of the Fahy permeameter*

AN EXPERIMENTAL STUDY OF THE FAHY PERMEAMETER

By Charles W. Burrows and Raymond L. Sanford

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I. INTRODUCTION

1. PURPOSE OF THIS INVESTIGATION

This permeameter was devised by F. P. Fahy, of the Pennsylvania Railroad Co., for the purpose of comparing the magnetic properties of two similar specimens of steel. In the fall of 1911 a cooperative investigation on the magnetic-mechanical properties

of steel was begun by the above-named company and the Bureau of Standards at the Bureau laboratories in which the Fahy apparatus was used for comparative magnetic measurements. Early results obtained with the apparatus, however, indicated the possibility of its development as a permeameter and this was immediately undertaken. In order to ascertain the degree of accuracy that has been attained, as well as its fitness for general laboratory use, a critical experimental study has been made of the instrument in its present improved form.

2. THE PROBLEM OF MAGNETIC MEASUREMENTS

The problem of magnetic measurements consists of the determination of simultaneous values of magnetic induction and magnetizing force. There is comparatively little difficulty in measuring the magnetic induction. In practically all methods of any reasonable degree of accuracy the magnetic induction is measured in terms of the impulsive electromotive force induced in a test coil surrounding the specimen when the direction of the magnetic flux is reversed. This method is simple and accurate.

The difficulties of measurement center about the determination of the magnetizing force. This magnetizing force has, in general, a number of components. The principal component is due to the magnetizing current flowing through a coil of wire. This component of the magnetizing force is susceptible of calculation and for certain forms of electric and magnetic circuits the calculation is quite simple. The other components are due to magnetic poles developed in the different parts of the magnetic circuit due to joints, yokes, nonhomogeneities, etc. These components are not susceptible of calculation. They can, however, generally be measured, and in some special cases may be neutralized by means of auxiliary magnetizing coils.

3. HISTORICAL

The method of compensating for the effect of the yokes and joints by means of auxiliary magnetizing coils has received considerable development. Searle¹ built the magnetic material into the form of a hollow square and applied the main magnetizing force through four solenoids surrounding the four sides of the square. Short coils at each corner were energized with sufficient current so that the added magnetomotive force was just equal to the magnetic potential drop due to the contact surfaces and those

¹ Searle, *Studies in Magnetic Testing*, *Inst. Elec. Eng. Jour.*, 34, pp. 55-113; 1904.

portions of the test specimens not surrounded by the main magnetizing coil. The compensating coils were in series with the main magnetizing solenoid and were adjusted once for all until a small magnetic needle placed near one of the joints showed no leakage effect. Esterline², in his direct-reading permeameter, applied the same principle.

In the permeameter of Picou,³ compensation is effected by means of auxiliary magnetizing coils wound on each of two yokes which span the specimen. During adjustment the coils are connected so that their magnetomotive forces are in series and the flux through the yokes crosses the specimen perpendicularly at each end. With this connection the flux in the yokes is determined. Then the current in one of the coils is reversed and the main magnetizing coil which surrounds the specimen is energized, and the current in this latter coil adjusted so that the flux in the yokes is the same as before. When this adjustment has been made, the mmf of the auxiliary coils is just sufficient to overcome the reluctance of the joints and yokes, and a direct measurement of magnetizing force can be made in terms of the current in the main magnetizing coil. This apparatus gives fairly good results but is extremely tedious in operation.

The Burrows⁴ permeameter has a magnetic circuit consisting of two similar rods and two short yokes. Solenoids surround the rods. Small coils wound around the rods near the joints carry a current just sufficient to overcome the reluctance of the joints and yokes. This is the precision method of the Bureau of Standards and has been officially adopted as the standard method by the American Society for Testing Materials.

Ewing⁵ attempted to eliminate the effect of the reluctance of the yokes and joints by experimental means. His magnetic circuit consists of two bars and two yokes as in the Burrows permeameter. Magnetic measurements are made first with short magnetizing coils and the yokes close up to them, then with double the length of coil. The additional magnetomotive force required to magnetize this second system is theoretically just sufficient to overcome the reluctance of the extra length of specimen. This method assumes several conditions which are not met. The leakage at the yokes is not zero nor is it the same in the two

² Treat and Esterline, *Magnetic Testing Apparatus*, *El. World*, **30**, pp. 696-697; 1897.

³ Picou, *Permeameter Universel*, *Bull. soc. internat. d'electr.*, **2**, pp. 828-834; 1902.

⁴ Burrows, *this Bulletin*, **6**, pp. 31-38 (Scientific Paper No. 117), 1909; see also *Magnetic Testing*, Circular No. 17 of the Bureau of Standards.

⁵ Ewing, *Magnetic Induction in Iron and Other Metals*, p. 362; 1900.

systems. Further, it is difficult to secure two test specimens which are uniform throughout their length and equivalent magnetically to each other. The authors of the present paper have been trying unsuccessfully for a number of years to get two rather long rods which were magnetically uniform along their length for the purpose of trying out this method with several widely different lengths. Our work, however, shows that with uniform material the Ewing method would give fairly good results.

The joint reluctance may be determined experimentally once for all by means of measurements in a given apparatus of a standard bar whose true magnetic constants have been determined in some other way. This is the procedure of the Physikalisch-Technische-Reichsanstalt, where a massive iron yoke is used to bridge the ends of the test specimen. The apparatus is calibrated in terms of the values obtained on an ellipsoid of revolution first measured magnetometrically and later machined down to a cylinder.

The last method mentioned consists in the measurement of the difference of magnetic potential between the ends of a test specimen. Provided there is no source of mmf between the ends of the specimen, this magnetic potential difference is the total mmf required to maintain the flux in the specimen. If there is such a source of mmf, then the potential difference measured between its ends is the algebraic sum of the impressed mmf and the drop in the specimen.

A number of investigators have used this method of potential difference in magnetic measurements.

Chattock⁶ described a method which he calls a "magnetic potentiometer," showing theoretically that the difference of magnetic potential between two points can be measured by means of a uniformly wound test coil of constant cross section.

Later Rogowski and Steinhaus⁷ described the same method of measurement and gave the results of a number of experiments made with the apparatus. Goldschmidt⁸ has described a method using a yoke having a gap with a magnetic needle in it similar to the arrangement of the well-known Ewing permeability bridge. On this yoke is a winding with which a mmf is produced opposing that to be measured. When the needle indicates a balance the mmf in ampere turns is equal to the product of current and turns in this winding. Iliovici⁹ has described a permeameter in which

⁶ Chattock, *Phil. Mag.*, 24, p. 94; 1887.

⁷ Rogowski and Steinhaus, *Archiv für Elektrotechnik*, 1, p. 141; 1912.

⁸ Goldschmidt, *Electrician*, 54, p. 207; 1904.

⁹ Iliovici, *Bull. Soc. Int. des Elec. 3d ser.*, 8, p. 581; 1913.

he uses a yoke with a test coil wound on it to detect a difference of magnetic potential. In this apparatus compensation is made for yokes and joints which is indicated by zero potential between the ends of the specimen. Goltze¹⁰ published data giving the results of permeability tests on a number of specimens magnetized between the poles of an electromagnet. He used an arrangement of the form described by Chattock and Rogowski and Steinhaus for the determination of magnetizing force.

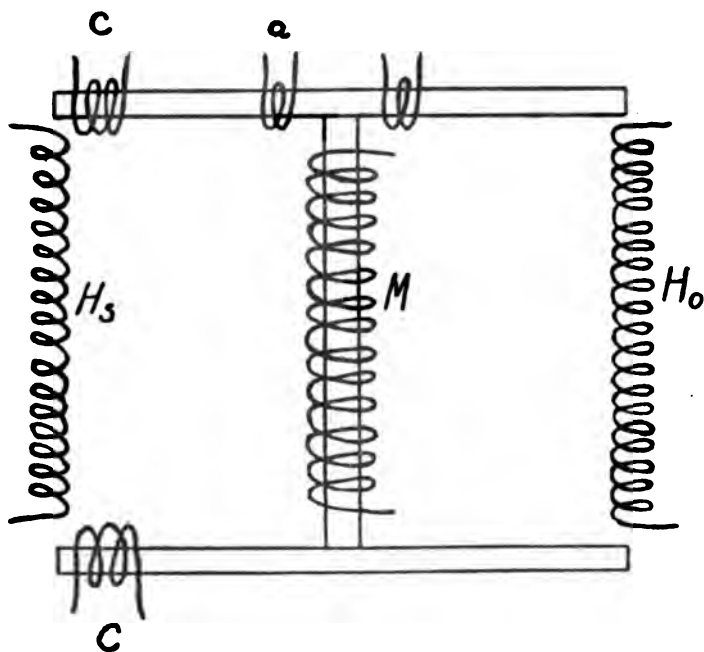


FIG. 2.—Magnetic and electrical circuits in their simplest form

The Fahy permeameter, which is the subject of the present paper, utilizes this latter principle. A short description of the apparatus has already been published.¹¹

II. THEORY

The basis of operation of this instrument is apparent from a consideration of Fig. 2, which shows the circuits of the instrument in their simplest form. The apparatus is H-shaped, with two test coils, H_o and H_s , spanning the free ends of the H and a magnetizing coil, M , wound on the crosspiece of the H .

By reason of the symmetry of the two parallel magnetic circuits thus formed it is evident that the magnetizing coil produces the

¹⁰ Goltze, *Archiv für Elektrotechnik*, 2, p. 303; 1914.

¹¹ Fahy, *El. World*, 60, p. 315; 1917.

same amount of flux in the two equal coils H_o and H_s and that either one may be used to measure the mmf drop between the yokes, and from this value the mean magnetizing force may be calculated.

If now we span the yokes by a steel bar placed within the coil H_s , the above conditions do not hold exactly. The H_o coil still measures the mmf drop between the pole faces which it spans. This drop, however, is not equal to the drop between the other pole faces. Due to the increased flux density in the yokes adjacent to the specimen, a greater proportion of the mmf is used to force the greater flux through one set of yokes than through the other set. Consequently, since the available mmfs in the two parallel magnet circuits are equal, the potential difference caught by the coil H_s is less than that caught by the coil H_o .

The magnetic potential differences between the ends of the coils can be reduced to equality by means of compensating coils, C , wound on the yokes near the pole faces. The criterion of proper adjustment for this compensating mmf is equality of leakage through the air on the two sides of the apparatus. The leakage on each side is measured by means of a short coil, a , wound on the yoke near the core and connected differentially in series with the main test coil H_s . The two test coils, of course, have the same number of turns.

After the adjustment is made the magnetizing force and magnetic induction may be measured in terms of the emfs induced in the coils H_o and H_s on reversal of the main magnetizing current.

A full description of the apparatus and detailed instructions for the manipulation are given in the appendix.

III. PRECISION AND ACCURACY

An experimental study of the permeameter has been made with a view to determining its precision, and also of the individual factors which might affect the accuracy of measurements such as variations in contact reluctance, length of specimen, position of specimen on pole face, etc.

1. CONSISTENCY ON REPETITION

Any measuring instrument to be of value must be capable of giving concordant results from repeated tests of the same specimen. It is essential, therefore, that all the factors which might

affect the accuracy of the measurements shall have so small an influence on the results as to be of negligible importance.

The ability of this apparatus to repeat readings on the same specimen under the same conditions is all that could be desired. This is shown very clearly by Fig. 3, where the solid curve represents a first determination and the circles represent points taken on the same bar after it had been removed from the apparatus

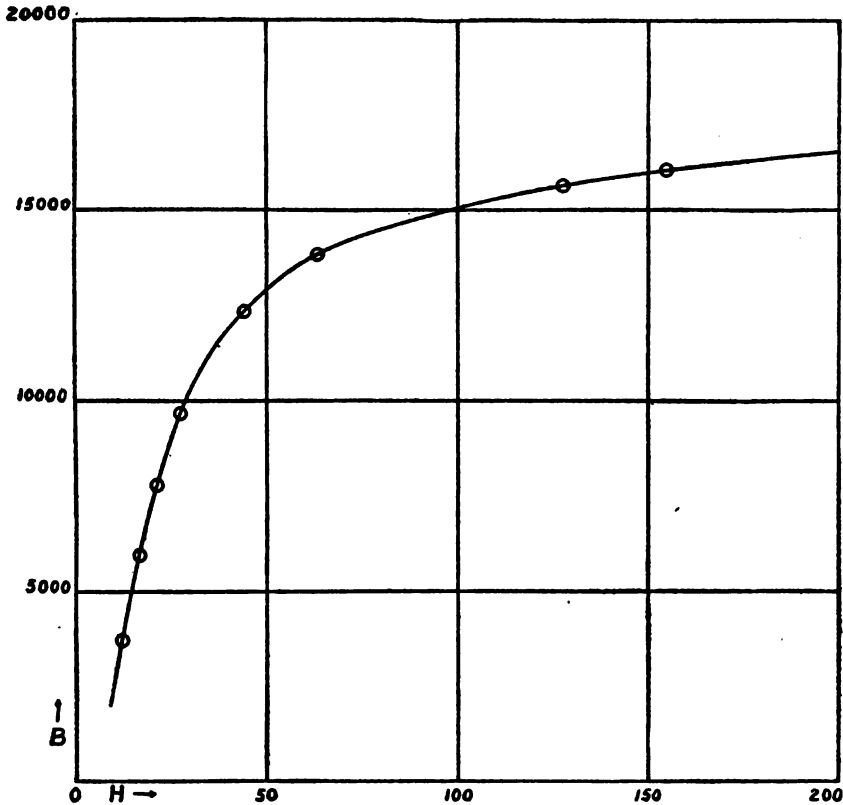


FIG. 3.—Showing consistency on repetition

The solid curve represents a first determination and the circles represent points taken after removal of the specimen and reinsertion.

and reinserted. The consistency is so good that this apparatus may be used to detect small changes in magnetic characteristics due to aging, mechanical strains, or differences due to small variations in heat treatment. The limit of consistency on repetition is probably determined by the reliability of the ballistic galvanometer and its ability to give the same deflection under the influence of the same impulse.

2. LENGTH OF SPECIMEN

This apparatus is designed to take specimens of a fixed length. The particular piece of apparatus under investigation was designed for a specimen 25.4 cm long. However, it frequently happens that it is necessary to test other lengths. Sometimes one wishes to know the magnetic characteristics of a portion of a long bar without cutting the bar. With this in view the end of a long bar was measured once with the whole bar intact and the free end projecting beyond the back pole face, and again with the excess portion of the bar removed. In each case the same material was between the pole faces, and one end of the test specimen was flush

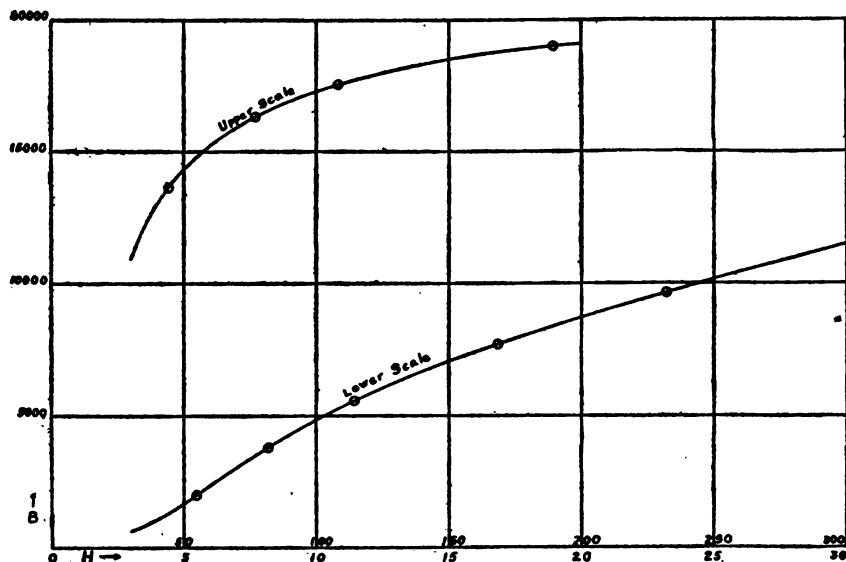


FIG. 4.—Showing the effect of projection of a long specimen

The solid curve shows normal induction for a specimen of proper length and the circles represent points taken with a projection of 150 cm.

with the test-coil pole faces. In Fig. 4 the solid curve shows the normal induction of the specimen cut to length, and the circles show points determined with a projection of 150 cm or six times the length of the specimen. The points fall very nearly upon the curve. It is difficult to tell whether the small difference is a real variation due to the projection or to a strain set up in the specimen by the weight of the projecting end. To reduce this latter the free end was supported.

As a check on the preceding, and also to make sure that this particular length is not a critical length at which several errors balance each other, a second bar was measured at one particular magnetizing force, after the bar had been reduced in length by

successive steps. In each case the same portion of material was tested, and the end of the test specimen was kept flush with the test-coil pole face. Table 1 shows the values of induction obtained with different lengths of the test sample projecting beyond the pole face. These results are for a magnetizing force of 58 gauss.

TABLE 1

Values of Induction Obtained for a Magnetizing Force of 58 Gauss with Different Lengths of the Test Sample Projecting Beyond the Pole Face

Projection in centimeters	Induction
125	15 140
25	15 120
10	15 130
5	15 210
2	15 210
0	15 190
-1	15 210
-2	15 230
-3	14 580

Here, again, it is not safe to say that there is any influence of the projecting end that may not be accounted for by strain in the material rather than by the mere presence of the ends. As the bar is reduced further in length so that it is less than the normal length, no appreciable change takes place until the reduction amounts to 3 cm. Since the full pole face width is 4 cm, this means a reduction in contact surface to one-fourth the normal amount. This reduces the induction by 600 in 15 000, or 4 per cent. This difference is removed when measurements are taken with this shortened test specimen placed not with one end flush with the test coil pole face, but symmetrically between the two pole faces. In this latter position the induction was 15 170, which is in good agreement with the normal length value of 15 190 gauss.

From the above it is quite evident that no appreciable error is introduced by the projecting end of the test specimen, provided this projection extends beyond the back yoke of the apparatus. Extension beyond the test coil pole face may be expected to introduce considerable error unless care is taken to compensate for the effects of this projection. This compensation is taken care of automatically by placing a test coil over the projecting end of the specimen close to the yoke and a similar coil in the corresponding place over the other end of the same yoke. Each of these test coils is connected in series with the other test coils on the same side of the core, and adjustments are made in the usual way.

3. PROJECTION OF SPECIMEN BEYOND THE TEST COIL POLE FACE

One factor that might affect the accuracy of measurements is improper insertion of the test piece so that one end projects beyond the test coil pole face. A test was made to determine the degree of accuracy with which the end of the specimen must be lined up with the edge of the test coil pole face. A test piece of the proper length was clamped with different amounts of projection beyond the pole face and the inductions read with three different values of magnetizing force. The results are given in Table 2.

TABLE 2

Observed Inductions with Various Projections Beyond Test Coil Pole Face

Projection in millimeters	Magnetizing force		
	10.5	23.5	54.0
6	5080	9740	14 770
4	5080	9770	14 780
2	5090	9790	14 800
0	5090	9800	14 810
-2	5110	9800	14 820
-4	5120	9800	14 820
-6	5150	9810	14 840

The effect is so slight that extreme care in lining up the specimen is not necessary. It is very easy to line up a bar to within 1 mm of the edge, so that no trouble is to be expected from this source.

4. RELUCTANCE OF JOINTS

It is to be expected that in apparatus of this type the reluctance of the surface of contact will affect the accuracy. To get some idea of the magnitude of the influence of the joint reluctance, measurements were made with paper inserted between the specimen and the pole faces. In Fig. 5, curves *A*, *B*, and *D* show the variation in induction with the width of gap for several magnetizing forces. The greatest effect of the gap seems to be in the region of maximum permeability as shown by curve *B*. These curves are of interest, since they show the effect of different width of gap. However, in actual practice there is no such air gap. Imperfect contact is due either to the warping of the specimens so that they do not seat well upon the pole faces or to the presence of scale. To approximate this latter condition thin sheet transformer iron was interposed between the specimen and pole faces. Curve *C*

shows the effect of one and of two such sheets upon the measured values of the induction in a region where a large effect was found for the paper separators.

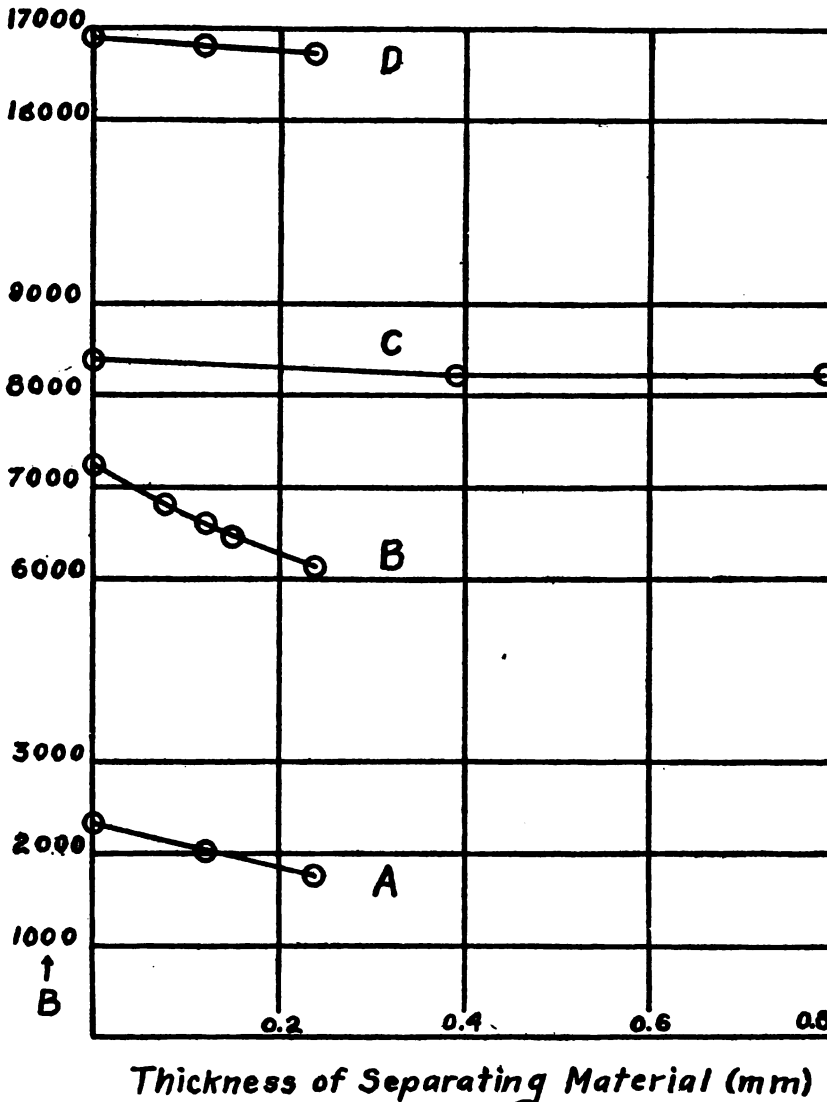


FIG. 5.—Showing the effect of added reluctance at the pole contacts

In curves A, B, and D the extra reluctance is due to paper inserted between the specimen and the pole face. In curve C the separating material is oxidized sheet steel

This transformer material as measured with a parallel-face micrometer had a thickness of 0.385 mm. Calculations based upon measured values of the mass, density, length, and breadth gave a mean thickness of 0.364 mm. This difference of 0.021,

or 5.8 per cent, is due to the irregularities of the surface. If we assume that the high permeability transformer steel adds only a negligible amount to the contact reluctance, the total added reluctance is due to the irregularity of surface and to the less magnetic coating of scale.

The paper separators undoubtedly cause a greater joint reluctance than would occur in practice. The close agreement in the comparison with the Burrows permeameter, in which this joint reluctance is compensated for, is evidence that the joint reluctance when the specimen alone is in place does not give rise to any serious error. However, the above results show that in precision work it is desirable to have fairly good magnetic contact at the pole faces.

5. EFFECT OF IRON IN THE NEIGHBORHOOD

Pieces of iron in contact with the test coil pole faces during measurement will cause an appreciable error. However, such a situation could arise only intentionally and need not be considered further. Pieces of iron placed otherwise than in very close proximity to the magnetic circuit do not produce any serious error. Ten kilograms of transformer steel strips 50 by 3 cm were placed in various positions near the apparatus and the effect on the measurements noted. With the transformer strips placed parallel to the yokes and as near to the side of the wooden base as possible, no result was noticed. With the strips placed parallel to the specimen and close up against the case on the same side as the specimen, a small decrease in the induction as measured was noted. The numerical values observed are given in Table 3.

TABLE 3

Effect of a Mass of Transformer Iron Placed Near the Permeameter on the Observations

H	B	Position of iron strips
27.6	11 040	Removed.
	11 040	Parallel to yoke.
	10 955	Parallel to specimen.
73.5	16 610	Removed.
	16 540	Parallel to yoke.
157	19 360	Removed.
	19 340	Parallel to specimen.

Even when a bar of iron is placed directly against the back yoke the effect is not very great, as is shown by the following data. The disturbing bar was 150 cm long and of the same cross section (1 by 1 cm) as the test specimen. The magnetic measurements were made at a magnetizing force of 56.2 gauss. The results are shown in Table 4.

TABLE 4

Inductions of a Test Specimen With a Second Rod 150 cm Long Placed in Various Positions

Induction	Position of second rod
15 285	Removed.
15 260	Butting against test specimen.
15 195	Lapping pole face by 1 cm.
15 210	Lapping pole face by 3 cm.

The general effect of this disturbing bar is very slight and is always in the direction of a diminished induction.

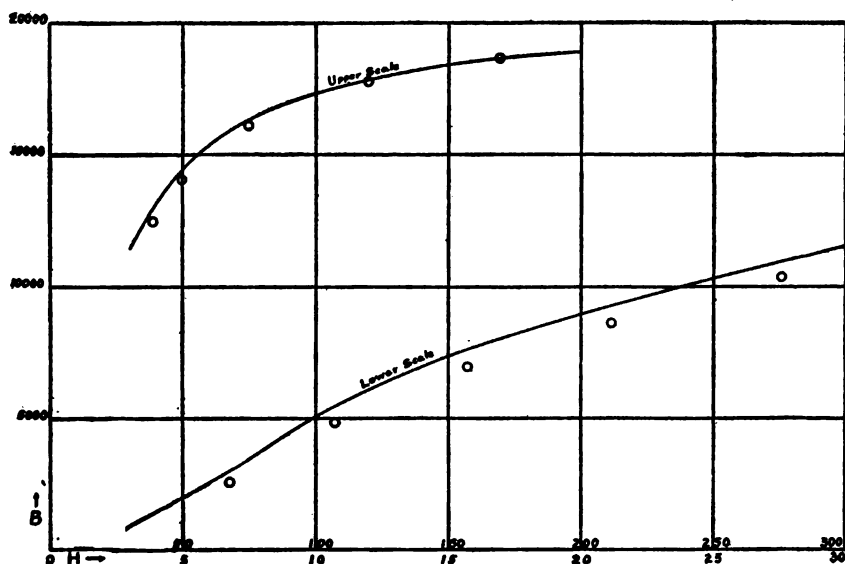


FIG. 6.—Showing the effect of strain in the specimen

The solid curve shows normal induction when specimen is free from strain and the circles represent points taken with the specimen slightly distorted

In general, it is safe to say that all masses of iron at distances from the permeameter greater than a few inches are without any appreciable effect.

6. POSITION OF SPECIMEN ON POLE FACE AND STRAIN EFFECTS

A very interesting point was brought out in an attempt to discover whether the exact position of the specimen on the pole face made any difference. With this in view, measurements were made with the specimen seated symmetrically on the pole faces and with the specimen making contacts on the lower portions of the pole faces. The results first obtained are shown in Fig. 6.

For these measurements the specimen was held in place by the clamps only. When the specimen was held against the middle portions of the pole pieces the clamps acted normally. However, with the specimen making contact on the lower portions of the pole faces the pressure exerted by the clamps was concentrated on one edge of the specimen. This lack of symmetrical pressure coupled with a slight warp causes a strain in the specimen. This strain is accompanied by a change in magnetic properties as indicated by Fig. 6. In order to make sure that this difference was due to strains in the test piece, the clamping device was so modified that it exerted a normal pressure for each position of the test specimen. These latter conditions gave substantially identical results for all positions of the test specimen on the pole faces.

Table 5 gives further data on the effects of strain in a specimen. The specimen was slightly warped and the strain was produced by tightly clamping in the apparatus.

TABLE 5

Differences in the Magnetic Data Due to Strain in the Specimen

Induction	Magnetizing force			
	Strain	No strain	Difference	Per cent difference
2000	1.62	1.58	0.04	2.5
4000	2.39	2.24	.15	6.7
6000	3.21	2.97	.24	8.1
8000	4.21	3.88	.33	8.5
10 000	5.63	5.13	.50	9.8
12 000	7.74	7.09	.65	9.2
14 000	12.1	11.8	.3	2.5
16 000	29.6	29.6	0	0
18 000	108.	112.	-4.	-3.6

The effect of strain may be either an increase or a decrease in permeability. Whether the change in permeability is an increase or decrease depends in all probability upon whether the tension or compression along the length of the specimen predominates. The observed differences can not be due entirely to an improvement in magnetic contact because the strained material shows a lower permeability in spite of the improved contact conditions.

7. ACCURACY

In order to determine the absolute accuracy of the measurements taken by this permeameter a number of bars were measured

in both the Burrows and the Fahy permeameters. The test rods for this investigation were taken from the magnetic standards of the Bureau. These standards have been carefully prepared and aged. They have been examined for magnetic uniformity and are the best magnetic standards at present obtainable.

The magnetic characteristics are shown in Figs. 7 and 8. It is to be noticed how completely these bars cover the full range of magnetic possibilities.

Tables 6 to 12 give the numerical values of the normal induction both by the Burrows permeameter and the permeameter under examination. These measurements for the seven materials

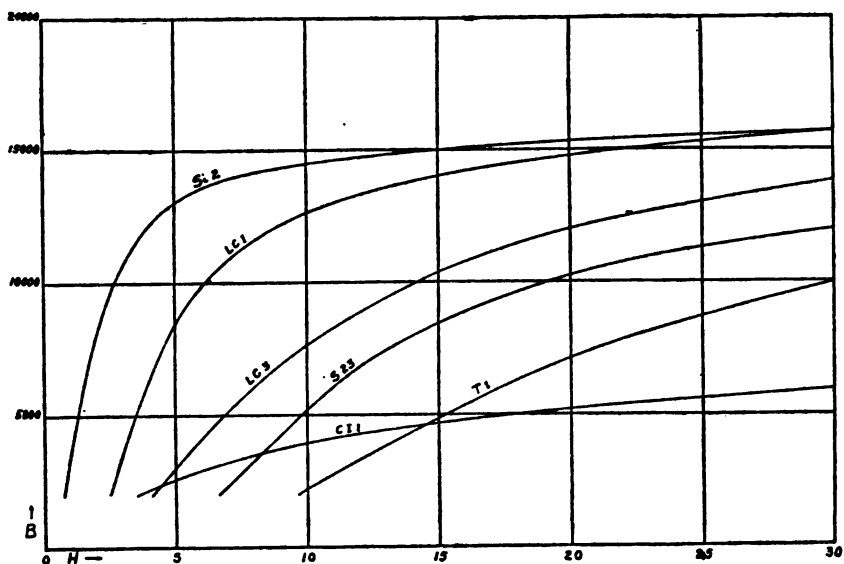


FIG. 7.—Normal induction curves for the standard bars for magnetizing forces up to 30 gauss

are in every case within 5 per cent of the true value. Such accuracy is all that is required by most commercial needs. In fact it is greater than one is justified in seeking in the case of many materials. The truth of this statement is evident when one considers that many materials differ from region to region by more than this amount. For example, the permeability of transformer steel differs from part to part even in a single sheet by as much as 10 per cent or 20 per cent. Rods which have not been prepared with especial reference to securing uniformity may show differences in permeability along their length of as much as 10 per cent or more.

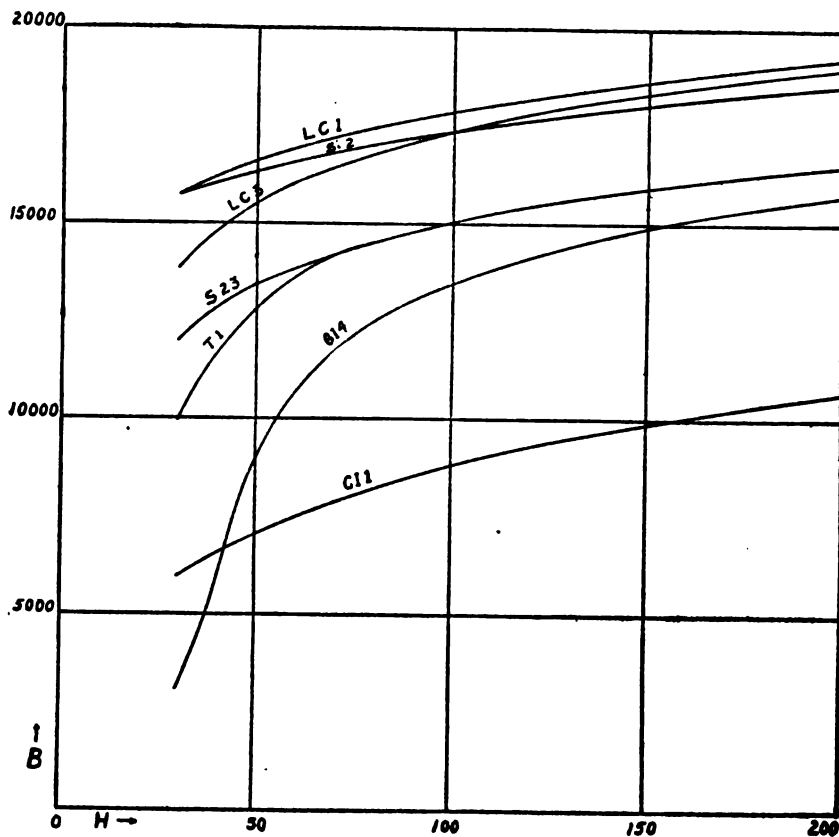


FIG. 8.—Normal induction curves for the standard bars for magnetizing forces from 30 to 200 gaussers

TABLE 6

Bar LC1, Machinery Steel, 1.27 cm Diameter

Induction	Magnetizing force			
	Standard	Faby	Difference	Per cent difference
2000	2.49	2.37	0.12	4.8
4000	3.15	3.12	.03	1.0
6000	3.82	3.91	-.09	-2.4
8000	4.72	4.91	-.19	-4.0
10 000	6.10	6.40	-.30	-4.9
12 000	8.61	8.96	-.35	-4.1
14 000	14.8	15.0	-.02	-1.4
16 000	35.0	34.0	1.0	2.9
18 000	106.	102.	4.	3.8

TABLE 7

Bar LC3, Machinery Steel, 1.27 cm Diameter

Induction	Magnetizing force			
	Standard	Fahy	Difference	Per cent difference
2000	4.08	4.07	0.01	0.2
4000	5.88	5.78	.10	1.7
6000	7.88	7.81	.07	.9
8000	10.5	10.3	.2	1.9
10 000	14.1	14.0	.1	.7
12 000	20.0	19.7	.3	1.5
14 000	31.3	30.9	.4	1.3
16 000	58.3	57.4	.9	1.5
18 000	127.	123.	4.	3.2

TABLE 8

Bar T1, Tool Steel, 1.27 cm Diameter

Induction	Magnetizing force			
	Standard	Fahy	Difference	Per cent difference
2000	9.60	9.27	0.33	3.4
4000	13.3	13.0	.3	2.3
6000	17.3	16.9	.4	2.3
8000	22.5	21.6	.9	4.0
10 000	30.1	29.0	1.1	3.7
12 000	42.4	41.2	1.2	2.8
14 000	68.5	66.7	1.8	2.6
16 000	156.	150.	6.	3.8

TABLE 9

Bar Si2, Silicon Steel, 1.27 cm Diameter

Induction	Magnetizing force			
	Standard	Fahy	Difference	Per cent difference
2000	0.76	0.74	0.02	2.6
4000	1.11	1.13	-.02	-1.8
6000	1.49	1.47	.02	1.3
8000	2.00	1.99	.01	.5
10 000	2.67	2.75	-.08	-3.0
12 000	3.81	3.99	-.18	-4.7
14 000	7.20	7.21	-.01	-.1
16 000	39.0	37.8	1.2	3.1
18 000	146.	140.	6.	4.1

TABLE 10

Bar S23, 1 Per Cent Carbon Steel, 2.97 by 0.886 cm

Induction	Magnetizing force			
	Standard	Fahy	Difference	Per cent difference
2000	6.60	6.70	-0.10	-1.5
4000	8.74	9.04	-.30	-3.4
6000	10.9	11.3	-.4	-3.7
8000	14.0	14.5	-.5	-3.6
10 000	19.1	19.9	-.8	-4.2
12 000	30.0	31.0	-1.0	-3.3
14 000	63.5	63.1	.4	.6
16 000	156.	162.	-6.	-3.8

TABLE 11

Bar 614, Tungsten Steel, 2.97 by 0.966 cm

Induction	Magnetizing force			
	Standard	Fahy	Difference	Per cent difference
2000	22.9	22.8	0.1	0.4
4000	34.0	34.8	-.8	-2.4
6000	40.2	41.4	-1.2	-3.0
8000	46.0	47.9	-1.9	-4.1
10 000	54.8	57.5	-2.7	-4.9
12 000	72.2	75.2	-3.0	-4.2
14 000	114.	117.	-3.	-2.6
15 000	153.	160.	-7.	-4.6

TABLE 12

Bar C11, Cast Iron, 3.00 by 0.950 cm

Induction	Magnetizing force			
	Standard	Fahy	Difference	Per cent difference
2000	3.50	3.61	-0.11	-3.1
4000	10.2	10.7	-0.5	-4.9
6000	30.0	31.1	-1.1	-3.7
8000	73.0	76.0	-3.0	-4.1
10 000	153.	156.	-3.	-2.0

TABLE 13
Comparison of Hysteresis Values with Standard Data

Bar	Method	H_{max}	B_{max}	B_r	H_c
LC1.....	Standard.....	150	18 620	12 200	2.9
	Fahy.....	150	18 760	12 120	2.8
LC3.....	Standard.....	150	18 320	7150	4.8
	Fahy.....	150	18 460	7580	5.3
T1.....	Standard.....	150	15 950	9650	12.6
	Fahy.....	150	16 000	9610	12.7
S23.....	Standard.....	150	15 920		9.4
	Fahy.....	150	15 840	10 630	9.8
614.....	Standard.....	150	14 930	10 220	39.3
	Fahy.....	150	14 780	9950	39.3
CH1.....	Standard.....	154.5	10 000	2800	2.8
	Fahy.....	156.0	10 000	3000	2.8

Table 13 shows a similar set of data on residual induction (B_r) and coercive force (H_c). The agreement here, except for the case of LC3, is within the requirements of commercial accuracy. The lack of agreement in the case of LC3 is as yet unexplained, but in view of the excellent agreement for materials magnetically both harder and softer it seems probable that the difference is not due to the instrument but to some peculiarity of the specimen, such as nonhomogeneity.

In order to show how the accuracy obtained with this permeameter compares with the behavior of other instruments, the corrections to be applied to the various permeameters are plotted in Figs. 9 and 10. These figures are taken with slight modification from a paper by one of the present authors. The only change is the addition of the curves showing the corrections to the Fahy permeameter.

While the absolute method is the one most generally employed, it is of interest to note the accuracy that is obtainable when readings are taken by comparison with a standard. In Fig. 11 the full-line curves represent the true normal induction, while the circles indicate readings on the Fahy permeameter by comparison with a standard. It will be seen that the agreement is exceedingly close, nowhere showing a difference as great as 5 per cent in the magnetizing force required for a given induction.

8. UNCOMPENSATED DATA

Fig. 12 gives two induction curves as obtained by this apparatus. In one the compensation is made while in the second there is no compensation. A comparison of these data with the curves of

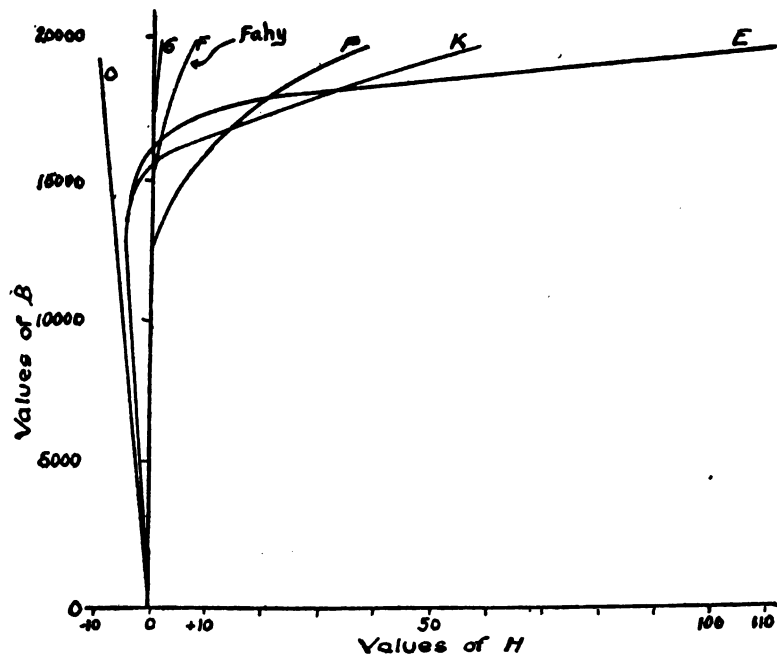


FIG. 9.—Shearing curves for a given quality of Norway iron as determined by various apparatus

E, Esterline; P, Picon; K, Koepsel; G and F from foreign research laboratories; O, a theoretical ovoid of 100 diameters

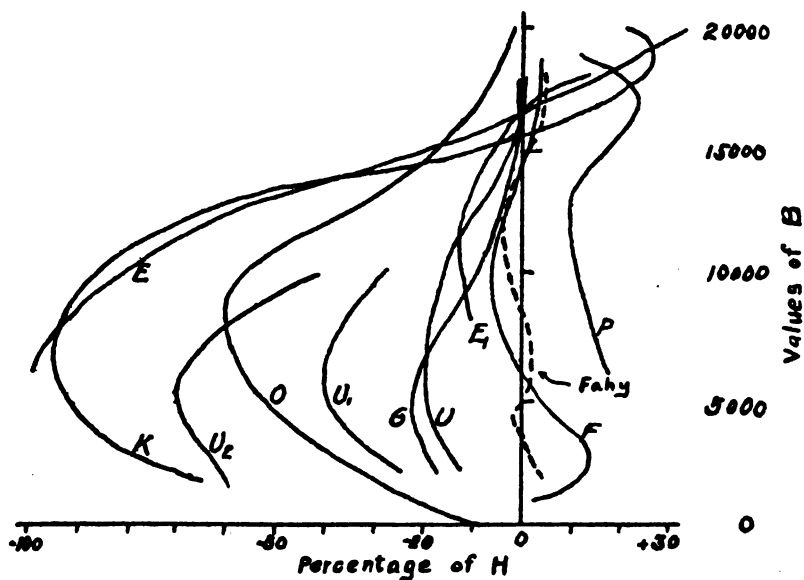


FIG. 10.—Showing the data of Fig. 18 on a percentage basis

E and E₁ are the extreme curves from four Esterline permeameters; U is from an uncompensated double yoke; U₁ is from a single yoke ($l=27.8$ cm); U₂ is from a single yoke ($l=16.8$ cm); O, G, P, F are as in Fig. 18

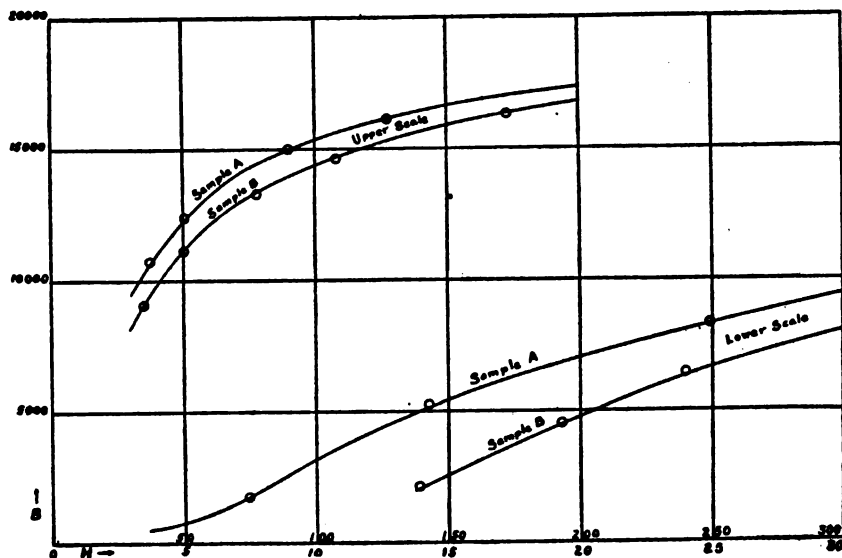


FIG. 11.—Showing values obtained by the method of comparison

The solid curve represents the normal induction by the standard method and the circles represent points taken by comparison with a standard bar

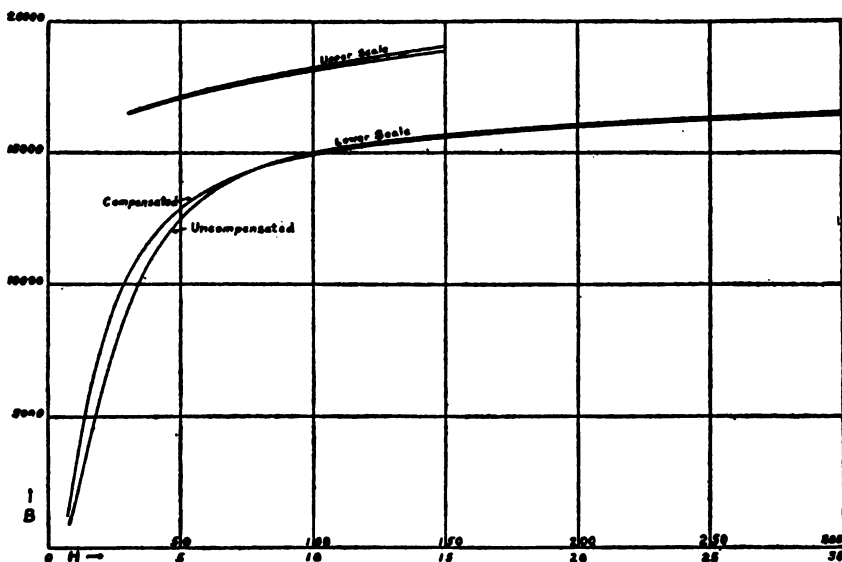


FIG. 12.—Showing normal induction curves for the same specimen with and without compensation

Figs. 9 and 10 will show that the error introduced by failure to compensate is no greater than the error existing in several of the types of permeameters now in use. This permeameter may be used without compensation in certain kinds of shop testing. Such tests may be desirable where comparative results only are desired. No erroneous conclusions as to the relative merits of two steels need be drawn. In materials very nearly alike the corrections would be nearly identical, so that the differences between the compensated and the uncompensated results would be the same.

The advantage, of course, in omitting the compensation lies in the greater speed of operation.

IV. CONCLUSION

The Fahy permeameter represents a distinct advance in the measurement of magnetic characteristics of steel and other magnetic materials. In accuracy it far excels the direct-reading permeameters which have been rather commonly used both in this country and abroad. Normal induction measurements of solid bars show errors no greater than 5 per cent of the magnetizing force required for a given induction. The consistency of its readings taken at different times on the same specimen is so close that comparative results on similar materials can be obtained to a much higher degree of precision. Commercial materials, however, are seldom uniform enough to warrant better precision than 5 per cent. Hysteresis measurements are accurate within the limits of commercial requirements and the uniformity of commercial materials.

The apparatus has not as yet been adapted to the measurement of sheet materials but preliminary experiments indicate that such adaptation is entirely possible.

WASHINGTON, March 23, 1917.

APPENDIX

DESCRIPTION AND OPERATION OF THE FAHY PERMEAMETER

In view of the fact that a more detailed description of the permeameter and its mode of operation may be of some interest, it seems desirable to present more descriptive matter than properly belongs in the body of this paper.

1. DESCRIPTION

(a) *The Magnetic Circuit.*—Fig. 13 shows the permeameter in plan and elevation. *B* represents the base of the apparatus upon which is mounted a magnetic core of H shape, consisting of a crossbar *J* formed at its ends with threaded portions, which screw into threaded sockets fashioned in the transverse core portions. The crossbar *J* is divided midway between its ends into two sections separated by a small air gap, which may be closed to a greater or less extent by a screw piston *P* of magnetic material, actuated through the knurled head *N*.

The ends of the transverse core portions form a part of each of the two parallel magnetic circuits and are adapted for bridging by specimens of magnetic material. In Fig. 13 a standard specimen *A*, the magnetic characteristics of which are known, bridges the core arms *Y*, and a test specimen *X*, the magnetic characteristics of which are to be determined, bridges the core arms *Y'*. Clamps *O* serve to hold the specimens in place. The ends of the core arms *Y* and *Y'* are slotted to permit the insertion of bushings *W*, which vary in form according to the shape of the material bridging the core arms. The adjustable air gap in the crossbar *J* is designed to create a demagnetizing force within the core itself, which may be varied to suit requirements. When the gap is open to its fullest extent, the magnetic poles created there when the magnetizing current is broken in the act of reversing the current are of material assistance in reducing the time constants of the magnetic circuits. This demagnetizing force may be adjusted to reduce to a negligible amount the magnetic field in the region occupied by the test specimen due to the residual properties of the core itself. Under this latter condition direct measurements of residual induction are simplified.

The magnetomotive force is applied in two sections, one over the crossbar *J*, by means of current in the magnetizing coil *M* and the other distributed in equal parts over the core arms *Y* *Y'* by means of current in the compensating coils *C*. The currents in these two sections are independently adjustable. The test coils indicated by *D*, *D'*, *S*, and *T* are each of the same number of turns and placed as shown in the figure; *D* and *D'* on opposite arms of the same transverse core portion and *S* and *T* uniformly wound on coil forms through which the standard and test specimens *A* and *X*, respectively, may be inserted. A test coil *H* of a relatively large number of turns is wound uniformly on the same form which carries the test coil *S*. The test coils *S* and *T* are of the same mean cross-sectional area.

Fig. 14 shows the flux paths within the magnetic core due to the magnetomotive forces generated by currents in the coils *M* and *C*.

The magnetic field due to current in the magnetizing coil has the direction indicated by the arrows or the reverse according to the direction of the current. The field due to current in the compensating coils has the direction indicated by the dotted line,

its direction at any time depending upon the direction of the current in the compensating coils. It will be seen that when current is flowing in both magnetizing and compensating coils, one magnetic circuit has the magnetomotive force due to its magnetizing coil augmented by the current in the compensating coils, while the other magnetic circuit has its magnetomotive force decreased by the same amount.

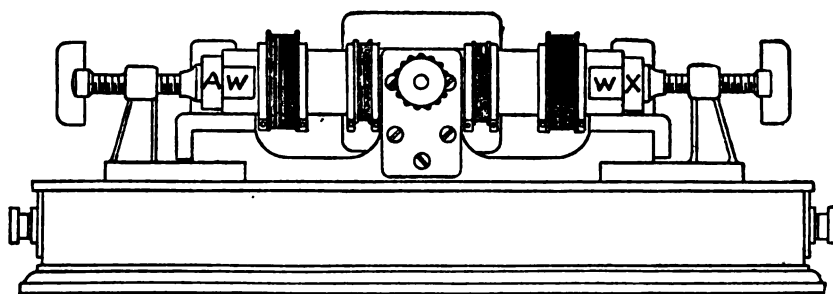
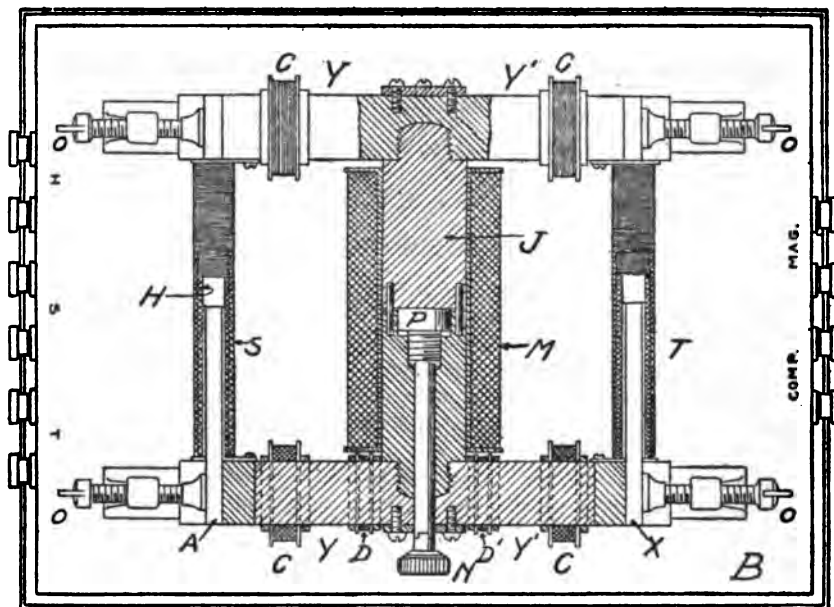


FIG. 13.—Diagram of the Fahy permeameter

It is obvious that by varying the current in the compensating coils $C C C C$ the difference in fluxes linking the test coils D' and T can be made equal to the difference in fluxes linking the test coils D and S .

When the compensation is adjusted so that these flux differences are equal, the leakage flux through the air path which is in parallel with the path of the coil T is equal to the leakage flux through the air path which is in parallel with the path of the coil S . Since these two leakage air paths are symmetrical and carry the same

magnetic flux, the same difference of magnetic potential must exist between the ends of the path encircled by the coil T that exists between the ends of the path encircled by the coil S . Consequently, the coil H , which is wound uniformly along the length of the coil S , and which may be used to measure the difference in magnetic potential over this region and hence of the mean magnetizing force for this region, gives also a measure of the mean magnetizing force along the path of the coil T within which the test specimen lies.

(b) *Internal Electrical Connections.*—Fig. 14 shows also the internal electrical connections of the permeameter. The magnetizing and compensating coils are connected to the binding posts marked "Mag" and "Comp," respectively. The test coils S , D , D' , and T are permanently connected in series, taps being brought out to binding posts marked S and T . This series connection is carried out in such a manner that

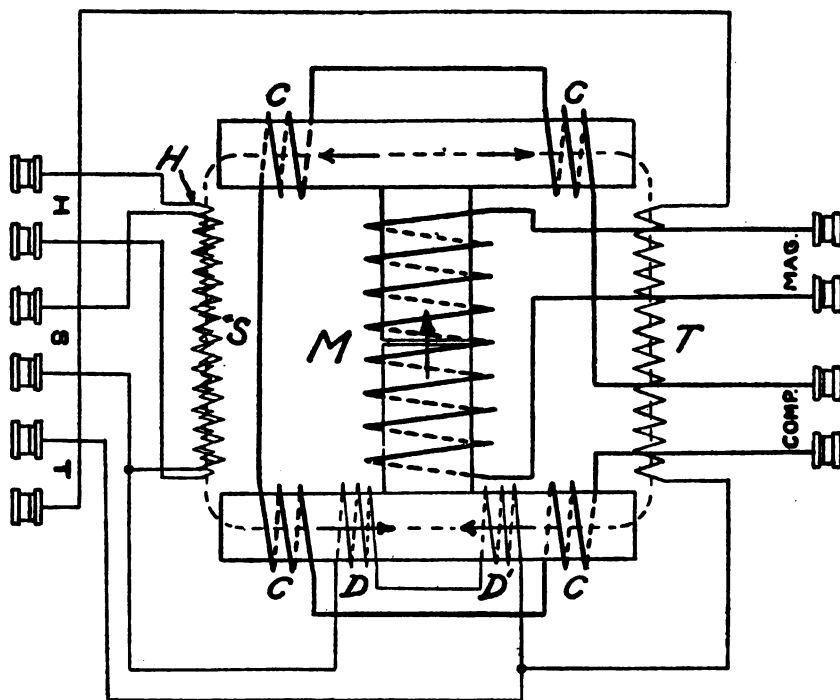


FIG. 14.—Internal electrical connections of the Fahy permeameter

upon reversal of the current in the magnetizing coil M the integrated electromotive forces induced in the coils S and D' act in the same direction and in opposition to the integrated electromotive forces simultaneously induced in the coils D and T . Connection to the adjacent binding posts of the pairs marked S and T enables observation of the differential effect of the coils D and D' to be made. The permeameter is initially adjusted, so that when no specimens are in the apparatus this differential effect is zero. Test coil H is connected to the binding posts marked H .

(c) *External Electrical Connections.*—Fig. 15 shows the external electrical connections of the magnetizing, compensating, and test circuits. M and C refer to the magnetizing and compensating coils, respectively, and MI is the primary of a mutual inductance; SM and SC are reversing switches for the corresponding circuits. When switch SMI is thrown downward, SM is the reversing switch for the mutual inductance. RM and RC are variable resistances in the magnetizing and compensating

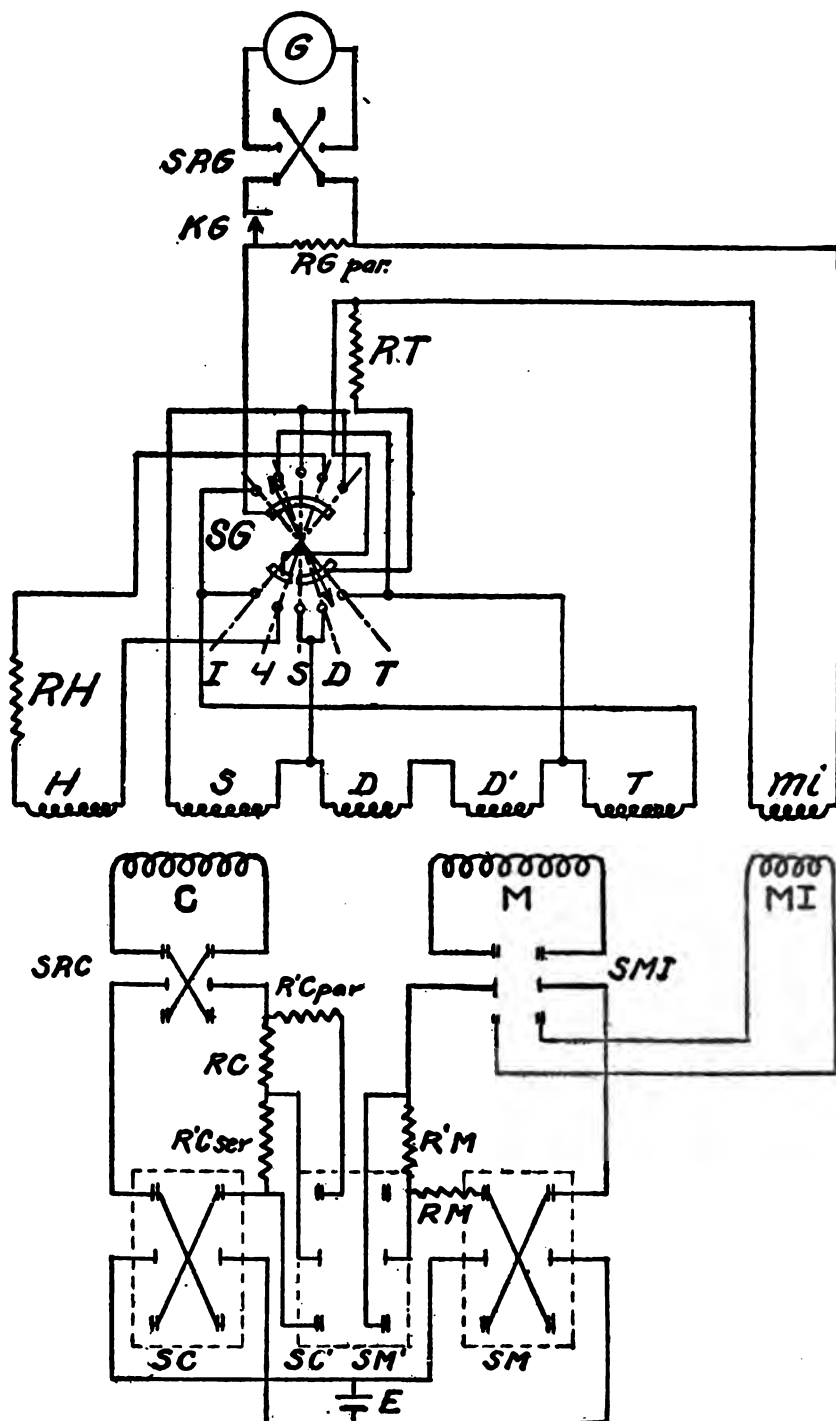


FIG. 15.—Complete electrical connections of the Fahy permeameter

circuits, respectively. Switch SM' when thrown upward inserts the variable resistance $R'M$ in series with the resistance RM . Switch SC' when thrown upward inserts the variable resistance $R' C_{\text{ser}}$ in series with RC and the variable resistance $R' C_{\text{par}}$ in parallel with RC . The resistances $R' C_{\text{ser}}$, $R' C_{\text{par}}$, and $R'M$ may each be made infinity or zero as may be required during the course of a test. The reversing switch SR controls the initial direction of the current in the compensating coils C . E represents a source of direct-current electromotive force.

The reversing switch SM , the combined switch $SC'-SM'$, and the reversing switch SC may be operated either singly or in any combination by a specially designed handle which engages the individual switches by means of latches. The normal position of these three switches is downward.

In the secondary circuits D , D' , S , T , and H refer to the test coils and mi is the secondary of the mutual inductance. G is a ballistic galvanometer and SRG a switch which controls the direction of throw of the galvanometer. RG_{par} is a fixed resistance which is placed in parallel with the galvanometer when the key KG is closed. SG is a five-point-dial switch through which connections are made between the galvanometer and the following test coils: Switch SG in position I connects the galvanometer with coils S , D , D' , and T in series; in position H with coil H ; in position S with coil S ; in position D with coils D and D' in series; and in position T with coil T .

RT is an adjustable resistance in series with the galvanometer when the switch RG is in the positions S , D , or T . RH is an adjustable resistance in series with the galvanometer when the switch RG is in the position H .

2. CALIBRATION

(a) *Ballistic Galvanometer.*—The galvanometer is calibrated by means of the mutual inductance MI , of fixed value, the secondary of which remains permanently in the galvanometer circuit. The reversal of a definite current in the primary of the mutual inductance produces a galvanometer deflection which is the same as that produced by the reversal of a definite number of gaussses in the test specimen. The formula used in the calibration of the galvanometer is

$$I = \frac{B N A}{M 10^3}$$

where

I = current in amperes in primary of mutual inductance.

B = gaussses (flux per square centimeter).

N = number of turns in test coil.

A = area of test specimen in square centimeters.

M = mutual inductance in henries.

Assuming, for example, the following values:

$B = 10\,000$

$N = 100$

$A = 3$

$M = 0.01$

then $I = 3$ amperes. If it is desired to have 10 cm galvanometer deflection correspond to the reversal of 10 000 gaussses, then with switch SG on point T the resistance RT is adjusted until a reversal of 3 amperes in the primary of the mutual inductance produces 10 cm deflection.

(b) *Coil H.*—The coil H is used to determine the magnetic potential difference existing between its ends when it encircles air only; that is, when no specimen is inserted within it. The average magnetic potential drop per unit length of the winding H is equal to the average magnetizing force. The area turns, NA , of the coil are determined by measurements of the coil in a uniform field such as is produced by a long solenoid. The product NA thus determined is subject to a correction if the

length of the coil H is not the same as the free length of the test specimen between contacts at the pole faces. Experiment has shown that, for the instrument investigated, this correction is 3 per cent. This corrected value is used in calculating the current to be reversed in the primary of the mutual inductance in adjusting the sensitivity of the galvanometer to read magnetizing force. Adjustment for this purpose is made by means of the adjustable resistance RH .

3. DETERMINATION OF NORMAL INDUCTION AND HYSTERESIS

The permeameter can be used for either relative or absolute magnetic measurements. It can be used (a) to determine the normal induction of a specimen of unknown magnetic characteristics by comparing it with a standard or known specimen, or (b) to determine directly the normal induction and hysteresis of a specimen. The method of procedure depends to a considerable degree upon the accuracy desired. Where approximate values of permeability only are required, the procedure is extremely simple, and the accuracy is sufficiently high for many practical purposes.

(a) *Normal Induction by Comparison.*—A standard specimen A , the normal induction of which is known, is inserted in the standard coil S and clamped in place. The unknown specimen X is inserted in the test coil T and clamped. Switch $SC'-SM'$ closed down and unlatched and remains so throughout the test; galvanometer switch SG is placed on the point S ; and switch SMI is thrown downward. The galvanometer is then calibrated as already described, its sensitivity being adjusted by means of the resistance RT . Switch SMI is then thrown up and the magnetizing coil M energized. Both bars are then demagnetized by repeated reversals of a successively decreasing magnetizing current. This demagnetizing current is gradually reduced, by means of the resistance RM , from a maximum value which brings the magnetization of the specimens well above the knees of their normal induction curves to a value lower than the lowest which is to be used. The frequency of reversal is about two per second. There should be no current in the C coils during demagnetization. After demagnetization, the lowest magnetizing current to be used is set and reversed several times to bring the specimens to a magnetically cyclic state. Switch SG is then placed on point I and the magnetizing current is reversed. A deflection is generally observed which is due to the fact that the two sides of the magnetic circuit are not exactly symmetrical, due to different areas of the two specimens or differences in their magnetic characteristics. These differences cause different magnetomotive force drops in corresponding parts of the parallel magnetic circuits which must be compensated for in order to eliminate errors due to this cause. If the difference in the fluxes linking the test coils D' and T is less than the flux difference between D and S , a deflection of the galvanometer is observed which may be reduced to zero by means of current in the compensating coils C . Switch SRC is closed in such a direction that when switches SC and SM are closed in the same direction the magnetomotive force in the magnetic circuit, which includes the test specimen, is augmented over that due to the main solenoid alone, while the magnetomotive force in the circuit including the standard specimen is decreased. The magnitude of this compensating magnetomotive force is adjusted by means of the resistance RC until upon simultaneous reversal of SC and SM the residual deflection of the galvanometer is zero. This adjustment is to be made for each point on the normal induction curve. The reversals of the magnetizing current during the adjustment are generally sufficient to reduce the specimen to a cyclic condition. When compensation is complete, switch SG is placed on point S , and the deflection due to the reversal of flux in the coil S is read upon reversal of switch SM . SG is then placed on point T , and the deflection due to the reversal of flux in the coil T is read upon a second reversal of switch SM .

Since the coils S and T have areas greater than the specimens A and X , the observed inductions are too great. The correction to be subtracted is $\frac{a-A}{A}H$ where a

is the area of the test coil, A the cross-sectional area of the specimen, and H the magnetizing force. From the normal induction curve for the standard specimen, the H corresponding to the observed induction is determined and the correction applied giving the true induction from which the true H can be derived. A similar "air correction" is to be applied to the induction observed in the coil T . When the specimen X is of the same area as A , the true induction of X thus obtained may be plotted against the value of H as taken from the curve of the standard specimen. When the area of X is different from A , since the galvanometer is calibrated for A , the true induction for X is

$$B_{\text{true}} = \frac{A}{A'} (B_{\text{obs}} - \frac{a - A'}{A'} H)$$

where A' is the area of specimen X . Other points on the induction curve are obtained in a similar manner. It is not necessary to repeat the demagnetization in the determination of successive points at magnetizing forces greater than the preceding.

If approximate values will be satisfactory, the above procedure may be simplified by omitting the compensations. In this case switch SR remains open throughout the test. The errors will depend upon the differences existing between the permeabilities and areas of the test specimen and of the standard.

When it is inconvenient to insert the test specimen in a test coil, a reading of the induction through the specimen may be made with switch SG on the point D . The galvanometer then reads the difference between the fluxes through the core arms Y and Y' . This differential flux, when algebraically added to the induction observed through the coil S , gives the induction in the specimen X . No air correction is necessary for the test specimen reading, but correction is made for differences in area of the test and standard specimens. When practicable it is best to have the standard and test specimens of equal areas.

(b) *Normal induction by Absolute Method.*—Absolute measurements of normal induction are made without the use of a standard bar. Compensation for unequal magnetomotive force drops in the two magnetic circuits should be made except when approximate results only are desired. With material of low permeability, results without compensation are, in general, fairly satisfactory. Normal induction readings of high permeability material are unsatisfactory unless compensation is used.

The adjustment of the compensation in the absolute measurement of normal induction is similar to that already described. The galvanometer resistances are adjusted as described under "galvanometer adjustment." The measurements of the magnetizing force and the induction are made by reversing simultaneously the switches SC and SM . The ballistic deflection of the galvanometer, when SG is on the point H , gives directly the magnetizing force acting on the specimen X . The induction of the specimen X is read when SG is on T as before.

(c) *Hysteresis.*—Measurement of hysteresis is made without the use of a standard bar, the magnetizing force being read from coil H and the induction from coil T .

The determination of a hysteresis loop can perhaps be best explained with reference to Fig. 16. When the specimen X has been magnetized to a desired degree and the cyclic state is established, its condition is represented by the point A , which is called the tip of the loop. The magnetizing force is then O_1 and the induction is OX . If the magnetizing force is reversed, the specimen is carried along the magnetic path represented by AEK to the point K , where the magnetizing force and induction are each of the same magnitude as at A , but opposite in direction. The change in induction, therefore, is twice the induction at the tip.

Instead of reversing the magnetizing force we may reduce it to the value represented by the point C . The reduced magnetizing force is then O_2 and the corresponding induction OD . The change in induction, which is the quantity actually measured by the test coil encircling the specimen, is that represented by XD . If the apparatus is calibrated to read induction, in taking normal induction by reversals

the reading of the test coil is to be multiplied by two in each case to give the true change in induction. The value of the induction at any point on the loop is the induction at the tip minus the change in induction. Points at the left of the OB axis are obtained by reversing the direction of the magnetizing force at the same time that it is reduced in value. If the change in induction is greater than the induction at the tip, it indicates that the flux has changed in sign as well as in magnitude as at the point G .

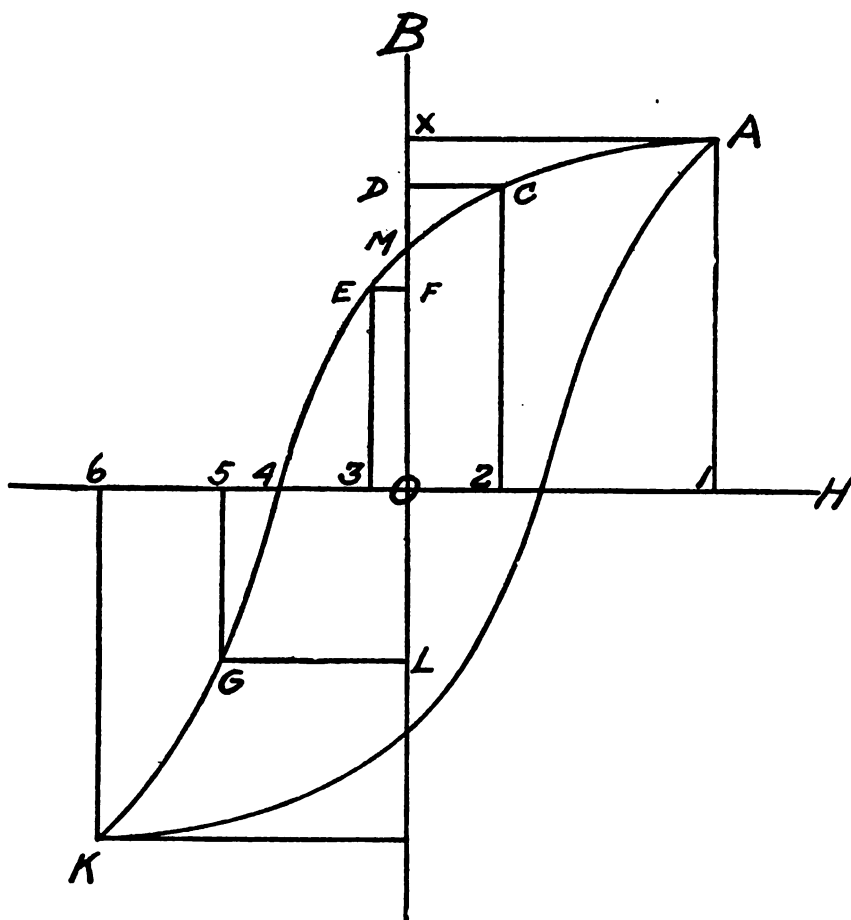


FIG. 16.—Typical hysteresis loop

In practice the tip of the loop is set either at a definite induction or at the induction corresponding to a definite magnetizing force. Switches SM , SC' - SM' , and SC are closed downward and compensation effected for the required tip as in 3. $R'C_{par}$ is increased to infinity and SG placed on the point 1. SC' - SM' is thrown upward and the galvanometer deflection observed. If the resistance RM' is properly chosen, the magnetizing current will be decreased so that the magnetizing force is, say, $O-2$, Fig. 16. Note that no currents are reversed in these operations. If the galvanometer deflection observed on throwing SC' - SM' upward is not zero, compensation must be adjusted. SC' - SM' is closed downward and SM and SC reversed several times to

regain the cyclic condition, being closed finally downward. $R'C_{\text{par}}$ is adjusted and $SC'-SM'$ again closed upward. If $R'C$ series is properly adjusted, the residual galvanometer deflection will be zero. If there is still a residual galvanometer deflection, bring the induction in the specimen to a cyclic state as before, and try again with a new adjustment of $R'C_{\text{par}}$. When compensation for the reduction of the induction from the tip is effected, SG is closed on the point T , $SC'-SM'$ is closed downward, and SM and SC reversed to regain the cyclic condition, being closed finally downward. Then, closing $SC'-SM'$ upward, the galvanometer deflection is noted. The observed induction corresponding to the reduced magnetizing force is equal to the induction at the tip minus twice the change in induction observed. Closing SG on H , and restoring the cyclic condition in the usual manner, observe the change in magnetizing force on closing $SC'-SM'$ upward. The observed magnetizing force is equal to the magnetizing force at the tip minus twice the change in magnetizing force observed.

The observed magnetizing force is the true magnetizing force as the galvanometer is calibrated for the area of the H coil. An air correction is applied to the observed induction, as in the case of normal induction by the formula

$$B_{\text{true}} = B_{\text{obs}} + K(H_1 - H_2)$$

$$\text{where } K = \frac{a - A}{A} \text{ as before}$$

and H_1 = magnetizing force at the tip

and H_2 = magnetizing force at the lower point.

If the magnetizing force is reduced to a point near the axis, but not reversed, it may be necessary to reverse the compensation at the same time it is reduced. This is accomplished by reversing SC at the same time as $SC'-SM'$ is operated. Again, it may be necessary both to reverse and increase the compensation. This is accomplished by reducing $R'C_{\text{par}}$ to zero and adjusting $R'C_{\text{par}}$, switch SC being reversed at the same time that $SC'-SM'$ is operated. To pass from the tip to negative values of the magnetizing force, as for instance to $O-3$, Fig. 16, SM is reversed at the same time as the other switches are operated.

For many purposes, as for instance in the determination of the constants of magnet steel, it is necessary to determine only the residual induction and coercive force of a specimen. These quantities are represented by OM and O_1 , respectively, in Fig. 16. This is accomplished by so adjusting the resistances in the manner described above that, for residual induction, the change in magnetizing force indicates that the magnetizing force at the lower point is zero, and, for coercive force, the change in induction is such that the true induction at the lower point is zero. The corresponding values of induction and magnetizing force determine the points desired.

4. MANIPULATION

During the course of a complete magnetic test where direct measurements are obtained, the individual switches may take the following positions:

Position 1. Switches SM , $SC'-SM'$, and SC are down and the corresponding electrical connections are as shown in Fig. 17.

Position 2. Switches SM and SC are up. Switch $SC'-SM'$ is down. Circuits are the same as 1, with currents reversed.

Position 3. $R'C_{\text{par}}$ placed on infinity point. Switch $SC'-SM'$ is up. Switches SM and SC are down. Circuits as shown in Fig. 18.

Position 4. $R'C_{\text{par}}$ placed on zero point. Switch $SC'-SM'$ is up. Switches SM and SC are down. Circuits as shown in Fig. 19.

Position 5. $R'C_{\text{par}}$ placed on infinity point. Switches SM and $SC'-SM'$ are up. Switch SC is down. Circuits as shown in Fig. 20.

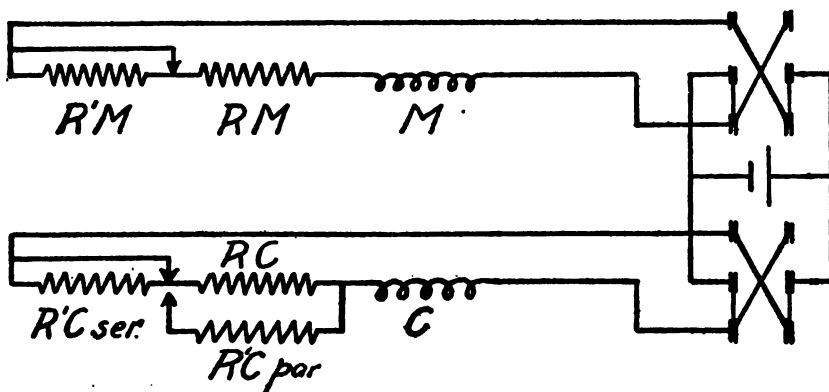


FIG. 17.—Electrical connections when switches are in position 1

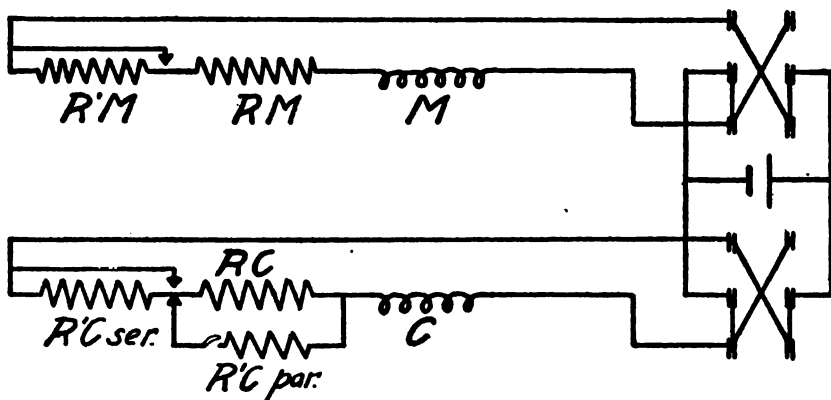


FIG. 18.—Electrical connections when switches are in position 3

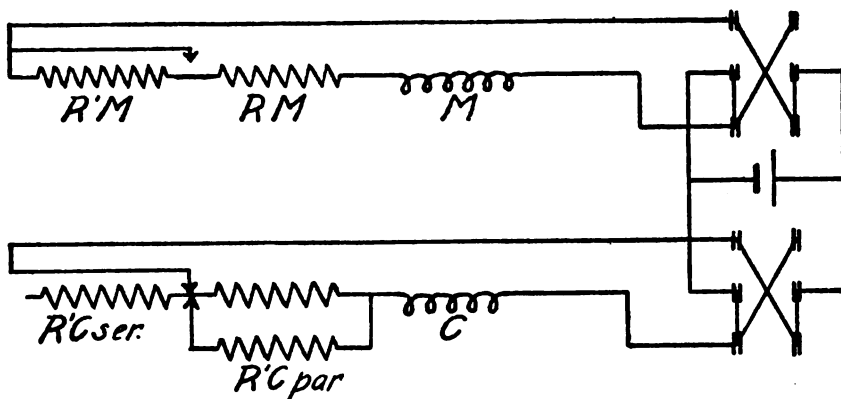


FIG. 19.—Electrical connections when switches are in position 4

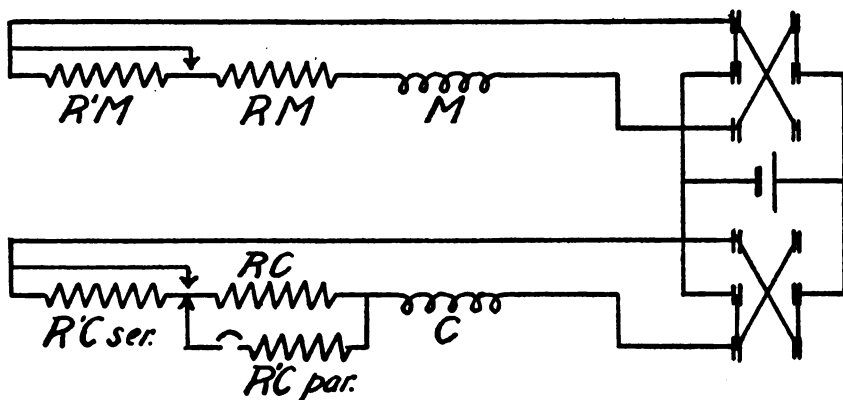


FIG. 20.—Electrical connections when switches are in position 5

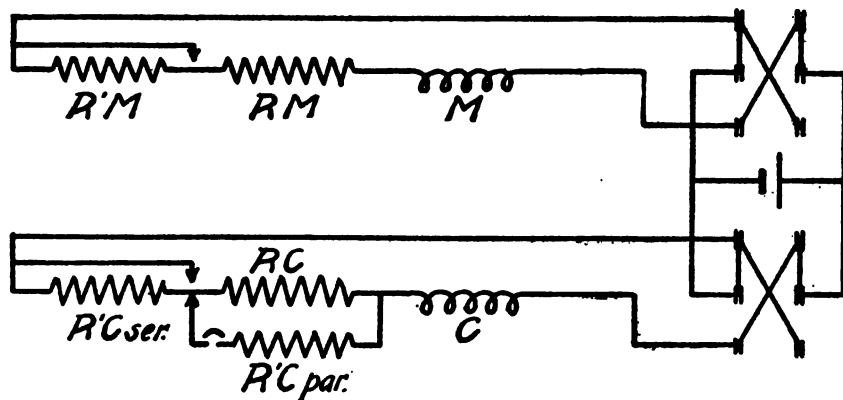


FIG. 21.—Electrical connections when switches are in position 6

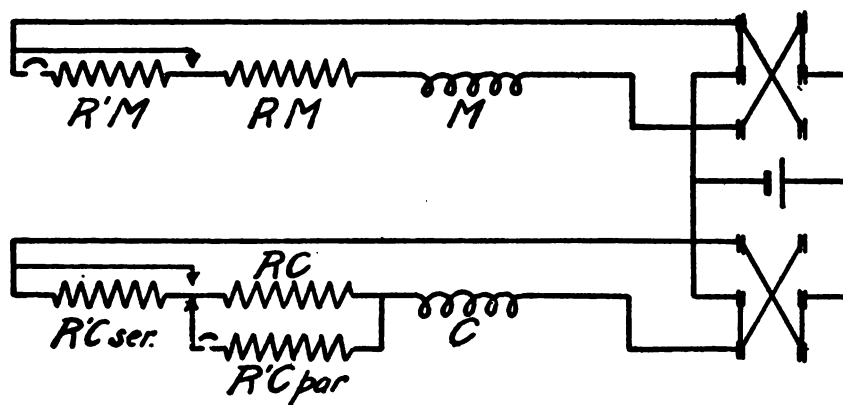


FIG. 22.—Electrical connections when switches are in position 7

Position 6. $R'C_{\text{par}}$ placed on infinity point. Switches SM , $SC'-SM'$, and SC are up. Circuits as shown in Fig. 21.

Position 7. $R'M$ and $R'C_{\text{par}}$ placed on infinity point. Switches SM and $SC'-SM'$ are up. Switch SC is down. Circuits as shown in Fig. 22.

5. PRECAUTIONS

In order to obtain the highest precision and accuracy from this apparatus, certain things should be kept in mind.

(a) *Magnetic Contacts.*—The magnetic contact between the specimen and the pole faces should be reasonably good. Poor contact may result from irregularities in the surface of the specimen, such as small elevations and depressions. The specimen may be covered with scale or rust, and thus produce the equivalent of an actual separation of the specimen from the pole face. A warped specimen may also give poor contact. Since the two pole faces lie in the same plane, if the specimen is bowed the corresponding contact surface reduces to a line contact. Difficulties of these kinds may occur in castings, hot-rolled bars, and in bars that have been heat treated. Machined specimens are generally free from contact troubles.

(b) *Position of Specimen.*—The specimen should be placed against the pole faces, so that there is no portion projecting beyond the test-coil pole faces. Specimens should be of sufficient length to span the pole faces. Longer lengths may be used, but are not recommended.

(c) *External Field.*—Since this apparatus has a large external magnetic field, care must be taken to see that this field does not exert an influence upon any other part of the circuit. The mutual inductance must be placed at some distance from the magnetic circuit and in such a position that no lines of magnetic flux link the secondary coil. The galvanometer should be separated from the magnetic circuit, so that there is no influence exerted either upon the permanent magnet or the coil of the galvanometer. Finally, all the leads of the secondary circuits should be brought near to each other or twisted, so as to eliminate the possibility of induced electromotive forces in these parts of the circuits. This arrangement of the apparatus is made once for all.

NOTE ON ELECTRICAL CONDUCTION IN METALS AT LOW TEMPERATURES¹

By Francis B. Silsbee, Assistant Physicist

The experiments of H. Kamerlingh Onnes on the resistivity of metals at liquid helium temperatures have shown that certain metals possess an enormously increased conductivity when the temperature, current density, and magnetic field are less than certain critical values. It is the purpose of this paper to point out that a definite relationship is to be expected between the values of critical current and critical field, and that this relation is in agreement with the experimental data available.

The present state of our experimental knowledge of this subject is somewhat as follows: Certain metals—mercury, tin, and lead—at the very low temperatures obtainable in a bath of liquid helium show a very greatly increased electrical conductivity, to which Onnes has given the name “superconductivity.” The actual resistivity of the metal in this state is too small to measure but has been shown to be less than $2 \cdot 10^{-11}$ times the resistivity at 0°C .² As the temperature of any of these metals is lowered from room temperature the resistance decreases uniformly with the normal coefficient of about 0.4 per cent per degree until the temperature is very low, when the rate of decrease becomes for a time less rapid. At a certain critical temperature, however ($4^\circ.2 \text{ K}$ for mercury, $3^\circ.8 \text{ K}$ for tin, and 6° K for lead),³ there is a

¹ This note, with some modifications, was published in the *Journal of the Washington Academy of Sciences*, 6, p. 597; October, 1916.

² *Kon. Akad. v. Wet.*, Amsterdam, XVII, 1, p. 280.

³ *Communications from Leiden Laboratory No. 133*, pp. 7, 52, 60.

sudden break in the curve connecting resistance and temperature, and within a temperature range of a few hundredths of a degree the resistance drops from about 10^{-8} times its value at 0°C to less than 10^{-10} times the same value. Other metals, such as gold, silver, platinum, iron, etc., do not show this phenomenon, and their resistivity tends to approach a constant value as the temperature is lowered to the lowest value ($1^{\circ}.\text{6 K}$) at which such measurements have been made. The critical temperature at which the change occurs is very definite when the current used to measure the resistance is small, but when the measuring current is very large the critical temperature is found to be definitely lower. Conversely, if the temperature of the bath be held constant some degrees below the critical value and the current be increased a certain "threshold" value of current will be found at which the resistance suddenly appears.⁴ The lower the temperature the greater the value of the critical current.

It is further found that when a superconductor is placed in a weak magnetic field it remains superconducting, but that as the field is increased the normal resistance appears suddenly at a certain critical value of the magnetic field, and for still higher values of field increases slowly with the field.⁵ The critical value is slightly less when the field is transverse to the direction of current flow than when it is longitudinal, but the difference is not great.

The particular point which is the subject of this note is that *the "threshold" value of current is that at which the magnetic field due to the current itself is equal to the critical magnetic field.* In other words, the phenomenon of threshold current need not be regarded as a distinct phenomenon to be explained by heating, etc., but is a direct result of the existence of the phenomenon of threshold magnetic field.

In case the specimen is in the form of a flat coil of wire it is evident that the inner turns are in a magnetic field, due to the current in the other turns, which is very similar to that due to an entirely external electromagnet. Consequently, when upon increasing the current this field reaches the critical value, first the inner turns will become resisting and, as the current is increased, more and more of the wire will cease to be superconducting. Because of the enormous factor by which the conductivity decreases from the superconducting to the normal state most of this decrease will take place when only a small fraction of a turn of the coil ceases to be superconducting. Owing to the cumulative

⁴ Leiden Comm. No. 133, p. 3.⁵ Leiden Comm. No. 139, pp. 65-71.

effect of the successive turns the field produced by a given current is much greater in the coil than in the same wire when straight and consequently the current required to give the critical field strength will be much less. This is verified by the results of Onnes on coils of lead and tin⁶ wire, which showed critical currents one-fifteenth and one-eighth, respectively, of those for the same wire when straight. No attempt has yet been made to measure the gradual further increase of resistance which would be expected on this theory as the current is further increased and more and more turns become resisting.

In the case of a straight wire of circular section the effect to be expected is rather more complicated. Consider a superconducting wire of radius r_0 carrying a current I , uniformly distributed over the cross section. The magnetic field intensity H at any point distant r from the axis but inside the wire is given by

$$H = \frac{2 I r}{r_0^2} \quad (1)$$

and that at the surface of the wire by

$$H_0 = \frac{2 I}{r_0} \quad (2)$$

If the current be increased to a value slightly greater than $\frac{H_0 r_0}{2}$,

where H_0 is the critical-field intensity for the material, the outermost layer of the wire will become resisting. Since this layer is shunted by the superconducting core, the whole current will tend to flow in this core. This, however, would make the field at the edge of the core even greater than that above computed since by equation (2) the field varies inversely as the external radius.

The system is, therefore, unstable, and the current will shift suddenly to a new distribution. This distribution will depend on the exact form of the relation connecting resistivity with magnetic field, and if this relation were known the current distribution might be computed from the usual electromagnetic equations.

If it be assumed, as a special case, that the resistivity changes by a large factor k at a definite field intensity H_0 , then for currents larger than the critical value there is a superconducting core, which, however, has such a small radius that only a negligible part of the total current flows in it, while the outermost portion becomes resisting.⁷ Prof. Langevin has kindly sent the author a complete

⁶ Leiden Comm. No. 133, pp. 57, 60.

⁷ As printed in the Journal of the Washington Academy of Sciences only these two layers were mentioned. This is, therefore, incorrect as noted below.

mathematical solution of this problem, which shows that between these regions there is a third layer in the cross section of the wire in which the current density varies so as to keep the magnetic field at the value H_c throughout the layer. Prof. Langevin finds that the radius of the surface separating the intermediate region from the small superconducting core is

$$r_1 = r_0 \frac{2}{k} \frac{I}{\left[\frac{I}{I_0} + \sqrt{\left(\frac{I}{I_0} \right)^2 - 1} \right]}$$

and that the radius of the surface separating the intermediate and outer layers is

$$r_2 = r_0 \frac{I}{\left[\frac{I}{I_0} + \sqrt{\left(\frac{I}{I_0} \right)^2 - 1} \right]}$$

where

I = total current.

$I_0 = \frac{H_c r_0}{2}$ = critical current.

The magnetic field in the three regions is as follows:

In the superconducting core, $H < H_c$.

In the intermediate layer, $H = H_c$.

In the outer layer, $H > H_c$.

The result of this distribution on the resistance of the wire as a whole is to cause a sudden increase of resistance by a factor $\frac{k}{2}$ as the current passes the critical value and a further increase, at first rapid and then slower, as the resistance approaches asymptotically to k times its original value.

For any other relation between resistivity and field there would be a corresponding current distribution. In general the abruptness of the increase of resistance with current would be similar to that of the increase of resistivity with field.

Owing to the great experimental difficulties of working at these extreme temperatures, the data available for an experimental verification of this theory are rather scanty. The following table contains in condensed form the observed values of threshold current for various wires at different temperatures as published by the Leiden Laboratory. Since the threshold values depend considerably on temperature, a comparison is possible only when observations were made on two wires at the same temperature, and the table contains the results of practically all such observations published.⁸

⁸ Leiden, Comm. No. 133.

In the last column is given the maximum value of magnetic field in any part of the conductor—that is, the field at the surface of a straight wire or at the inner turns of a coil (the computations for the latter case being only approximate)—due to its own threshold current. It is seen from the table that at each temperature this magnetic field is much more nearly a constant of the material than either the current or current density. In the case of mercury the effect of a magnetic field on the resistance in the superconducting state has not been measured. For tin the threshold value at 2° K. is about 200 gauss, which is in good agreement with the slightly larger values computed from the threshold current corresponding to a slightly lower temperature. In the case of lead the agreement of the observed critical field (600 gauss at 4° K.) with the computed values is not so good, particularly in the case of the straight wire. Any discrepancy here, however, is easily explained by the possibility (frequently referred to by Onnes) of the existence of thin spots in the wire where the field intensity would be much greater for a short length.

Critical Values of Current for Various Metals and Temperatures

[From data by H. K. Onnes]

MERCURY

Temperature, degrees K	Area	Threshold current	Threshold-current density	Maximum magnetic field
	mm ²	amp.	amp./mm ²	gausses
4.1.....	0.0016	0.17	107	15
	.0025	.17	69	12
	.0035	.23	42	11
	.0055	.32	58	15
3.6.....	.0016	1.00	625	89
	.0025	1.07	427	76
	.004	>1.04	>260	>59
	.0052	.78	151	39

TIN. $H_{crit.} = 200$ at 2° K

1.6.....	.0143	1.0	70	a 430
	.0143	8.0	560	b 240

LEAD. $H_{crit.} = 600$ at 4° K

4.25.....	.025	9	680	b 385
	.014	>4	>300	c >110
	.014	.6	41	a 375
1.7.....	.014	.84	60	a 550
	.014	11.1	790	b 330

a Coll.

b Straight wire.

c In vacuo.

Further experiments which immediately suggest themselves are measurements on the critical magnetic field for mercury. The relation here advanced would indicate a critical field of only about 15 gauss at $4^{\circ}.1$ K and less than 100 gauss at $3^{\circ}.6$ K. It would also be of interest to observe the threshold value of current when the material is in very thin films. In this case for a given section of material the magnetic field resulting from a given current density is less than in the case of a straight wire, and the threshold-current density would consequently appear larger.

The theories thus far proposed to account for superconductivity by Onnes,⁹ Lindemann,¹⁰ and Thomson¹¹ do not specifically indicate the existence of a critical magnetic field, and only the latter accounts for a threshold-current density (by assuming a saturation effect). If it is true, as indicated in this paper, that the magnetic effect is the more fundamental, it would seem that this fact might afford a valuable clue leading toward a more satisfactory theory of the superconducting state and perhaps of metallic conduction in general.

WASHINGTON, February 12, 1917.

⁹ Onnes, *Leiden Comm.* No. 119.

¹⁰ Lindemann, F. C., *Phil. Mag.*, **29**, p. 127; 1915.

¹¹ Thomson, J. J., *Phil. Mag.*, **80**, p. 192; 1915.

REFLECTING POWER OF TUNGSTEN AND STELLITE

By W. W. Coblentz, Associate Physicist, and W. B. Emerson, Laboratory Assistant

I. INTRODUCTORY STATEMENT

In a previous paper¹ it was shown that tungsten, in common with all the pure metals thus far investigated, has a low reflecting power in the visible spectrum, which rises quite abruptly to a high value beyond 2μ in the infra-red. Unfortunately in assembling the data for publication a smooth curve was drawn through the observations in the region of 0.8μ , on the supposition that the irregularities in the data were due to errors of observation. As a result, an interesting characteristic which has an important bearing upon the production of light escaped attention. This characteristic, as will be shown in the present paper, is a marked depression at 0.8μ in the reflectivity curve.

In a subsequent investigation² of the radiation from incandescent tungsten filaments the energy curves showed elevations in the smooth curve in the region of 0.8μ to 0.9μ , which could not be attributed entirely to experimental errors. If tungsten has an indentation in its reflectivity curve, in the region of 0.8 to 0.9μ , similar to the indentations in the reflectivity curves of gold and of copper in the visible spectrum, then selective emission must occur in this region of the spectrum similar to the selective emission of incandescent gold and copper in the visible spectrum.

A further examination of the spectral reflectivity of tungsten was, therefore, made recently in order to verify these observations.

¹ Coblentz, this Bulletin, 7, p. 197; 1910.

² Coblentz, this Bulletin, 14, p. 115; 1917.

II. APPARATUS AND PROCEDURE

The apparatus used was a mirror spectrometer, fluorite prism, and a vacuum thermopile of bismuth silver, as described in previous papers. The source of radiation was a Nernst glower, *N*, Fig. 1. An image of the glower was projected upon the spectrometer slit, *S*, by means of a concave silvered glass mirror of 50 cm focal length.

The samples of tungsten examined were in the form of plane mirrors, obtained from the Research Laboratory of the General Electric Co. The surfaces were flat to within less than a wave

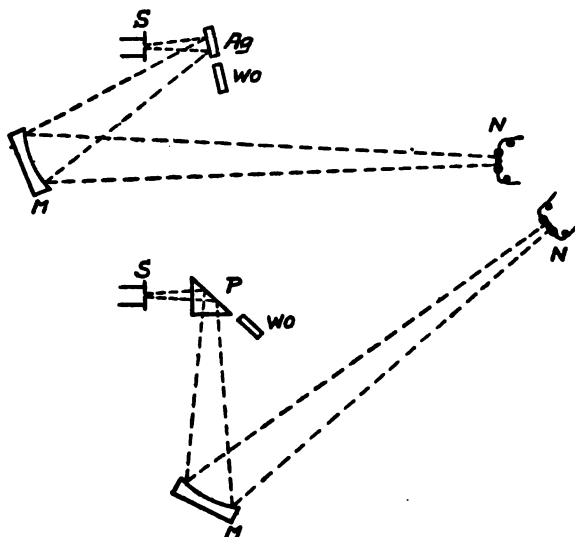


FIG. 1.—Apparatus for reflectivity measurements

length of light, and they were highly polished. The reflecting power of the tungsten mirrors was determined by comparison at a small angle of incidence with a freshly silvered glass mirror, *Ag*, Fig. 1, having an optically plane, highly polished surface. These mirrors were mounted upon a slider which permitted interchanging the samples in the optical path, as described in previous papers.

In order to determine the absolute reflecting power, the observed data were corrected for absorption by the silver mirror.³

For the region of the spectrum from 0.5μ to 1.2μ the absolute reflectivity of tungsten was determined also by a new method in which the silvered glass mirror was replaced by a right-angled glass prism, *P*, Fig. 1. In this arrangement the hypotenuse face

³ Using the reflecting power data given in this Bulletin, 10, p. 43; 1913.

of the glass prism replaced the reflecting surface of the silvered glass mirror, and hence was in the same plane with the surface of the tungsten mirror which was under investigation, as shown in the lower part of Fig. 1. When using this arrangement, the absolute reflectivity of tungsten was obtained by correcting the observed reflectivity for losses by reflection at the surfaces of the right-angled glass prism, the absorption being negligible in the region of 0.5 to 1.2 μ .

The change in focus due to change in length of the optical path, caused by the presence of the glass prism was not sufficient to

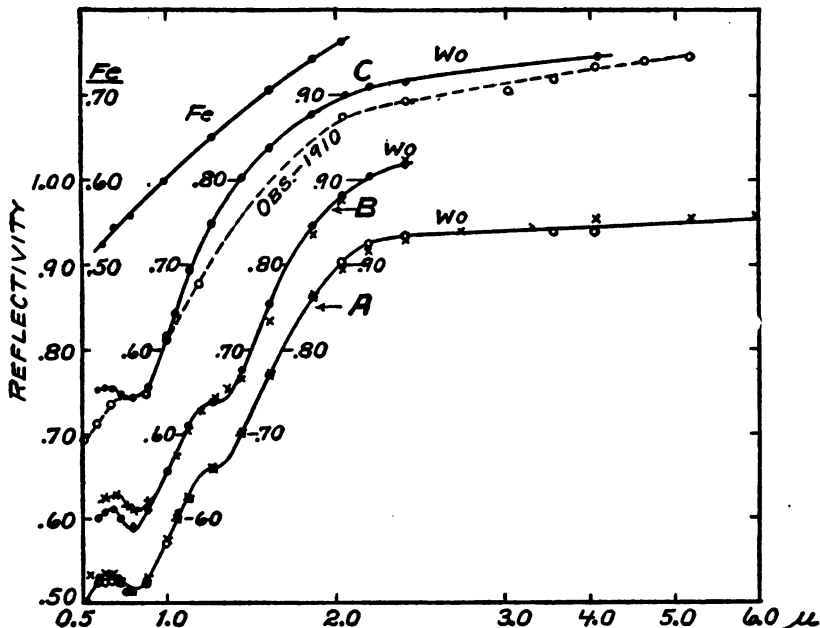


FIG. 2.—Reflecting power of various samples of tungsten

affect the observations by a measurable amount. Hence no attempt was made to have the optical path the same as for the tungsten mirror.

The reflectivity data of tungsten obtained with the right-angled glass prism are in excellent agreement with those obtained by comparison with the silvered mirror.

III. EXPERIMENTAL DATA

The experimental data presented in this paper are based upon observations on four samples of tungsten, one of which was examined seven years ago.

Sample No. 1 consisted of a tungsten X-ray target, 18 mm in diameter, mounted in a heavy block of copper. The first measurements, using a silvered glass mirror for comparison, are shown by the circles ($\odot \odot \odot$) in curve A, Figs. 2 and 3. In order to ascertain whether the indentations in the reflectivity curve of tungsten in the region of 0.8 and 1.3μ were influenced by the comparison mirror, another silver mirror⁴ was substituted. As shown by the crosses ($\times \times \times$) in curve A, Figs. 2 and 3, the observations are in close agreement with those obtained with the first silvered mirror.

In order to further verify the depression at 0.8μ in the reflectivity curve, the silvered mirror was replaced by a total reflection glass prism, as illustrated in the lower part of Fig. 1. The obser-

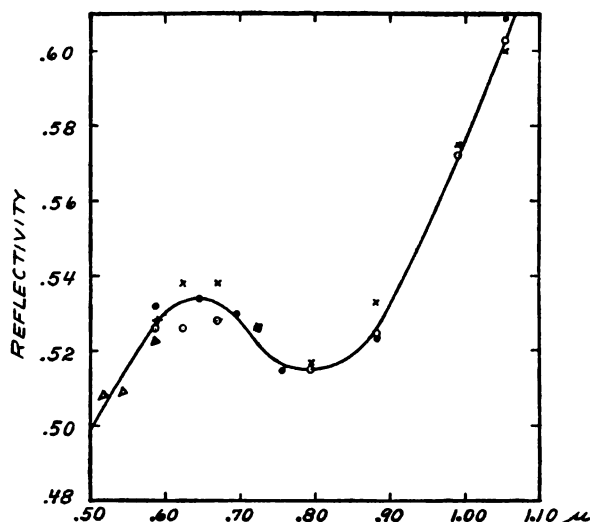


FIG. 3.—Selective reflection of tungsten

uations are in excellent agreement with the preceding observations. These observations are illustrated by the black dots ($\bullet \bullet \bullet$) in curve A, Figs. 2 and 3.

The reflecting-power curve of this sample of tungsten is of interest in showing indentations at 0.8μ and 1.3μ . At 5μ the reflecting power is 95.3 per cent while beyond 6.2μ it is about 95.8 per cent, which is close to the calculated value obtained from a knowledge of the electrical conductivity of tungsten as shown in the paper just cited. These indentations are of especial interest in view of the fact that this is the first metal yet investigated having reflection minima which occur outside of the visible and the ultra-violet parts of the spectrum.

⁴ Made several years ago and preserved as described in this Bulletin, 7, p. 227; 1910.

Sample No. 2 consisted of a sheet of tungsten, 30 by 14 by 0.5 mm, obtained several years ago. It was flat, but did not seem to take such a high polish as the first sample, owing to its brittleness. The surface showed pits of microscopic size, owing no doubt to the lack of plasticity of the material. This pitting reduces the absolute value of the reflecting power by 2 to 4 per cent, especially in the visible spectrum. The reflecting-power observations, obtained by comparison against silver, are illustrated (by dots; •••) in curve *B* of Fig. 2. The depressions at 0.8μ and 1.3μ coincide with those found in the first sample.

Sample No. 3 was the specimen of tungsten examined with a vacuum spectrolometer seven years ago.⁵ This sample was part of a rod of the metal used in making lamp filaments. A further examination was deemed desirable, in view of the fact that the first observations showed no depression at 1.3μ in the reflectivity curve. For this purpose the parallelepiped of tungsten (6 by 6 by 22 mm) previously examined was cut lengthwise into two pieces, which were cemented side by side upon a glass plate. This formed a wide reflecting surface (dimensions, 22 by 12 by 2 mm) which could be made planer than is possible with a long, narrow surface. The surface, one part of which had been previously examined, was reground and polished. This specimen was not as hard as the first and second samples, and the surface appeared more highly polished and free from minute pits.

The reflecting power is about 55.5 per cent at 0.62μ to 0.67μ , as compared with a reflectivity of 53.5 in sample No. 1. As shown in curve *C*, Fig. 2, the reflectivity curve has no indentation at 1.3μ . It is in agreement with the observations of 1910 as shown by the dotted curve in Fig. 2. The high reflectivity of this sample of tungsten, at 1.3μ , may be due to the presence of some other metal which is said to be used with the tungsten employed in lamp filaments.

Sample No. 4 was very generously supplied by Dr. Langmuir in reply to our inquiry concerning the purity of the first three samples. The sample No. 1 is considered of exceptionally pure material, but in view of the fact that it had not been heated to a very high temperature, it may contain small traces of tungstic oxide.

The present sample was a piece of a rod, 20 by 20 by 8 mm, of tungsten which had been heated in hydrogen nearly to its melting point, thus producing the purest material that has yet been made.

⁵ Coblentz, this Bulletin, 7, p. 197; 1910.

The reflecting surface had a fairly high polish, but it was not entirely free from fine pits, which seemed to be due to the brittleness and perhaps the structure of the material. This pitting caused a slight scattering of light, thus lowering the absolute values of the reflectivity as compared with sample No. 1, which seemed to be more compact and which was quite free from the minute pits present in this sample.

In the region of the spectrum from 0.6μ to 1.5μ the reflectivity was determined by using the glass prism reflecting device, *P*, shown in Fig. 1. The observations at 1 to 1.5μ were verified and extended to 2.5μ by comparison with a silver mirror.

The reflectivity data of this sample of tungsten are illustrated by the crosses ($\times \times \times$) in curve *B*, of Fig. 2. They coincide exactly with the measurements on sample No. 2, except in the visible where the reflectivity is somewhat higher, due, no doubt, to a higher polish. It is of interest to note that all these samples of pure tungsten have indentations in common at 0.8μ and 1.3μ in the reflectivity curve.

The reflecting power data of pure tungsten are given in Table 1. It is to be understood, of course, that these data are average values for the samples examined and that the absolute values for perfect mirrors might be 1 to 3 per cent higher, especially in the visible spectrum.

TABLE 1
Reflecting Power of Tungsten

Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity
μ		μ		μ		μ		μ		μ	
0.500	0.501	0.675	0.533	0.850	0.518	1.15	0.640	1.60	0.770	2.50	0.938
.525	.508	.700	.530	.875	.525	1.20	.655	1.70	.810	2.75	.940
.550	.517	.725	.523	.900	.533	1.25	.662	1.80	.845	3.00	.943
.575	.525	.750	.518	.950	.554	1.30	.665	1.90	.875	3.50	.945
.600	.531	.775	.516	1.00	.576	1.35	.670	2.00	.900	4.00	.948
.625	.534	.800	.515	1.05	.604	1.40	.685	2.10	.918	5.00	.953
.650	.535	.825	.516	1.10	.622	1.50	.726	2.25	.930	6.00	.958

Sample of Pure Iron.—In order to obtain a further check on the present data on tungsten, the sample of pure iron previously examined* was reexamined. The surface was not repolished, which no doubt explains the lower absolute reflecting power as

* Coblenz, this Bulletin, 7, p. 197; 1910.

compared with that previously observed. However, the reflectivity curve is smooth and free from the indentations which occur in the reflectivity curves of tungsten.

IV. PRACTICAL APPLICATIONS

These data on the reflecting power of tungsten are useful in explaining the irregularities which, as already mentioned, were found in the spectral radiation curves of incandescent tungsten lamp filaments. These irregularities consisted in a spectral energy distribution which is practically a straight line throughout the greater part of the visible spectrum, followed by a marked protuberance at 0.8 to 0.9μ in the infra-red, or what might be considered a depression at 0.7μ in the energy curve. The trend of the energy curve was found to be entirely different from that of other metals, e. g., platinum, which had been previously investigated.

In the spectral radiation curves previously published the wavelength scale is so small that these irregularities are not very conspicuous. In the present illustration, Fig. 4, the wave-length data are plotted on a large scale, which emphasizes the irregularities under discussion.

In Fig. 4 curve *A* gives the (relative) energy distribution in the spectrum of a straight wire "hairpin" filament of tungsten in a nitrogen filled bulb, observed three years ago. Curve *B* gives the spectral distribution of radiation from the outside of the turn of a helical filament of tungsten in a nitrogen-filled lamp.⁷ In these two curves the dotted lines indicate the energy distribution which would be obtained if the reflecting power increased uniformly between 0.7μ and 1μ , instead of having a depression at 0.8μ .

Curve *C* gives the spectral distribution of radiation of tungsten computed from the distribution of radiation from a black body at 2200° Abs., using the reflectivity data given in Fig. 2. The dotted line, as in curves *A* and *B*, indicates the energy distribution on the basis of a uniformly increasing reflectivity between 0.7 and 1μ . This curve is very similar to the observed curves, and it would no doubt fit the observed curves still closer if the exact temperature had been known, and if corrections had been applied for variation of reflectivity with change in temperature.⁸ This would tend to depress the curve at 0.5μ and make it straighter at 0.55μ to 0.6μ

⁷ Coblentz, this Bulletin, 14, p. 115; 1917.

⁸ Hulburt, Jour. Franklin Inst., 182, p. 695, 1916; Weniger and Pfund, Jour. Franklin Inst., 183, p. 354, 1917.

as observed. As a result of this peculiar reflectivity in the visible spectrum, the temperature computed from the slope of the spectral-energy curve is different from that computed from observations in the infra-red, as found in a previous paper.⁹

A further application of the reflecting-power data of tungsten is in connection with the question of increasing the luminous efficiency of incandescent lamps. As mentioned in previous papers, one way to increase the luminous efficiency is to obtain a substance having a high reflecting power in the infra-red, a low reflecting power in the visible spectrum, and a high operating

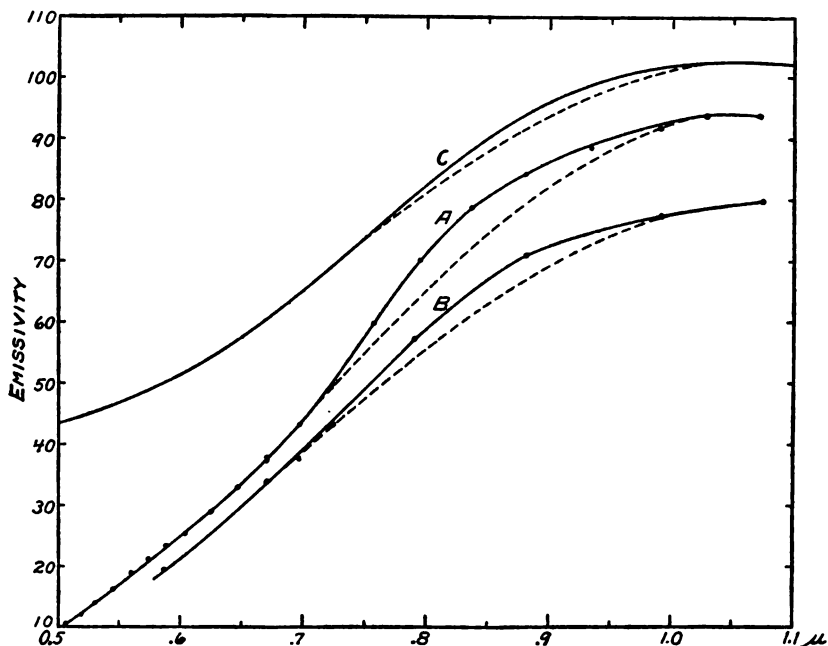


FIG. 4.—Observed (A and B) and computed (C) spectral radiation of incandescent tungsten

temperature. The reflectivity curves of tungsten indicate that this condition for obtaining high luminous efficiency is fulfilled, to some extent, by using pure tungsten, and that the luminous efficiency of an alloy of tungsten in which the reflectivity minimum at 1.3μ is obliterated is still higher than of pure tungsten. Unfortunately, among the contrary things in nature is the reflectivity minimum of tungsten at 0.8μ , which lies just outside the range of the spectrum to which the eye is sensitive, whereas gold with its low melting point has, as is well known, a band of strong selective

⁹ Coblenz, this Bulletin, 14, p. 115; 1917.

emission in the visible spectrum. If this reflectivity minimum at 0.8μ in tungsten occurred at 0.55 to 0.60μ the luminous efficiency would be increased by an appreciable amount.

V. REFLECTIVITY OF STELLITE

Stellite is an alloy of chromium, cobalt, and molybdenum.¹⁰ The sample examined was a plane, highly polished plate 30 by 20 by 5 mm. The reflectivity data given in Table 2 were obtained by comparison with silver. In view of the fact that this material does not deteriorate on exposure to the air, it is applicable for mirrors. It is, therefore, proposed to examine concave mirrors of stellite in order to obtain more accurate data on its absolute reflecting power.

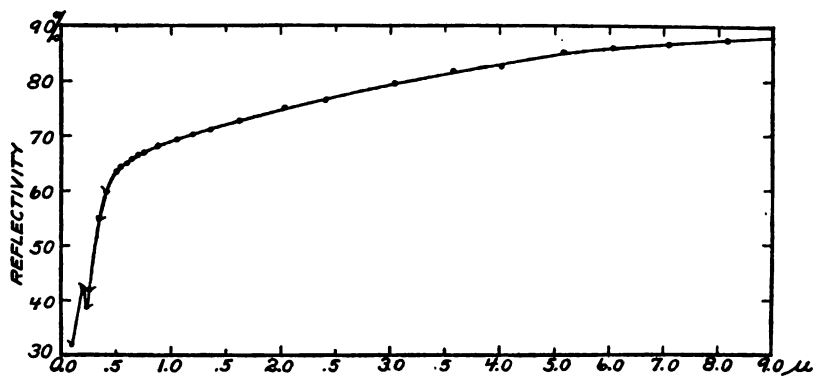


FIG. 5.—Reflectivity of stellite

TABLE 2

Reflecting Power of Stellite

Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity	Wave length	Re-flec-tivity
μ		μ		μ		μ		μ		μ	
0.15	32.0	0.45	62.0	1.00	68.9	1.75	73.3	3.25	80.1	7.00	86.8
.20	42.0	.50	63.6	1.10	69.6	2.00	74.7	3.50	81.0	8.00	87.3
.245	39.0	.60	65.2	1.20	70.1	2.25	76.0	4.00	82.5	9.00	88.0
.30	50.0	.70	66.5	1.30	70.9	2.50	77.1	4.50	83.8		
.35	55.0	.80	67.5	1.40	71.2	2.75	78.1	5.00	84.8		
.40	60.0	.90	68.3	1.50	71.9	3.00	79.2	6.00	86.0		

The observations on stellite are illustrated in Fig. 5, from which it may be seen that the reflectivity rises abruptly from 30 per cent at 0.15μ to 65 per cent at 0.6μ , beyond which point the reflectivity

¹⁰ Obtainable from the Stellite Works, Kokomo, Ind.

increases gradually to 87.5 per cent at 8μ . The data for the region from 0.15μ to 0.4μ were taken from a paper by Hulburt.¹¹ They appear to be an exact continuation of the present observations.

VI. SUMMARY

This paper gives the results of a critical examination of the reflecting power of tungsten in the region of the spectrum from 0.5 to 6.0μ .

The reflecting power of tungsten in the form of plane, highly polished mirrors was determined by comparison with silver and by a new method employing a total reflection prism.

The results obtained are based upon an examination of three samples of pure tungsten and a sample of unknown purity.

All samples show a marked depression in the reflectivity curve at 0.8μ . A similar indentation occurs at 1.3μ in the reflectivity of the pure metal, but is absent in a sample of impure tungsten.

The reflecting-power curve of pure tungsten rises abruptly from 50 per cent at 0.5μ to 90 per cent at 2μ , beyond which the reflectivity increases gradually to about 96 per cent at 6μ .

The bands of selective reflection at 0.8μ and 1.3μ render tungsten conspicuous as being the only pure metal thus far investigated which has bands of selective reflection in the infra-red. The minimum of reflection at 0.8μ causes a perceptible protuberance in the emission spectrum of an incandescent tungsten filament.

The reflection power of stellite rises gradually from 65 per cent at 0.6μ to 88 per cent at 9μ .

WASHINGTON, March 24, 1917.

¹¹ Hulburt, *Astrophys. Jr.*, 42, p. 205; 1915.

A METHOD FOR TESTING CURRENT TRANSFORMERS

By Francis B. Silsbee

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INTRODUCTION

Several precise laboratory methods are now available for the determination of the ratio and phase angle of current transformers.¹ These, however, all require a considerable amount of special apparatus, such as carefully calibrated noninductive shunts and very sensitive alternating-current detectors, and are therefore not suited for use under shop or central-station conditions.

The task of comparing the constants of one transformer with those of a second transformer taken as a standard is much less difficult. The standard transformer should, of course, have the same nominal ratio, and its constants should have been determined by one of the precise laboratory methods. A method for such a comparison of two voltage transformers has been published by Brooks,² and another method applicable to either voltage or current transformers by Agnew³ and by Knopp.⁴ The method developed in this paper is somewhat analogous to the first of these and will be found rather more rapid than the second.

As in the other comparison methods the detector may be much less sensitive than in the absolute methods, and it is therefore

¹ Agnew and Fitch, this Bulletin, 6, p. 281, 1909; Electrical World, 54, p. 1042, 1909; E. Orlich, E. T. Z., 20, p. 435, 466, 1909; L. T. Robinson, Trans. Am. Inst. Elec. Eng., 28, p. 1005, 1909; F. A. Laws, Electrical World, 55, p. 223, 1910; Sharp and Crawford, Trans. Am. Inst. Elec. Eng., 29, p. 1517, 1910; Agnew and Silsbee, Trans. Am. Inst. Elec. Eng., 31, p. 1635, 1912; Schering and Alberti, Archiv für Elektrotechnik, 2, p. 263, 1914.

² H. B. Brooks, this Bulletin, 10, p. 419, 1914 (Scientific Paper No. 217); Electrical World, 62, p. 898, 1913.

³ P. G. Agnew, this Bulletin, 11, p. 347, 1914 (Scientific Paper No. 233).

⁴ O. A. Knopp, Electrical World, 67, p. 92, 1916.

possible to use a more rugged type of instrument, such as a commercial wattmeter. Multiple-range transformers are particularly useful as standards, since they show practically proportional ratios and identical phase angles on the various primary connections. Two or three such transformers, the ratio and phase angle of which have been accurately determined, suffice for testing a considerable range of transformers.

GENERAL PRINCIPLES

The principle of the method is illustrated by Fig. 1. *S* and *X* are the standard and the unknown transformer, respectively. The primary windings are connected in series and supplied with current from a suitable source. The secondary windings are also

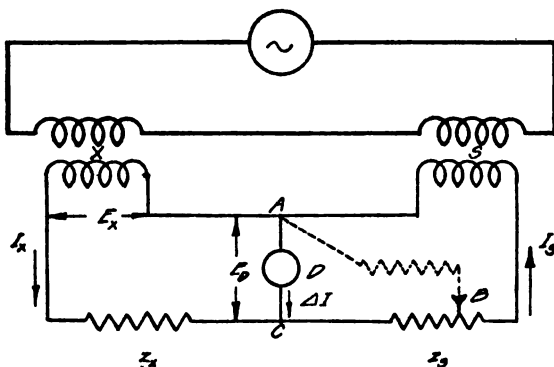


FIG. 1

connected in series, with such polarity that both tend to send current in the same direction, and any desired impedance loads such as z_s and z_x complete the circuit. A suitable detector *D* is then connected so as to bridge across between the transformers.

It is evident that the current ΔI through the detector (which we shall assume for the present to have a negligible impedance) is necessarily equal to the vector difference of the secondary currents I_s and I_x of the transformers. Consequently, if the magnitude and phase of ΔI are measured, the difference in performance of the two transformers can be computed. This measurement of ΔI may be made either directly by using as a detector one winding of a separately excited wattmeter, as is described below as the "Deflection method," or the measurement may be made indirectly by using an additional compensating circuit between *A* and *B*, as indicated by the dotted lines. By proper arrangement of the impedances all of ΔI may be made to flow in this

compensating circuit, and the current in the detector reduced to zero. This is described below as the "Null method." There are, of course, a great many other possible arrangements for measuring ΔI , but the two here described in detail will be found the most convenient.

The performance of a current transformer depends to a considerable extent upon the impedance of the apparatus and wiring connected in the secondary circuit. It is therefore of importance to determine what impedance is introduced into the circuit of each transformer by the arrangement of circuits shown in Fig. 1. Now transformer X is carrying a current I_x at a terminal voltage E_x , and the equivalent impedance load is

$$Z_x = \frac{E_x}{I_x} = \frac{I_x z_x - E_d}{I_x} = z_x - \frac{E_d}{I_x} \quad (\text{vectorially})$$

In the deflection method E_d (the voltage between the terminals of the detector) is

$$E_d = \Delta I z_d$$

And hence

$$Z_x = z_x - z_d \frac{\Delta I}{I_x} \quad (\text{vectorially})$$

and similarly

$$Z_s = z_s + z_d \frac{\Delta I}{I_s} \quad (\text{vectorially})$$

It is therefore evident that when the secondary currents are nearly equal, so that $\frac{\Delta I}{I_s}$ is very small, a detector of considerable impedance and therefore of high-current sensitivity may be used.

On the other hand, when $\frac{\Delta I}{I_s}$ is large, then the use of a detector of high impedance may shift the load from the transformer of higher ratio to that having the lower and thus make the difference in ratio appreciably less than the correct value.

In the null method, on the other hand, E_d is zero when a balance is reached, so that

$$Z_x = z_x$$

and the performance of each transformer is the same as if the other transformer were replaced by a short-circuiting link across $A C$. Although very evident theoretically, this fact was also tested experimentally, and the behavior of the transformer was found to be the same as when the detector was replaced by a wire of negligible resistance.

The principal limitation on the sensitivity attainable in any method of testing current transformers is due to the condition that the measuring circuit must not affect the performance of the transformer. This means that the measuring apparatus must not increase the equivalent connected secondary load by more than a certain resistance which we may denote by r .

The gain in sensitivity of a comparison method, such as is described in this paper over an absolute method, is seen by comparing the power available for operating the detector in the two cases. In the usual absolute method the secondary current I is passed through a noninductive resistance which is preferably equal to r . The voltage drop Ir is balanced by the drop due to the primary current flowing through a proportionately smaller resistance. With such an arrangement the impedance of the circuit external to the detector is approximately r , and for the maximum sensitivity the detector itself should also have this same impedance. If under these conditions the secondary current should differ by a small amount δI from that required for an exact balance, then the unbalanced voltage acting in the detector circuit will be only δIr . Since the resistance of the complete circuit is $2r$, the current flowing in the detector will be $\frac{\delta I}{2}$, and the voltage across the detector

will be $\frac{\delta Ir}{2}$. Consequently the power available is $\frac{\overline{\delta I}^2}{4}r$.

On the other hand, in the case of the deflection method outlined in this paper, the total difference ΔI between the two secondary currents flows through the detector and produces a voltage at its terminals equal to $\Delta I z_d$. Here z_d is the impedance of the detector, and is limited by the fact that this voltage $\Delta I z_d$ must not exceed the permissible value Ir for the largest value of ΔI . Consequently, we have $z_d \leq \frac{I}{\Delta I}r$ and if we use a detector having this impedance, the volt amperes available for a difference δI in the currents is $\overline{\delta I}^2 \frac{I}{\Delta I}r$. Since the two transformers will usually differ by only a few per cent the factor $\frac{I}{\Delta I}$ is fairly large and the detector required may be 50 or 100 times less sensitive than in the precise laboratory methods. It is this relation which brings the method within the range of sensitivity obtainable with commercial pivoted instruments.

NULL METHOD

The connections of what is probably the most satisfactory form of null method based on the general arrangement outlined above are given in Fig. 2. $ABCD$ is a slide wire of about 0.2 ohm total resistance. M is a mutual inductance of about 600 microhenrys, the primary of which can carry 5 amperes without excessive heating. r_1 is a resistance of about 30 ohms, preferably capable of being set at several other values down to 2 ohms. As the value of r_1 appears in the denominator in the equations below it is convenient to have the total resistance between F and C , including the resistance of the secondary winding of M , an exact integer for each setting of r_1 . M , r_1 and r_2 need be calibrated only to the per cent accuracy which is desired in the *difference* of the ratios; that is, to about 1 per cent.

The detector shown in the figure is a separately excited electro-dynamometer instrument. A commercial wattmeter of low-current range may be adapted for this work by bringing out taps directly from the moving coil without using any of the series resistance. The moving coil is connected as shown and the current coil excited by its full rated current in either of two phases, which preferably are in quadrature, through the double-throw switch G . Any other form of alternating-current detector sensitive to 0.00005 ampere might be used.⁵

As a precaution a 10-ampere ammeter should be connected in parallel with the detector, so that the transformers will not be damaged if they have inadvertently been connected in opposition instead of aiding. If on closing the circuit this ammeter shows no appreciable current the polarity of the transformers is correct, and the ammeter should then be disconnected.

The procedure is to adjust the position of the slider C and the value of the mutual inductance M until no deflection is obtained on closing G in either direction. When a balance is thus obtained all of the differential current ΔI is flowing through r_1 , and also the difference of potential between points B and F is zero. Consequently, the voltage drops between C and F and between C and B must be equal and in phase with one another. From this relation the differences in the ratios and phase angles of the two transformers may be computed.

⁵ A new type of vibration galvanometer has been recently developed by Agnew which is very well suited for this work. A description of this sensitive yet rugged instrument is to be published shortly in this Bulletin.

If R_s and R_x are the ratios of the transformers S and X , respectively, and α_s and α_x are the corresponding phase angles, then we have⁶ for the case where the slider C is to the right of B (Fig. 2)

$$\frac{R_x}{R_s} = 1 + a - \frac{b^2}{2} - bc$$

and $\tan (\alpha_x - \alpha_s) = b + ac$

⁶ The equations given below may be deduced as follows: For the connections as drawn in Fig. 2

we have $\Delta I = I_s - I_x$ (1)

and $\Delta I(r_1 + j\omega L_1) + I_s \omega M = I_x r_2$ (2)

by Kirchhoff's laws, where the currents are to be regarded as vector quantities.

Hence, eliminating ΔI , we get

$$I_s(r_1 + j\omega(L_1 + M)) = I_x(r_2 + r_1 + j\omega L_1) \quad (3)$$

or

$$\frac{I_s}{I_x} = \frac{1 + \frac{r_2}{r_1} + j\omega \frac{L_1}{r_1}}{1 + j\left(\frac{\omega L_1}{r_1} + \frac{\omega M}{r_1}\right)} \quad (4)$$

Substituting

$$a = \frac{r_2}{r_1}, b = \frac{\omega M}{r_1}, c = \frac{\omega L_1}{r_1}$$

$$\frac{I_s}{I_x} = \frac{1 + a + jc}{1 + j(b + c)} = 1 + a - bc - b^2 \dots + j(-b - ac - ab \dots) \quad (5)$$

This quotient must now be determined in terms of the current ratios and phase angles of the two transformers. The ratio of a current transformer is simply the ratio of the magnitude of the primary current, I_p , to that of the secondary. But since we are now regarding the currents as vector quantities, we must write for the ratios

$$R_s = \frac{I_p}{I_s}(\cos \alpha_s + j \sin \alpha_s) \quad (6)$$

$$R_x = \frac{I_p}{I_x}(\cos \alpha_x + j \sin \alpha_x) \quad (7)$$

where the complex factors in the parentheses have been introduced to make the ratios themselves (R_s and R_x) simple numbers instead of vector quantities. (The complex factor in each case merely turns the vector I_p through the angle α , into coincidence with the secondary current.) It is to be noted that this assumes that the phase angle, α , is positive when the reversed secondary current leads the primary current.

From (6) and (7) we have

$$\frac{R_x(\cos \alpha_s + j \sin \alpha_s)}{R_s(\cos \alpha_x + j \sin \alpha_x)} = \frac{I_s}{I_x} \quad (8)$$

or rationalizing

$$\frac{R_x\{\cos(\alpha_s - \alpha_x) + j \sin(\alpha_s - \alpha_x)\}}{R_s} = \frac{I_s}{I_x} \quad (9)$$

Equating (5) and (9) and separating the real and imaginary terms, we get

$$\frac{R_x}{R_s} \cos(\alpha_s - \alpha_x) = 1 + a - bc - b^2 \dots \quad (11)$$

and

$$\frac{R_x}{R_s} \sin(\alpha_s - \alpha_x) = -b - ac - ab \dots \quad (12)$$

Solving these equations simultaneously, we have

$$\frac{R_x}{R_s} = \left\{ (1 + a - bc - b^2)^2 + (-b - ac - ab)^2 \right\}^{1/2} = 1 + a - \frac{b^2}{2} - bc \quad (13)$$

and

$$\tan(\alpha_x - \alpha_s) = b + ac \dots \quad (14)$$

which are the formulas desired.

The deduction for the case when the slide C is to the left of B (Fig. 2) is similar to the above with slight differences in the second-order terms.

For the case where C is on the same side of B as transformer X , then

$$\frac{R_x}{R_s} = 1 - a + a^2 - \frac{b^2}{2} - bc$$

$$\tan (\alpha_x - \alpha_s) = b - ac - ab$$

where

$$a = \frac{r_2}{r_1}, \quad b = \frac{\omega M}{r_1}, \quad c = \frac{\omega L_1}{r_1}$$

and $\omega = 2\pi \times$ frequency.

r_1 = the total resistance between C and H through r_1 and M (Fig. 2), in ohms.

r_2 = the resistance of the slide wire between B and C , in ohms.

L_1 = the self-inductance of the secondary coil of the mutual inductance M , in henrys.

M = the value of the mutual inductance, in henrys. (This is to be taken as positive if α_x is greater than α_s as shown by the test described below.)

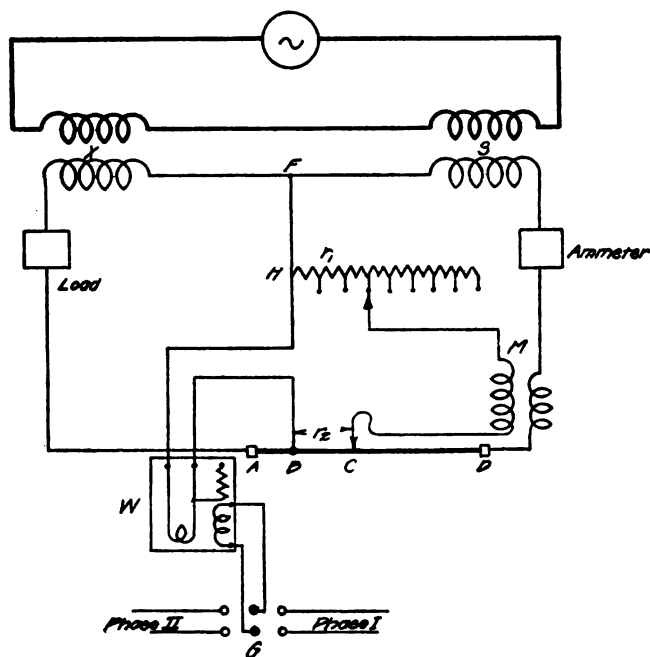


FIG. 2

The second-order terms are usually very small and to an accuracy of a tenth per cent we have in either case

$$R_x = R_s (1 \pm a)$$

$$\alpha_x = \alpha_s + 3438 b$$

if the angles are expressed in minutes.

If the balance is obtained with the slider between *B* and *D* (Fig. 2), then (provided the second-order terms are less than *a*), the standard transformer is supplying the greater secondary current and consequently has the smaller ratio. The question which secondary current leads the primary current by the greater angle can best be determined once for all by adding some resistance to the load on transformer *X*. This will always make *I_x* lead more than before and if the value of *M* required for a balance is increased then *I_x* was leading originally and *M* and *b* are to be taken as positive with this connection of the mutual inductance.

In the case when the slider *C* is to the right of *B* (Fig. 2) and the part *AB* of the slide wire is omitted, then the measuring circuits introduce no impedance whatever into the equivalent secondary circuit of transformer *X*. It is therefore possible by this arrangement to test a current transformer under the condition of zero external load. The load on transformer *S* or, in case *C* is to the left of *B*, on both transformers, is practically the same as the impedance of the slide wire and the primary winding of *M*. The slight change in this, due to the fact that *r₁* in effect shunts *r₂*, is of the order *ar₁*, and its effect on the performance of either transformer is entirely negligible.

It is evident that if *S* always had a smaller ratio than *X* the slider would always be on the standard side of *B* and the part *AB* of the slide wire would be unnecessary. This arrangement is very desirable since it does away with the necessity of allowing for the resistance of *AB* in making the total impedance load on transformer *X* equal to the desired value. The ratio of the standard transformer could be reduced by omitting a few turns from the secondary winding in the process of manufacture. In completed transformers of the hole type this can still be done by passing a few secondary turns through the hole in a direction to oppose the main secondary winding.

Since at the point of balance no current flows through the moving coil of the detector the self-inductance of this coil has no effect on the results. The mutual inductance between the two

coils of the wattmeter, however, does produce an electromotive force between the points *B* and *F* even when no current flows and therefore affects the setting. This source of error can easily be eliminated by shifting the position of the control springs so that the normal zero of the instrument corresponds to the position of zero mutual inductance.⁷

If the current in Phase I of the exciting circuit is in phase with the current in the transformers and Phase II is 90° from this, then the slider *C* may be adjusted with *G* closed to the right, and *M* with *G* to the left, and the two settings will be independent of each other. It is not at all necessary, however, that these phase relations be exact and Phase II may be 60° from Phase I,

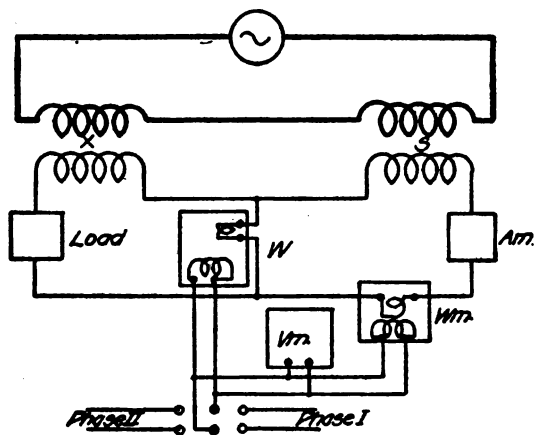


FIG. 3

as in a 3-phase system. If a polyphase supply is not available, current for Phase I may be drawn through a lamp bank and Phase II from the same supply through a reactance coil or transformer winding. With this arrangement it is well not to work the iron of the core much above its normal rating or the harmonics in the exciting current will become excessive.

DEFLECTION METHOD

In cases where a variable mutual inductance and slide wire are not available, but a polyphase supply is at hand, the deflection method (the connections for which are shown in Fig. 3) may

⁷ This position can be determined by short-circuiting the moving coil (or better still by connecting it to a large inductance of low resistance, such as the 220-volt winding of a transformer) and passing full-rated current through the fixed coil. Under these conditions the pointer deflects toward the position of zero mutual inductance, and by suitably moving the control springs the desired position can soon be located. An alternative method is to connect a telephone receiver to the moving coil and pass full-rated current through the fixed coil. The springs can then be adjusted until no sound is heard in the telephone.

be used. The detector W in this method must be of the dynamometer type, and arranged so that one coil may be supplied from two circuits, one giving a current in phase with the current in the transformers and the other giving a current having a known phase relation, preferably quadrature with the former. If D_1 and D_q are the deflections in divisions, observed with the excitation in phase, and in quadrature, respectively, we have

$$\frac{R_x}{R_s} = 1 + \frac{K D_1}{I_s}$$

$$\tan (\alpha_x - \alpha_s) = \frac{K D_q}{I_s}, \text{ approximately,}$$

where K is the constant of the instrument in amperes per division. If the wattmeter is excited by passing a current I_o through its current coil,

$$K = \frac{k}{I_o R_v}$$

where k is the constant in watts per division for a given range and R_v is the resistance of the voltage circuit of the same range. If the wattmeter is excited by applying a voltage E_o to the voltage circuit, then

$$K = \frac{k}{E_o}$$

In case the second phase available as a source of excitation is not in quadrature with the first, but gives exciting current which lags behind the current in the transformers by an angle θ and produces a corresponding deflection D_θ , then we have

$$\frac{R_x}{R_s} = 1 + \frac{K D_1}{I_s}$$

$$\tan (\alpha_x - \alpha_s) = \frac{K}{I_s} \left(\frac{D_\theta - D_1 \cos \theta}{\sin \theta} \right)$$

As before, the question which secondary current is the greater and which leads the primary current by a greater angle can best be answered by changing the load on one transformer and noting the effect on the deflections, remembering that an increase in secondary resistance makes the secondary current smaller and also makes it lead the primary current more.

The principal limitation of this method is, as mentioned above, the effect of the impedance of the detector in shifting the load from one transformer to the other. The change in the equivalent impedance load on either transformer due to this is given by

$$\Delta Z = \frac{z_d \Delta I}{I_s} = \frac{z_d K \sqrt{D_1^2 + D_q^2}}{I_s}$$

Since this varies with the deflection it is impracticable to allow for this impedance in arranging the secondary loads on the transformers and it is necessary to reduce ΔZ not to a comparatively small and definite value, as in the null method, but to a value which can be entirely neglected. If the moving coil of a commercial wattmeter is used as a detector, it must therefore be shunted by a rather low resistance and the sensitivity correspondingly reduced. The current circuit of a 1-ampere wattmeter has about the desired impedance and will be found satisfactory as a detector for this work. The use of the voltage circuit for the excitation is also convenient, since it avoids the necessity of a separate resistance.

The mutual inductance between the coils also introduces an equivalent load on the transformers, but if over the part of the scale used the coil is reasonably near the position of zero mutual inductance, this error will be less than that due to the impedance.

As a specific example of the various factors entering into the choice of a detector, let us consider the commercial wattmeter used in the experiments described below, which has the following constants:

Nominal current.....	2. amperes.
Maximum current.....	3. amperes.
Nominal voltage.....	75. volts.
Maximum voltage.....	110. volts.
Resistance of fixed coil.....	0.27 ohm.
Resistance of moving coil.....	42. ohms.
Resistance of voltage circuit.....	1,178. ohms.
Watts per division.....	.5

When this instrument is used with excitation on the voltage circuit we have

$$K = \frac{0.5}{110} = 4.5 \times 10^{-3} \text{ amperes per division,}$$

$$z_d = 0.27 \text{ ohm, approximately.}$$

When used with excitation on the current circuit the moving coil must be shunted to give a resistance of about 1 ohm so that

$$K = \frac{0.5}{3 \times 1178} \times \frac{42}{1} = 6.0 \times 10^{-8} \text{ amperes per division,}$$

$z_d = 1 \text{ ohm, approximately.}$

If the two transformers differ by 1 per cent in current, then at the 5-ampere point we have in the first case the deflection

$$D = 8.3 \text{ divisions}$$

and $\Delta Z = 0.003 \text{ ohm,}$

and in the second case

$$D = 11. \text{ divisions}$$

and $\Delta Z = 0.01 \text{ ohm.}$

A further error arises if the exciting voltages are not in the correct phase. The magnitude of this effect varies greatly with the difference in constants of the two transformers. If this difference is 1 per cent in ratio and 30 minutes in phase angle, a shift in phase of 7 degrees in the exciting voltage will produce an error of about 0.1 per cent in ratio and 3 minutes in angle. An auxiliary 5-ampere wattmeter connected as shown at *Wm* in Fig. 3 is a convenient means of determining these phase relations.

EXPERIMENTAL RESULTS

Both of the methods described above have been tried experimentally and the results are plotted on a large scale in Fig. 4. Curves *A* and *B* give the ratio factor and phase angle, respectively, of a 25-ampere, 40 volt-ampere portable current transformer at 60 cycles. The curve is drawn through values obtained by a precise laboratory method. The crosses are points observed by the null method using as a standard a similar transformer carrying considerably less load. Curves *C* and *D* give the ratio factor and phase angle of another transformer which was tested by the deflection method. As before, the curves are drawn through the points obtained by a precise laboratory method and the circles show the values obtained by the method here proposed.

SUMMARY

A general method has been outlined for the determination of the ratio and phase angle of current transformers in terms of the constants of previously calibrated standard transformers of the

same nominal ratio. It has been shown that such methods are essentially more sensitive, or conversely may be used with much less sensitive instruments, than the laboratory methods now in use for the absolute determination of the ratio and phase angle of a single transformer. Two of the most convenient of the many possible modifications of the general method are described in

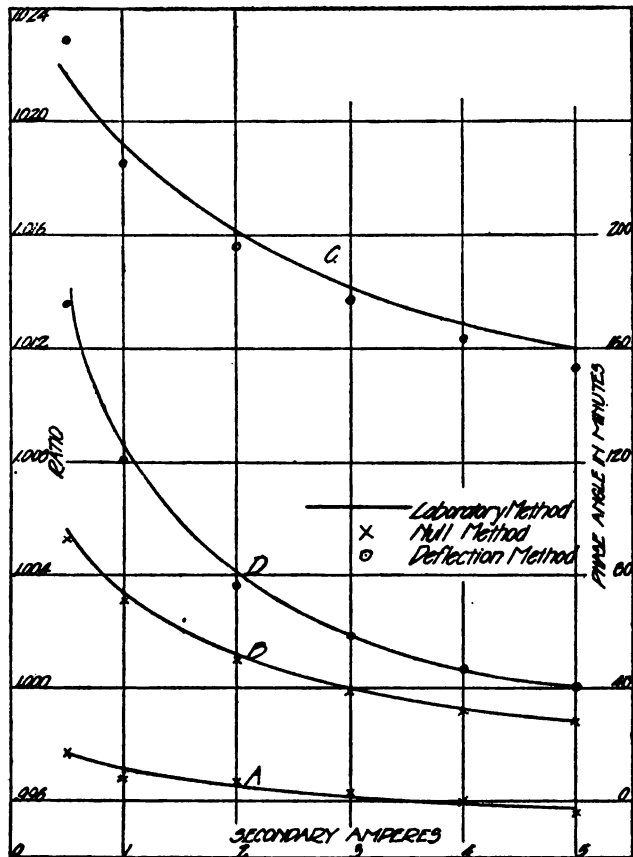


FIG. 4

detail. The experimental results obtained on two pairs of transformers by these methods agree very satisfactorily with the values obtained by absolute methods. It is hoped that the methods will be found useful in commercial plants where delicate laboratory equipment is not available, and where large numbers of transformers must be tested rapidly and with moderate accuracy.

WASHINGTON, July 1, 1917.

BUREAU OF STANDARDS

S. W. STRATTON, Director

ISSUED JUNE 24, 1916

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SOME ELECTRICAL PROPERTIES OF SILVER SULPHIDE

By George W. Vinal

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1. INTRODUCTION

In the course of another investigation the author was led to examine some of the electrical properties of silver sulphide. It was found that the sulphide could be rolled into thin strips or drawn into short wires like a metal. The electrical behavior of the sulphide is peculiar. It is properly to be classed as a pyro-electric¹ conductor, but it possesses certain characteristics which further study may show to be useful in various practical applications. In the form of a strip it is both a metallic and electrolytic conductor at the same time, but it may be prepared as a metallic conductor of negligible temperature coefficient. Observations have been made on its resistance to alternating and direct current, its volt-ampere characteristics, its specific resistance, its electrochemical decomposition, etc. The results of only a part of the experiments which were planned are contained in this paper, which is published now because it does not appear possible to continue the investigation on account of the pressure of other duties.

2. PREPARATION OF SPECIMENS

The native silver sulphide, argentite, is seldom pure, but silver sulphide may be prepared chemically in the form of a black powder. This powder may be fused in a porcelain crucible by means

¹Steinmetz, General Electric Review, 19, p. 362; 1916.

of a blast lamp. The melting point is about 825°C , according to Pelabon.² It is necessary to keep the crucible tightly covered during the fusion to prevent the oxidation of the sulphur by the air, with the consequent liberation of metallic silver, which becomes plainly visible if any considerable amount is formed. The fused mass after cooling resembles gun metal in color. It is sufficiently malleable to permit the fused button to be rolled by small steps into a strip. In doing this the sulphide "cries" like tin and cracks along the edges somewhat worse than crystalline zinc. At a higher temperature (about 200°) it becomes very malleable and may be hammered out on a hot plate or drawn through a heated drawplate into short wires. Such working of the material changes its electrical properties, as will be shown later. Analysis of the fused sulphide, by Mr. Scherrer, showed about 3 per cent free silver over that corresponding to the formula Ag_2S , and slight traces of silica and iron.

For making experiments on the sulphide it was necessary to have some means of making good electrical and mechanical contact between the sulphide and copper wires. It has not been found possible to solder to the sulphide directly, and brass clamps are not desirable, but it is possible to silver plate the ends of a piece of the sulphide in a silver-potassium-cyanide solution. After this is done the sulphide can be soldered to the copper wire with comparatively little difficulty. The principal precaution to be observed in the soldering is to avoid overheating the sulphide.

3. METALLIC AND ELECTROLYTIC CONDUCTION

Bädeker³ found that below 175° the sulphide is an electrolytic conductor and above this temperature it is a metallic conductor, the transition being marked by an abrupt change in the electrical resistance as the temperature passed this point in either rising or falling. His specimens were prepared by sputtering thin films of silver on glass or mica and then converting the silver into sulphide. In this respect Bädeker's work has little bearing on the present investigation. Other papers dealing with this sulphide note that it has a high temperature coefficient of resistance.

(a) *Temperature Coefficient.*—In the present investigation it was found that a strip of the sulphide rolled at room temperature showed a large negative temperature coefficient of resistance, but it was difficult to repeat the readings with exactness, although

² Comp. Rend., 148, p. 594.

³ Ann. d. Phys., 327, p. 758; 1907.

the strip was in an oil bath. A short wire of the sulphide drawn through a drawplate at about 200° was observed at eight temperatures in the range 18° to 31° , with the result that the temperature coefficient was found to be zero. The explanation of this difference, as deduced from other observations, appears to be that the strip rolled at room temperature conducts the current both metallically and electrolytically at the same time, while the wire is a metallic conductor. The large temperature coefficient of the strip was due to the electrolytic conduction.

It is not easy to explain why mechanical working of a substance such as this makes so radical a change in electrical properties, but it seems likely that rolling the sulphide at room temperature may produce minute cracks. The wire which was drawn hot is probably homogeneous, due to the mechanical working. It was thought that if a strip of the sulphide were put in a vacuum it might lose its electrolytic properties, but this did not prove to be the case. A strip, with fine copper conducting leads, was sealed in a desiccator over phosphorous pentoxide and evacuated as far as possible with an oil pump. After several hours in the vacuum measurements were made, but no change in the peculiar behavior (described in the next section) of the strip could be detected. After standing over night it was found that the resistance had increased greatly and visible decomposition of the sulphide had begun. It had lost its metallic luster entirely.

(b) *Comparison of Alternating and Direct-Current Resistance.*—One of the earliest experiments made was a comparison of the resistance of the specimens on alternating and direct current. It was found that the resistance of the strip rolled at room temperature was usually higher for the alternating-current measurements than for the direct current. To eliminate the possibility that this effect might be due to temperature changes, the specimens were mounted in a thermostatic oil bath controlled to 0.01° and vigorously stirred. A special bridge was arranged to make measurements on either alternating or direct current by merely throwing a switch and interchanging a telephone and galvanometer. The measurements could therefore be made under similar conditions and nearly simultaneously, but the result was the same as before. Measurements made on the hot-drawn wire did not, however, show these peculiarities of the strip. Its resistance with direct and alternating current was always the same.

Although the bridge currents were very small, it was found that the use of the alternating current produced steadily increas-

ing values of resistance while the direct current decreased the values of the resistance. As successive measurements were made the resistance with alternating and direct current differed more and more. This effect was also produced when a small auxiliary alternating (60 cycles) or direct current was passed through the specimen for a few seconds or a minute and then the resistance measurements made. An increase in resistance always resulted from the passage of the alternating current and a decrease from the direct current. Fig. 1 shows the effect of 8.8, 14, and 18.6 milliampere, 60-cycle alternating current applied for one-minute intervals, as indicated by the double lines. During the periods

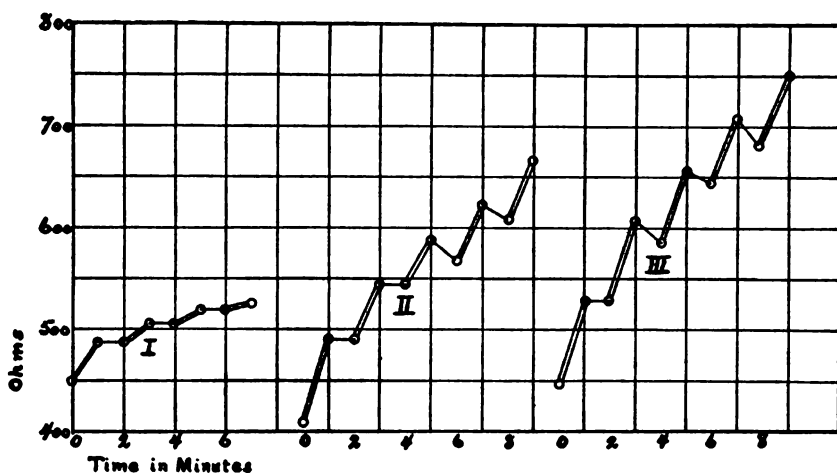


FIG. 1.—Resistance of a strip of silver sulphide as affected by a small alternating current. An Alternating current of 60 cycles was passed through the sulphide for one-minute intervals as indicated by the double lines. Resistances were measured on an alternating-current bridge. The current for curve I was 8.8 ma; for II, 14.0 ma; for III, 18.6 ma

denoted by the single lines only the bridge current of about 5 milliamperes was flowing through the specimen. The downward direction of the single lines in the upper part of curves II and III of Fig. 1 and the upward direction of the single lines in the D. C. curve of Fig. 2 show that the effect produced is not a permanent one. Complete recovery is often a matter of considerable time.

After making these observations it was found that Fitz Gerald ⁴ had previously observed similar differences in resistance on alternating and direct current, but the explanation which he gives does not seem to fit the present observations noted at top of page that successive measurements of resistance made alternately and

⁴ Trans. Am. Electrochem. Soc. 25, p. 393, 1914.

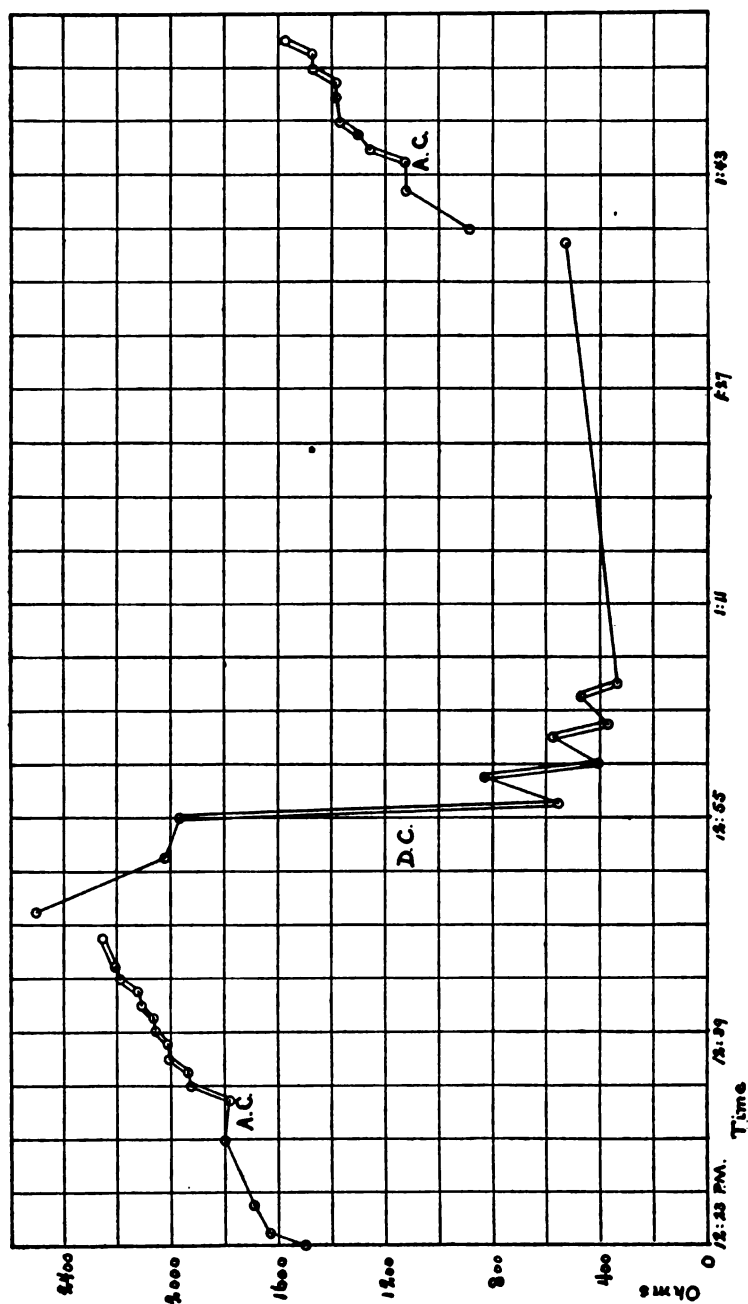


FIG. 2.—Resistance of a strip of silver sulphide as affected by small alternating and direct currents. A 60-cycle alternating current of 19 ma was passed through the strip for 10-second periods during the latter part of the intervals indicated by the double lines of the 1st and 3rd curves. Similarly a direct current of 30 ma was passed through the strip for 10-second periods of the middle curve

in rapid succession with alternating and direct current differed more and more as each new measurement was made.

Fig. 2 gives somewhat similar results for another specimen (strip) of the sulphide. These results do not show quite the perfect regularity of those in Fig. 1, but are given to illustrate the contrasting effects of alternating and direct currents on the sulphide. This specimen was in the large oil bath, thermostatically controlled. The measurements on alternating and direct current were made in quick succession, as will be noted from the time scale, time being plotted as abscissa.

Experiments similar to the foregoing were tried with the wire which was drawn hot, but no such effects were observed. The alternating and direct current measurements were in agreement, and no effect was produced when auxiliary currents were passed through it. The wire behaved like a metal.

It is possible that the decrease in resistance of the strip when a direct current is passed through it may have some relation to the "voltage effects" ⁵ noted in the case of selenium and stibnite cells. Steinmetz ⁶ says that some pyroelectric conductors have the property that their resistance increases permanently, often by many hundred per cent, when the conductor is for some time exposed to high-frequency electrostatic discharges. The observations of the present paper indicate that silver sulphide is sensitive to low frequency and that it may recover its former value.

(c) *Volt-Ampere Characteristic.*—The volt-ampere characteristic of a strip in air (4.7 by 0.4 by 0.007 cm) was measured with both alternating and direct currents. It was found to be similar to that of a pyroelectric conductor. Some difficulty was encountered in making the alternating-current measurements because of the peculiar behavior of the sulphide. When the current passing through the specimen was small the resistance was very high, but when relatively large currents were forced through it the resistance became very low. As a result the low-scale alternating-current voltmeters took most of the current, and it was not found practicable to make corrections. Mr. H. B. Brooks very kindly arranged and operated an alternating-current potentiometer to overcome this difficulty. A small correction had to be made for the current through the potential coil of the wattmeter that was also included in the circuit. The direct-current measurements were easily made. A portable galvanometer with resistance

⁵ Elliott, *Phys. Rev.*, 5, p. 53; 1915.

⁶ Loc. cit.

in series was used for reading the voltage. The results of these measurements are given in Fig. 3. It was not possible to repeat the whole series of measurements on the wire, but direct-current measurements were made with the same apparatus as was used for the strip. The wire was in the thermostat bath, where it had been kept from the beginning of the experiments. The maximum current density in this specimen was 43 amperes per square centimeter. The curve (Fig. 4) is characteristic of metallic conduction, and taken in connection with the agreement of the bridge

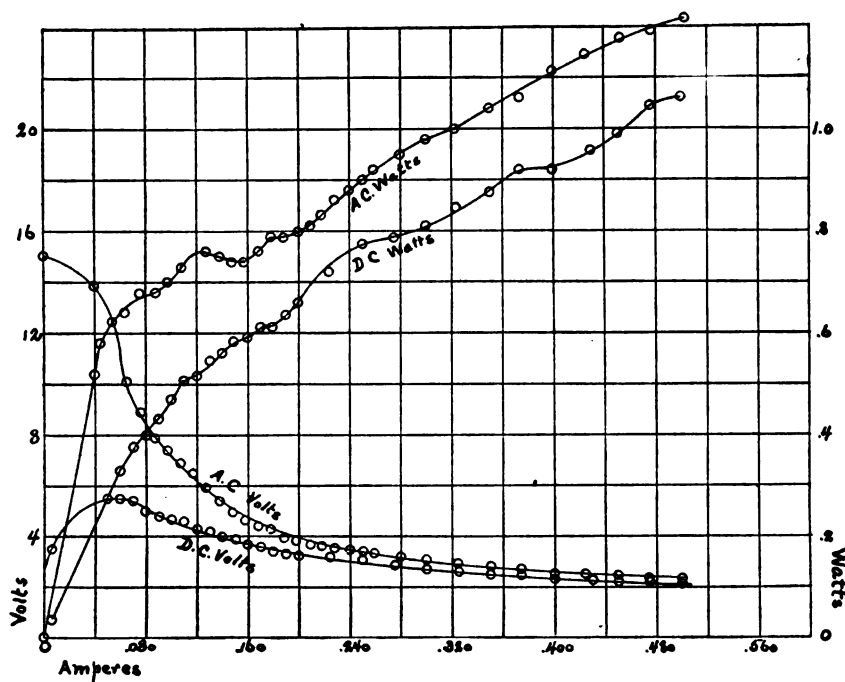


FIG. 3.—Volt-ampere characteristic of a strip of silver sulphide on alternating and direct current

measurements for both alternating and direct current it indicates that the sulphide drawn hot is a metallic conductor at ordinary temperatures, within the range of these experiments. The conditions for these experiments were quite different from those of the bridge measurements, the results of which are given in Figs. 1 and 2. Probably they are not as correct at the very low currents.

4. SPECIFIC RESISTANCE

Measurements were made on the wire to determine its specific resistance at 25°C. It was found to be 17 300 microhm-centimeters, or about 10 000 times the specific resistance of copper.

5. ELECTROCHEMICAL DECOMPOSITION

Experiments were made to find the electrochemical decomposition due to the electrolytic conduction of the strips of sulphide. A strip in air (5.5 by 0.3 by 0.01 cm) with silver-plated ends (Fig. 5) was soldered to copper leads and put in a direct-current circuit. The initial current of 25 milliamperes was passed through it for nearly an hour without visible change. The current was increased by steps of 50 milliamperes at 10-minute intervals until with 200 milliamperes a discoloration of the plating at the anode

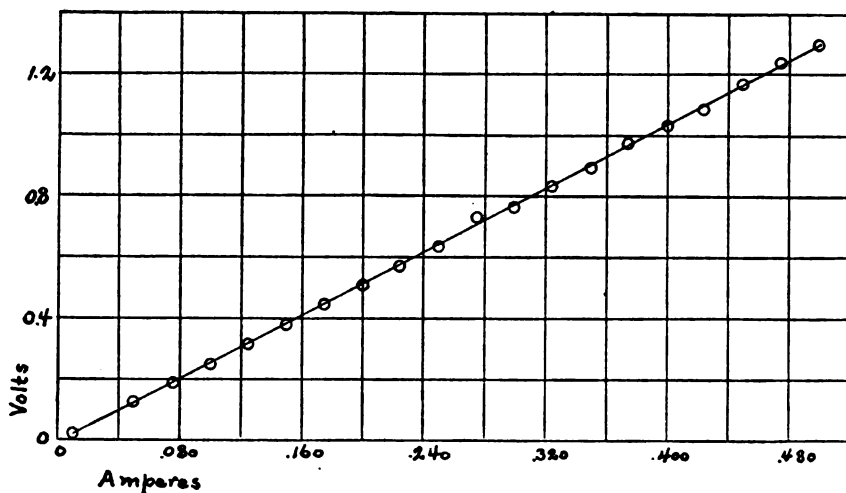


FIG. 4.—Volt-ampere characteristic for a wire of silver sulphide drawn hot, measured with direct current

end was noticed. A further increase to 300 milliamperes completed the destruction of the silver plating at the anode end and finally burned off the terminal, but before this happened a myriad of little shiny silver crystals appeared on the black surface of the sulphide, as shown in Fig. 6. The appearance of these crystals was carefully studied under the microscope, and it was found that they occurred even to within a small fraction of a millimeter of the anode terminal. (Fig. 7.)

They appeared in various forms, some of which suggested that they had been expelled from the interior of the sulphide by considerable force. The strip of silver sulphide appeared to be made up of a multitude of tiny electrolytic cells between which metallic conduction occurred. The appearance of the silver shown in Fig. 6 suggests a possible cause for the occurrence of native silver



FIG. 5.—Silver plating at cathode end of strip. No visible change took place in the plating at this end. This picture may properly represent also the appearance of the plating at the anode end before it was destroyed by electrolysis. $\times 50$



FIG. 6.—Outcropping of silver in middle of strip due to the electric current which flowed approximately parallel to the direction of the lines of silver. $\times 50$



FIG. 7.—Anode end showing where the terminal burnt off. After the silver plating was destroyed by the sulphur, the outcropping of silver took place, even within a small fraction of a millimeter of the anode end. $\times 50$

in argentite⁷ as an electrochemical product. The lines of silver in Fig. 6 suggest veins of silver in the earth.

An experiment similar to this was carried out with alternating current, but no visible decomposition of the sulphide was observed, although the current was carried to 600 milliamperes, which was twice the direct current that completed the destruction of the other strip of sulphide.

6. MISCELLANEOUS EXPERIMENTS

Experiments were also made to determine the thermoelectromotive force of the silver sulphide against copper, but the results were inconclusive. This was probably due to the fact that a strip instead of a wire was used. The latter, if available in a long enough piece to make a suitable thermocouple, would probably show more constant values. Other experiments indicated some microphonic action when two pieces of the sulphide were pressed together and in circuit with a telephone receiver and battery.

No data have been obtained as to the stability of silver sulphide. Clarke⁸ says that it is less stable than the sulphides of copper, lead, and mercury, and quotes Thomsen's values for the heat of formation of these sulphides, showing that the heat of formation of silver sulphide is only about one-fourth of that for lead sulphide.

7. SUMMARY

Silver sulphide may be prepared in the form of short wires or thin strips like a metal. The wire which must be drawn hot, has been found to conduct electricity like a metal of high specific resistance and practically zero temperature coefficient. The strip of sulphide, rolled at room temperature, has a large temperature coefficient and shows both metallic and electrolytic conduction at the same time. It has a volt-ampere curve characteristic of a pyroelectric conductor. The resistance of these strips has been examined with both alternating and direct current, with the result that the alternating-current resistance was nearly always found to be higher than that with the direct current, and the passage of a small alternating current of a frequency as low as 60 cycles increased temporarily the resistance of the sulphide, while a small direct current produced the opposite effect.

WASHINGTON, June 26, 1917.

⁷ Clarke, *Data of Geochemistry*, p. 539.

⁸ *Data of Geochemistry*, p. 561.

AXIAL ABERRATIONS OF LENSES

By E. D. Tillyer and H. I. Shultz

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I. INTRODUCTION

The definition in the center of the field of a lens system is generally assumed to be good, but its relative importance in different lens systems varies greatly. In a telescope objective, where only the center of the field is used, the errors there require strict correction, while those which affect only the edge of the field can be neglected. In a projection lens the same is true, but to a lesser degree, since more of the field is used than in a telescope objective. Here the central corrections are important, though often neglected because of the low cost of the lens and because the observer sitting at some distance from the projected picture does not see small imperfections in the image.

In photographic lenses a large field is very desirable—really a requirement—and more attention must be paid to the definition at the edge of the field than in the previously mentioned cases.

Here a compromise must often be made between the center and the edge of the field. In these systems the color must be well corrected to avoid chemical focus or the blurring of the image, due to the actinic rays focusing nearer to (or farther from) the lens than the visual. Especially is the color correction important in lenses used for color photography and in process lenses.

In astrographic lenses, those used in stellar photography, a field is required which is large relative to telescopic systems but small compared with lenses for ordinary photography. Over this field they are required to be corrected very accurately with regard to all aberrations, though color is of secondary importance, since a color screen, or plates which are sensitive only to a short range of the spectrum, may be used.

In portrait lenses the central definition is of greater importance than that of any other part of the field, since focusing on the eyes of the sitter will give sharp definition there and produce the effect of concentrating the attention on the expression without directing it to less important details.

In all cases the central definition is very important and the determination of the errors in the center of the field will generally give a comparative value of the lens. The central corrections show, with very few exceptions, the highest quality of definition a lens will give. Even in a lens with a large field, when a compromise must be made between the center and edge of the field, the central definition must be at least as good as the edge, if not better.

II. PURPOSE OF INVESTIGATION

The following investigation was undertaken to determine the errors of a complex lens system which affect the definition of the image near the center of the field, and to compare the various types of lenses on the market with respect to their corrections for central definition and their applicability to various purposes for which they might be used. The lenses investigated were in the following principal classes: Photographic lenses of relatively large aperture; projection lenses for stereopticon and motion picture work; telescope lenses of short focal length and large aperture and telescopic systems.

III. ERRORS AFFECTING AN OPTICAL IMAGE.

It seems advisable to define and describe the errors found in optical systems before proceeding to discuss the method used in determining them. Each one of these errors is called an aberra-

tion, there being seven principal aberrations, namely: (1) Axial chromatic, (2) oblique chromatic, (3) spherical aberration, (4) zonal variation of equivalent focal length (sine condition), (5) curvature of field, (6) astigmatism, and (7) distortion.

1. AXIAL CHROMATIC ABERRATION.—When a ray of white light (Fig. 1) passes through the lens LL at Q a prism action is exerted on the ray and the white light is broken up into its component elements. The red component, or the ray of longer wave length, is brought to a focus at a point P_1 , farther from the rear lens surface than the violet component or the ray of shorter wave length, which is focused at P_2 . The intervening wave lengths are focused between the points P_1 and P_2 . This error is known as axial

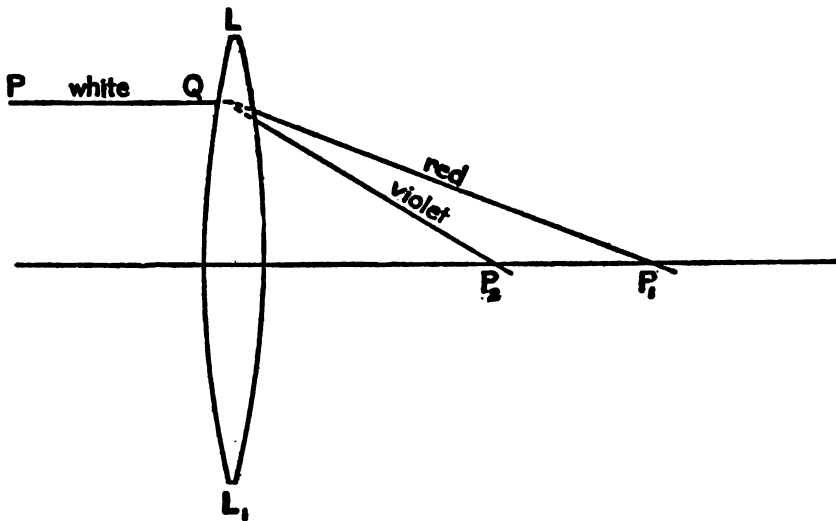
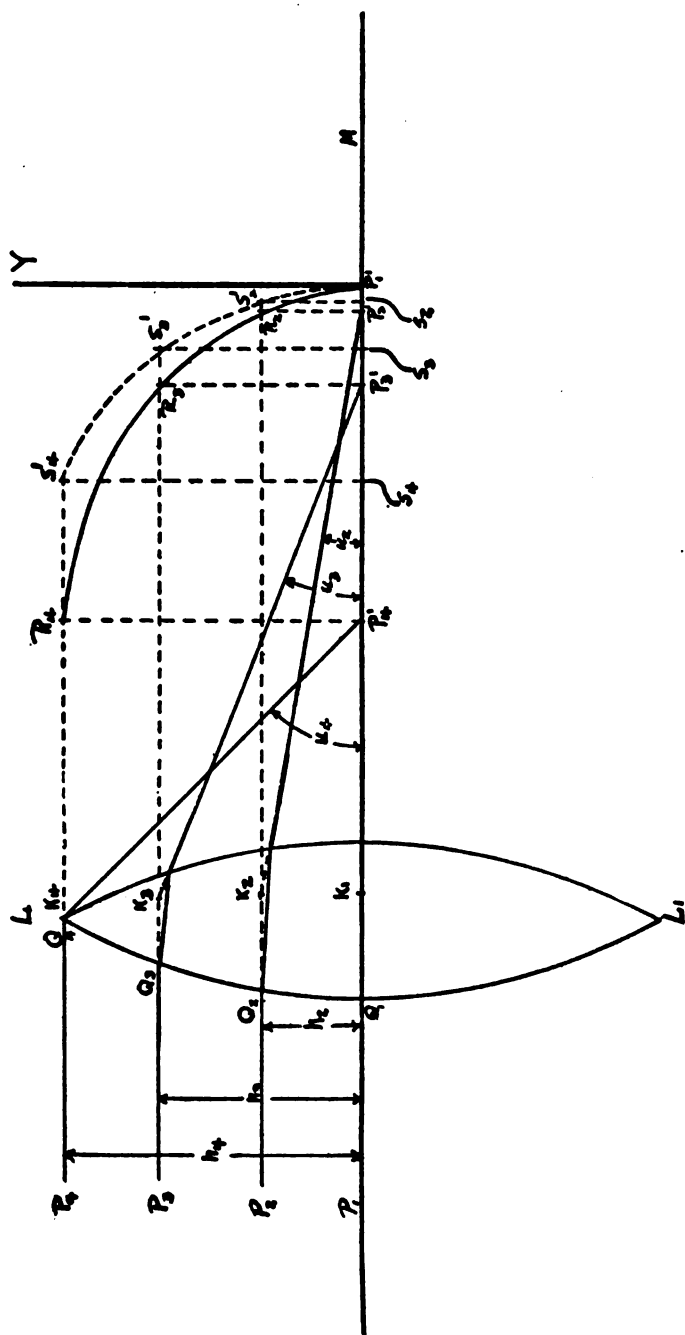


FIG. 1.—Diagram illustrating axial chromatic aberration

chromatic aberration. The amount of chromatic aberration, or the distance $P_1 P_2$, is dependent upon the focal length of the lens and the dispersion of the glass. By combining two lenses of proper focal lengths and glasses of suitable refractive indices and dispersions the axial chromatic aberration can be minimized—that is, two colors such as red and blue—may be brought to a focus at the same point and the other colors be made to focus very close to this point.

2. OBLIQUE CHROMATIC ABERRATION.—Even when all the colors are focused at the same point the equivalent focal lengths of the various colors may not be the same, and each color may give a different sized image, causing the familiar colored fringes to large



objects. A particular example of this is found in the lenses used in three-color photoengraving, where if the focal length for the red is slightly longer than that for the blue, there will be a red fringe to the image. This error is known as oblique chromatic aberration.

3. SPHERICAL ABERRATION.—Let LL_1 (Fig. 2) be a convex lens, with the axis P_1M , and P_1, P_2, P_3 , and P_4 be four parallel rays striking the lens at points Q_1, Q_2, Q_3 , and Q_4 at heights h_1, h_2, h_3 , and h_4 from the axis. The rays, after being refracted by the lens, will cut the axis at some point. Let the ray from P_1 traveling infinitely near the axis cut the latter after refraction at P_1^1 ; then P_1^1 will be the focus for a paraxial ray (ray very near the axis). The ray P_2 , striking the lens at a height h_2 from the axis will,

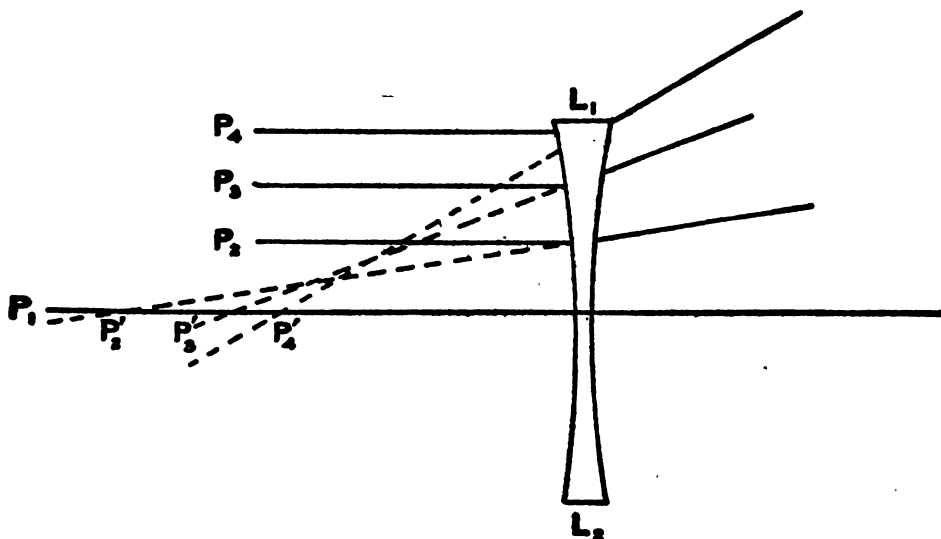


FIG. 3.—Diagram illustrating spherical aberration in a negative lens

after refraction, be brought to a focus at P_2^1 . The ray from P_4 , striking the lens at the extreme edge at a height h_4 from the axis, will be brought to a focus at P_4^1 , nearest to the lens. If V represent the paraxial focal length, then $P_1^1P_4^1 = \Delta V$ is known as the axial spherical aberration.

If we draw Q_2R_2 , Q_3R_3 , and Q_4R_4 parallel to the axis, and erect the perpendiculars at P_2^1, P_3^1 , and P_4^1 , and draw a curve through the intersections P_1^1, R_2, R_3 , and R_4 , we obtain a curve which represents the spherical aberration along the axis for any zone of the lens at a distance h from the axis.

In a negative lens (Fig. 3) the ray through the edge zone, if prolonged backward, would cut the axis at a point P_4^1 nearer the lens than the point P_1^1 , which is the focus for paraxial rays.

But here the spherical aberration $P_1'P_1' = \Delta V$ is opposite in sign from that of the positive lens. The size of ΔV is dependent on the shape of the lens, for a given focal length and kind of glass. By combining a positive and a negative lens of the proper shapes and of glasses and focal lengths previously determined by other conditions (axial chromatic, Petzval, and oblique chromatic in photographic lenses, but only axial chromatic in telescope objectives), the spherical aberration of the negative lens may be made to counteract that of the positive lens, and thus be eliminated. In the absence of spherical aberration the curve $P_1'R_1$ will coincide with the line $P_1'Y$.

Variation of Spherical Aberration with Color.—The spherical aberration curves for all colors do not coincide nor do they run parallel to each other, since the axial chromatic aberration for different zones is different. Hence, if curves are drawn for light of different wave lengths, each curve would represent the axial spherical aberration for that wave length, the variation in the shape of the curves representing the variation of the axial spherical aberration with the color. The abscissa of each curve for any definite ordinate is the spherical aberration of the zone represented by that ordinate for the given color. The difference between two such abscissæ is the axial chromatic aberration of that zone for the two colors. Thus, the curves show spherical and chromatic aberrations, and the variation of the spherical with the color.

4. ZONAL VARIATION OF THE EQUIVALENT FOCAL LENGTH (E. F. L.) (SINE CONDITION).—Assume a lens $L L_1$ (Fig. 4) entirely corrected for spherical and chromatic aberrations. Then all zones will have the same axial focal point for all colors. If all the zones have not the same focal length, then the aberration called coma is present.

Let $P Q$ (Fig. 4) be a small object perpendicular to the axis. Then $P Q$ will be imaged in $P' Q'$ by the central zone of the lens $L L_1$. If the lens is free from spherical and chromatic aberrations, the point P will be focussed by the edge zone as well as by all other zones at P' ; but if the E. F. L. (equivalent focal length) of the different zones be different, the edge zone will have a different magnification from the central zone and the object $P Q$ will be imaged in $P' Q'$ by the central zone, and in $P' Q''$ by the edge zone. In other words, the edge zone will focus the point Q in Q'' in the same plane but in a different position from Q' , the image by the central zone. If Q be a finite point, then Q' will be a very small disk of somewhat larger diameter than the disk pro-

duced at Q'' . Hence the point Q will be imaged in a figure shaped like a comet, having a bright point at one end and trailing off in a broad fuzzy circle at the other. The sharp point will be directed toward or away from the axis depending on the relative sizes of the focal lengths for the edge and central zones.

The graphical representation of zonal variation of E. F. L. is shown in Fig. 2. Prolong the rays to P_1' , P_2' , P_3' , and P_4' backward into the lens till they meet the rays P_1 , Q_1 , P_2 , Q_2 , P_3 , Q_3 , and P_4 , Q_4 prolonged at K_1 , K_2 , K_3 , and K_4 (K_1 is obtained by computing the axial focal length and measuring back from P_1'). The points K_1 , K_2 , K_3 , and K_4 are the principal points for the corre-

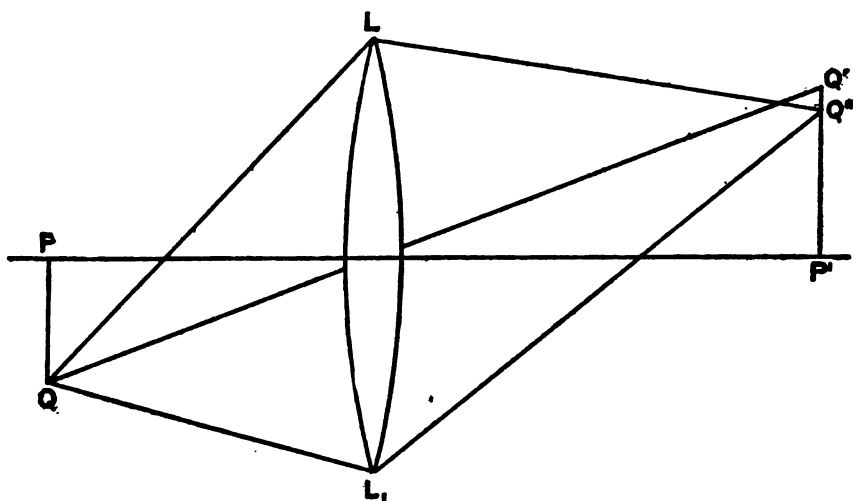


FIG. 4.—Diagram illustrating image by lens free from spherical aberration, but having zonal variation of E. F. L.

sponding zones, and the surface on which all these points lie is the principal surface of the lens.

Let h be the height at which the incident ray strikes the lens and u be the angle which the refracted ray makes with the axis.

Then $\frac{h}{\sin u} = E$ is defined as the E. F. L. of the zone at a distance h from the axis. In Fig. 2,

$$\frac{h_1}{\sin u_1} = K_1 P_1'$$

$$\frac{h_2}{\sin u_2} = K_2 P_2'$$

$$\frac{h_3}{\sin u_3} = K_3 P_3'$$

$$\frac{h_4}{\sin u_4} = K_4 P_4'$$

will be the equivalent focal lengths of the corresponding zones. If the lens is absolutely free from spherical aberration, then the condition

$$\frac{h}{\sin u} = \text{constant} \quad (a)$$

means also freedom from zonal variation of the E. F. L.; hence freedom from coma for points near the axis. Equation (a) is commonly known as the "Sine condition" for rays from an infinitely distant object. For artificially stopped systems and at large angles with the axis other additional terms have to be considered.¹

In practice, no lens is entirely corrected for spherical aberration. Some of the latter is left in to balance the higher order aberrations and to compromise for the best condition for the elimination of the other errors. In this case, as is shown below, the condition for freedom from coma is

$$\frac{h}{\sin u} - f = 0 \quad (b)$$

where f for the zone h is the distance from the axial principal point K_1 to the intersection of the ray from that zone with the axis. In Fig. 2

$$\begin{aligned} f_1 &= K_1 P_1' \\ f_2 &= K_1 P_2' \\ f_3 &= K_1 P_3' \\ f_4 &= K_1 P_4' \end{aligned}$$

Let $L L$ (Fig. 5) be a lens with spherical aberration present. Then the rays from the edge and center will intersect the axis at P' and P'' , respectively. Assuming $\frac{h}{\sin u} = \text{constant}$, every zone will have the same magnifying power, and there will be a series of images of the object $P Q$ at all points from P' to P'' , all of the same size. But the position of the plate will be the position of the average image somewhere between P' and P'' , say, $G G$. The image at $P'' Q''$ for the edge zone, when produced forward to the plate $G G$, will become a large disk in the position mm' . The image for the central zone produced back to $G G$ will be a small disk at m'' . At Q^n there will be a sharp image produced by some zone between the center and the edge of the lens. Following the

¹ The greatest effect on the definition is due to these seven errors; but there are certain unnamed aberrations, consisting of higher order terms, which affect the image to a smaller extent. Czapski gives 12 terms of the fifth order and 20 terms of the seventh order, besides the five terms of the third order.

same procedure, the other zones will produce disks of diameter smaller than mm' . The figure on $G G$ will consist of a bright point at m'' trailing off into the broad fuzzy perimeter of the disk mm' , giving the true shape of coma.

In order to obtain a symmetrical image in a lens with spherical aberration present, the series of out-of-focus images in any position $G G$ (Fig. 6), formed by producing the rays of the various images

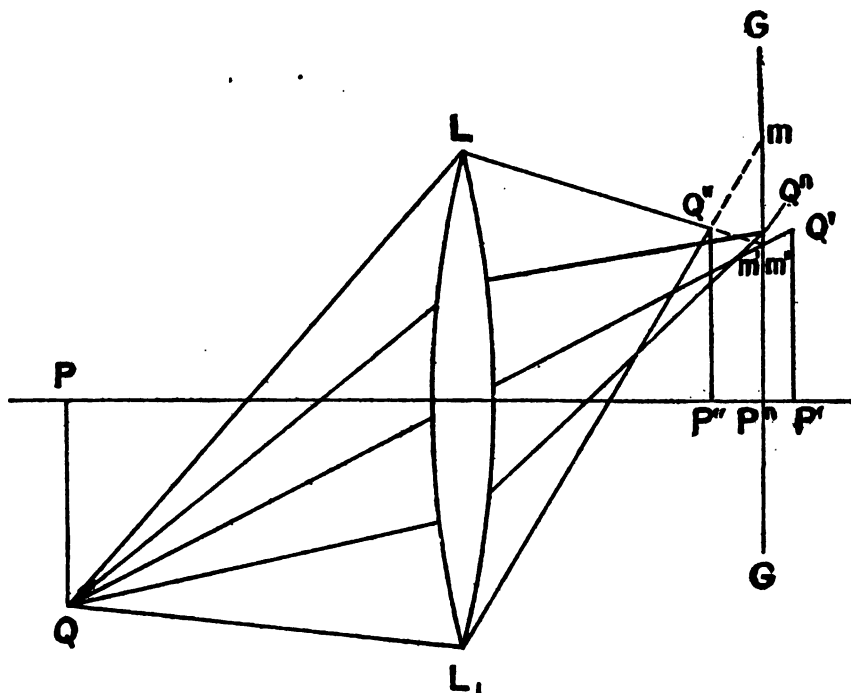


FIG. 5.—Diagram illustrating image by lens possessing spherical aberration, but free from zonal variation of E. F. L. (Condition $\frac{h}{\sin u} = \text{const. fulfilled}$)

Q' to Q'' , must all be symmetrically placed with respect to the ray $K Q'$, through the central zone. In Fig. 6,

$$\frac{K P^n}{K P'} = \frac{P^n Q^n}{P' Q'}$$

but

$$\frac{P^n Q^n}{P' Q'} = \frac{\frac{h}{\sin u_n}}{\frac{h_1}{\sin u_1}} = \frac{E_n}{E'} = \frac{E_n}{E_{ax}}$$

and

$$K P' = F_{ax}, K P^n = f_n$$

where the subscript "ax" refers to the axial zone.

Hence,

$$\frac{f_n}{f_{ax}} = \frac{E_n}{E_{ax}}$$

For the axial zone,

$$E_{ax} = f_{ax} = K_1 P_1'.$$

Hence $E_n - f_n = 0$ is the condition for a symmetrical image free from comatic flare in a lens with spherical aberration present.

If we plot along the axis $P'Y$ (Fig. 2) the height h , and along $P'P$ the variation ΔE in the equivalent focal lengths of the zones from that of the central zone, we obtain the curve $P'S'$ repre-

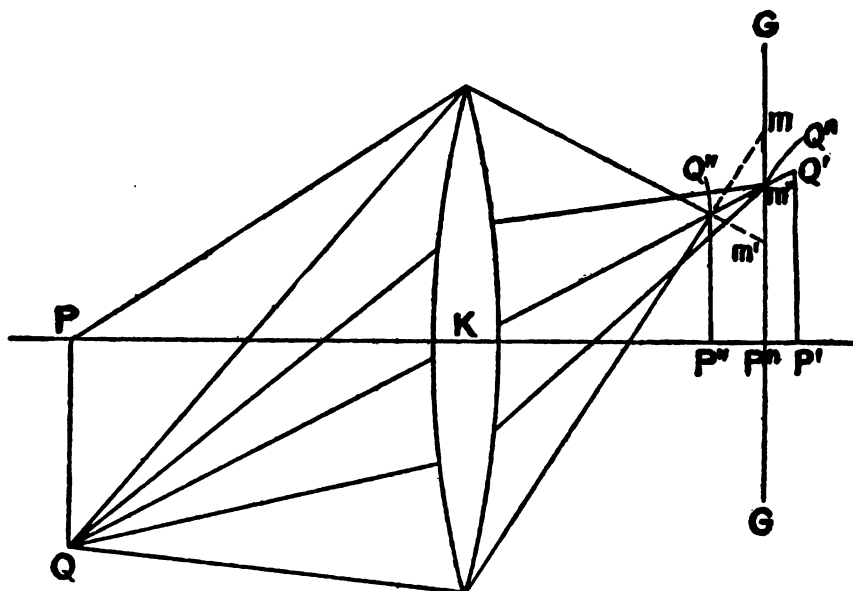


FIG. 6.—Diagram illustrating image by lens with condition $\left(\frac{h}{\sin u} - f = 0\right)$ fulfilled

sents the zonal variation of the E. F. L. If we represent the E. F. L. of the zone h by E_h and that of the axial zone by E_{ax} , then

$$\Delta E = E_h - E_{ax}$$

Similarly, if f_{ax} is the distance from the paraxial principal point to the paraxial focal point and f_h is the distance from the same principal point to the focal point for the zone h , then the spherical aberration $\Delta V = f_h - f_{ax}$.

When expressed in terms of the aberrations ΔE and ΔV , the condition $E_h - f_h$ becomes

$$E_h - f_h = (E_{ax} + \Delta E) - (f_{ax} + \Delta V) = \Delta E - \Delta V,$$

since $E_{ax} = f_{ax}$.

In Fig. 2 the full line represents ΔV and the dotted one is ΔE , hence the condition for freedom from coma for points near the axis, $\Delta E - \Delta V = 0$, means that the two curves coincide.

Variation of Sine Condition with Color.—If the zonal variation of the E. F. L. be determined for different wave lengths, the curves may not coincide, showing a variation of ΔE with the color. Since the size of the image depends on the E. F. L., the zonal variation of the E. F. L. with color gives the variation of the size of the image with the color and hence the oblique chromatic aberration.

5. CURVATURE OF FIELD.—The preceding aberrations are the ones to be given most consideration when examining the central definition of a lens. When we depart somewhat from the center of the field and go toward the edge, we find that the image of a plane object does not lie in a plane. If we plot the points of best focus, we find them forming a surface of a saucerlike shape. This defect is known as curvature of field.

6. ASTIGMATISM.—Toward the edge of the field we find that there may not be a position of good focus. The image may consist of a series of streaks running radially, and somewhere in a plane other than that of the radial image another series of streaks in a direction perpendicular to the first. This is due to the fact that a pencil of rays from a point striking the lens in one plane focuses at a point different from the focus of a pencil of rays from the same point but in a plane perpendicular to that of the first pencil. This error is known as astigmatism, and is nonexistent in the center of the field of a lens with good surfacing and not strained in the mounting.

7. DISTORTION.—Another aberration which affects the image toward the edge of the field is distortion. This is the deformation of a straight line in the object into a curved line in the image due to a variation of the angular magnifying power of a lens for rays incident at different angles.

Besides the preceding aberrations, we must take into account the so-called hybrid aberrations, or the effect of some of the previously mentioned errors on the others. These affect the definition at the edge of the field and are exceedingly important in photographic lens systems where a large field is very desirable.

For the purpose of ascertaining the best use to which a lens of a given description may be put it is necessary to determine the magnitude of these aberrations, and the region of best correction. The significance of the axial aberrations from the point of view of

general definition has been discussed in a previous paragraph. Hence the determination of these axial aberrations will give us a comparative value of the lens. To do this the method described below was devised.

IV. METHOD

The method used is an extension of Hartmann's method of objective testing.³ His work is fully described in the *Zeitschrift für Instrumentenkunde*. Hartmann uses a screen to isolate rays along four diameters, 45° apart. He takes shadowgraphs at points very near the focus and from these he obtains the spherical aberration and the astigmatism along the axis. For the zonal variation of the E. F. L. he places the lens with the mount in contact with a scale and measures the apparent length of the scale divisions through the lens with a micrometer. Hartmann's method has been applied to many large telescope objectives and small lenses. The method is tedious and complicated and requires two totally independent sets of measurements to obtain both the spherical aberration and the zonal variation of the E. F. L. Moreover, for comparatively thick lenses it is not possible to use the method for the zonal variation of the E. F. L. The extension described later gives both the errors with a single measurement and can be used for all kinds of lenses of relatively large aperture and short focal length and for complete telescopic systems.

1. DESCRIPTION.—Fig. 7 is a diagrammatic sketch of the method. Parallel rays from a distant source fall on a metal screen SS pierced with small holes which isolate single rays. These rays are then refracted by the lens system under test, LL , and come to a focus at F , or near F , in the presence of spherical aberration. A photographic plate is placed in the position A and a shadow photograph taken. The plate is then moved to the position B on the other side of the focus and another shadow photograph taken. The shadow photographs have center diffraction disks which can readily be measured if care is taken to choose a screen with holes of proper diameter and distance apart. If the plate is near the focus, as at B , sharp disks are obtained; at A , farther away from the focus, the disks are less sharp.

Consider a ray, R (Fig. 7), passing through the screen and striking the lens at a height, h , from the center. It is refracted by the lens system making a disk on the plate A at m , and on the plate B at n . If M and N are the positions of the disks made by

³ *Objectivuntersuchungen., Zeitschrift für Instrumentenkunde*, 24, pp. 1, 33, 97; 1904.

the ray through the center of the lens, then a and b will be the distances from the center of the plates A and B to the intersections of the ray R with the plates. The position F will be given by the expression $V = \frac{ad}{a+b}$ where V is the distance MF from the fixed point M , and d is the distance between the two positions of the plates. Assuming that V_0 is the distance of the paraxial focal

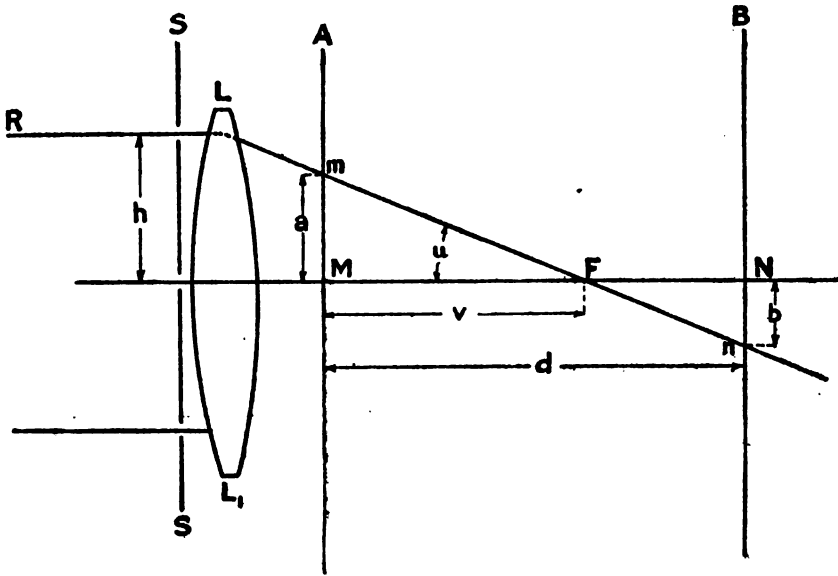


FIG. 7.—Diagrammatic sketch of methods

point from M , then the spherical aberration for the zone at a distance, h , from the lens axis is $\Delta V_h = V_h - V_0$.

In Fig. 7 u is the inclination of the refracted ray with the axis and

$$\tan u = \frac{a+b}{d}$$

from which $\sin u$ is easily determined. Hence, $\frac{h}{\sin u}$ for each zone can at once be determined if h is known. For this purpose, the screen used was calibrated and the position of the holes was known to within 0.005 mm.

The apparatus used consisted of the following: R (Fig. 8) is a monochromatic source of light, which in this case was a Fuess monochromator giving a fairly pure band of the spectrum of about $4\mu\mu$ in width in the violet ($425\mu\mu$) to about $21\mu\mu$ in the red ($650\mu\mu$).

The illuminant was a Nernst filament close to the slit. The width of the band of spectrum was determined with a Hilger wave-length spectrometer, which had been checked up by means of a mercury arc. The monochromator was placed about 20 feet away from the screen, and corrections applied to reduce the results to infinity. The screen *S* was a metal plate accurately centered with respect to the lens holder. The holes used for fairly large-sized lenses were 1 mm in diameter and spaced at intervals of 3 mm apart. For smaller lenses a screen with smaller holes spaced closer together was used, permitting a suitable number of zones to be determined. If the holes are too small, or too near together, the diffraction disks will not be clearly defined and it will be impossible to measure them. The lens holder, *L*, is of the iris diaphragm type, so that the lens is always centered with respect to the holder, and therefore with respect to the screen. The lens holder rotates about a ver-

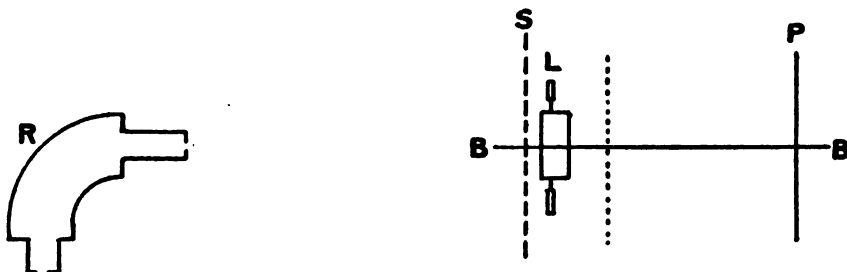


FIG. 8.—Diagrammatic sketch of apparatus

tical axis, permitting the lens to be centered with respect to the photographic plate. The photographic plate *P* slides on an optical bench *B B* along a scale. The plate can be raised or lowered, permitting several exposures to be made on the same plate.

2. METHOD OF MANIPULATION.—The lens is placed in the holder, care being taken that it is fairly well centered. It is then accurately centered by reflection from the back of the lens. For this purpose a ground-glass plate with a little 90° prism cemented to it (leaving a clear space in the ground glass surrounding the prism) is placed in the plate holder. The prism is used to send a beam of light to the rear lens surface. When the beam is reflected on itself by the lens and comes back to the observer through the clear space around the prism, then the lens is normal to the direction of motion of the plate, and the centering is accurate.

The monochromator is then set to give a beam of wave length $425\mu\mu$, the plate placed at a considerable distance in front of the

focus and an exposure made. The plate is then lowered, the monochromator changed to give a beam of wave length $475\mu\mu$ and another exposure made. This is repeated with the monochromator set at $550\mu\mu$ and $650\mu\mu$. The plate is then moved through a distance d , previously determined, so that the plate now comes behind the focus. The exposures are repeated through the wave lengths $650\mu\mu$, $550\mu\mu$, $475\mu\mu$, and $425\mu\mu$. The length of exposure is timed so as to get uniform density on the plate for all colors, giving disks which can easily be measured on a Zeiss horizontal com-

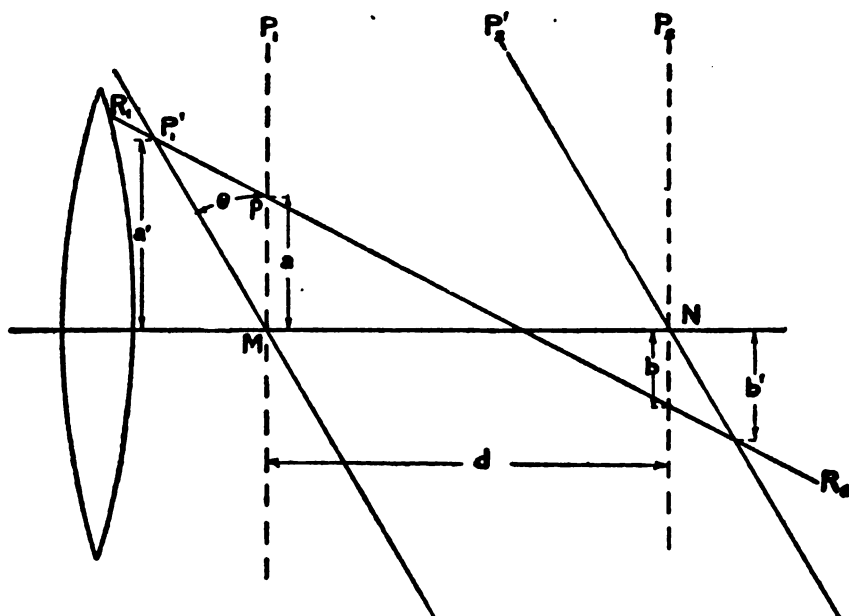


FIG. 9.—Diagram illustrating a tilt of the plate to the lens axis

parator or measuring engine with sufficient accuracy for rays somewhat removed from the axis of the lens.

3. SOURCES OF ERROR.—The lens was carefully centered in the method described above. A slight tilting of the plate would produce an error in ΔV and ΔE of at least the second order.

To find the effect of a slight tilting of the photographic plate on the value of V , let us assume that θ be the angle that the plate $P'1$ (Fig. 9) makes with the perpendicular to the lens axis $P1M1$, which position the plate should occupy. The distance $MF = V = \frac{a}{a+b} = \frac{a'}{a'+b'}$, which shows that there is no error produced in V

by a tilt in the plate through an angle θ to the perpendicular to the axis, and hence no errors in ΔV .

The effect of the tilt θ of the plate on the value of E can be determined as follows:

$$E = \frac{h}{\sin u} u, \text{ but for purposes of discussing the errors } \tan u = \frac{a+b}{d}$$

can be substituted for $\sin u$ so that $E = \frac{h}{\tan u} = \frac{h d}{a+b}$

Let the plate be in the position $P_1' M$ (Fig. 9) making an angle θ with the position $P_1 M$ where the plate ought to be. The distance $d = M N$ will not be affected. The only quantities entering in E which will be affected by the angle θ will be a and b .

Let $P_1' M = a'$, as measured, and

$p M = a$, the distance from M to the point p where the ray $R_1 R_2$ would strike the plate in its perpendicular position.

Then, in the triangle, $P_1' P_1 M$

$$\angle P_1' M p = \theta$$

$$\angle M p P_1' = (90 + u)$$

$$\angle P P_1' M = 180 - (90 + u) - \theta = 90 - (u + \theta)$$

$$M P_1' = a$$

$$P M = a'$$

$$\text{then } \frac{a}{a'} = \frac{\sin (90 + u)}{\sin [90 - (u + \theta)]} = \frac{\cos u}{\cos (u + \theta)}$$

$$a = a' \left(\frac{\cos u}{\cos (u + \theta)} \right)$$

A similar expression can be derived for b .

The angle u in any case is never large, the largest angle obtainable in an $f:2$ lens is one whose $\tan$ is one-fourth, which is about 13° . The angle θ is very small, due to care taken in centering. Hence, $\cos u$ is not very different from $\cos (u + \theta)$ and a may be taken equal to a' . Similarly, $b = b'$. Hence, the error in E due to a tilt of the photographic plate may be neglected.

The error in ΔE is still smaller, as ΔE is the difference between two quantities in which the error is systematically in the same direction.

The error in ΔE , due to an error in the known value of h , is one of the first order, as $E = \frac{h}{\sin u}$. The metal screen used was calibrated so that the value of h was known to within 0.005mm, corresponding to an average error in E of less than 0.05 per cent.

An error in the distance d affects both V and E systematically in the same direction.

$$V = \frac{a}{a+b} d$$

$$E = \frac{h}{a+b} d$$

Assuming an error in d , equal to δ , then the spherical aberration, ΔV and the zonal aberration ΔE of any two zones will be, respectively,

$$\begin{aligned}\Delta V &= \frac{a_1(d+\delta)}{a_1+b_1} - \frac{a_2(d+\delta)}{a_2+b_2} \\ \Delta V &= \left(\frac{a_1}{a_1+b_1} - \frac{a_2}{a_2+b_2} \right) d + \left(\frac{a_1}{a_1+b_1} - \frac{a_2}{a_2+b_2} \right) \delta \\ &= \text{spherical} + \text{spherical} \cdot \frac{\delta}{d} \\ \Delta E &= \frac{h_1(d+\delta)}{a_1+b_1} - \frac{h_2(d+\delta)}{a_2+b_2} \\ \Delta E &= \left(\frac{h_1}{a_1+b_1} - \frac{h_2}{a_2+b_2} \right) d + \left(\frac{h_1}{a_1+b_1} - \frac{h_2}{a_2+b_2} \right) \delta \\ &= \text{zonal variation} + \text{zonal variation} \cdot \frac{\delta}{d}\end{aligned}$$

Hence the percentage error in the spherical aberration and in the zonal variation of the E. F. L., due to an error in d , is the same, as the percentage error in d . The value of d in all the lenses measured lies between 200 and 400 mm, and the error in d is about 0.1 mm. This gives a percentage error of from 0.05 to 0.02 per cent, which is negligible. In the following discussion of errors we will neglect the effect of an error in d .

For a discussion of the effect of an error in a and b on the final results, we have the formulas

$$V = \frac{a \cdot d}{a+b} \quad \tan u = \frac{a+b}{d} \quad E = \frac{h}{\sin u}$$

$$\partial V = \frac{a\partial a + b\partial a - a\partial a - a\partial b}{(a+b)^2} d + \frac{a}{a+b} \partial d$$

Neglecting ∂d and substituting $\tan u$ for $\frac{a+b}{d}$

$$\partial V = \frac{b\partial a - a\partial b}{(a+b)} \cdot \frac{1}{\tan u} \quad (1)$$

From (1) we see that ∂V is dependent on ∂a and ∂b . In practice ∂a and ∂b are smaller, the smaller a and b , but negligibly so, so that for an approximation ∂a and ∂b may be considered independent of the size of a and b . Hence, we may assume $\partial a = \partial b$, and $a = b$, approximately; then, since ∂a and ∂b may be either plus or minus,

$$\partial V = \frac{1.4a\partial a}{2a} \cdot \frac{1}{\tan u} = \frac{1.4\partial a}{2\tan u} = \frac{1.4\partial b}{2\tan u}$$

Hence ∂V is dependent on $a + b$, only as the latter is contained in $\tan u$.

$$E = \frac{h}{\sin u} = \frac{h}{\tan u} = \frac{hd}{a+b}$$

$$-\partial E = h \cdot d \left(\frac{\partial a + \partial b}{(a+b)^2} \right) = \frac{\partial a + \partial b}{(a+b)} \cdot \frac{1}{\tan u} \cdot h$$

If, now, we assume $\partial a = \partial b$ and $a = b$, then

$$\partial E = \frac{1.4\partial a}{2a} \cdot \frac{h}{\tan u} \text{ or } \frac{1.4\partial b}{2b} \cdot \frac{h}{\tan u}$$

Hence, the larger $(a+b)$ the smaller ∂E , the error in E .

$$(E - V) = \frac{hd - ad}{(a+b)} = \frac{d(h-a)}{a+b}$$

$$\partial \frac{(E - V)}{d} = \frac{-(a+b)\partial a - (h-a)(\partial a + \partial b)}{(a+b)^2}$$

After expanding and collecting terms, this becomes

$$\partial \frac{(E - V)}{d} = \frac{-(h+b)\partial a - (h-a)\partial b}{(a+b)^2}$$

Again assuming $\partial a = \partial b$, and $a = b$

$$\partial \frac{(E - V)}{d} = \frac{-h\partial a - a\partial a - h\partial a + a\partial a}{(2a)^2} = \frac{1 \cdot 4(-h\partial a - a\partial a)}{4a^2}$$

$$\frac{\partial(E - V)}{d} = \frac{-1 \cdot 4(h+a)\partial a}{4a^2}$$

h is about twice the size of a , hence

$$\partial(E - V) = 4 \cdot 2 \frac{d \cdot (-a\partial a)}{4a^2} = -\frac{d}{a}\partial a, \text{ approximately.}$$

Hence, an error of δa in a or δb in b will cause an error in $(E - V)$ equal to $d\frac{\partial a}{\partial a}$ or $d\frac{\partial b}{\partial b}$.

The errors due to measurement and to irregularities in the photographic plate were examined by taking two plates of the same lens and having two observers measure the same plate. The following table gives the results thus obtained. P_1 and P_2 are the measurements on plates 1 and 2, $P_1 - P_2$ being their difference. In the column $T - S$ is given the variation between two observers. Because of the physical indeterminateness of the values for the first two zones, they may be disregarded, and the probable error of a single observation, then, is 0.06 mm in V and 0.04 mm in E for two plates. There is a slight systematic difference between the two plates, which may be due to different intensities in the shadow-graphs or to a speck of dust changing the shape of the shadow-graph.

The probable error of a single observation for two observers as obtained from the columns $T - S$ is 0.02 mm for V and 0.05 mm for E . The systematic error here is very slight.

Comparison of Measurements

V (spherical aberration)				E (sine condition)				
P ₁ (first plate)	P ₂ (second plate)	P ₁ -P ₂	T-S difference between 2 observers	P ₁ (first plate)	P ₂ (second plate)	P ₁ -P ₂	T-S difference between 2 observers	
mm	mm	mm	mm	mm	mm	mm	mm	
425 μ	138.91	138.80	-0.11	-0.10	212.82	212.61	+0.21	+0.17
	8.44	8.66	-0.22	-0.00	2.92	3.03	-0.11	+0.13
	8.26	8.40	-0.14	-0.04	2.44	2.60	-0.16	+0.03
	7.94	8.09	-0.15	-0.00	2.36	2.40	-0.04	-0.14
	7.74	7.90	-0.16	+0.01	2.06	2.16	-0.10	-0.10
	7.84	8.02	-0.18	+0.02	2.18	2.32	-0.14	-0.02
	8.62	8.78	-0.16	+0.01	3.02	3.15	-0.13	-0.07
475 μ	8.74	8.88	-0.14	-0.02	2.36	2.96	-0.63	-0.50
	8.50	8.55	-0.05	-0.07	2.78	2.95	-0.17	+0.18
	8.14	8.35	-0.04	-0.03	2.44	2.67	-0.23	-0.10
	7.83	8.00	$\pm$ 0.00	-0.04	2.24	2.24	0.00	-0.08
	7.52	7.70	-0.01	+0.03	1.88	1.98	-0.10	-0.08
	7.52	7.65	+0.04	-0.04	1.92	1.96	-0.04	-0.04
	8.06	8.23	$\pm$ 0.00	-0.02	2.41	2.58	-0.17	-0.01
550 μ	8.92	9.36	-0.44	+0.17	2.74	2.66	+0.08	+0.62
	8.90	8.74	+0.16	+0.13	3.04	2.95	+0.09	+0.38
	8.33	8.56	-0.23	-0.03	2.69	2.76	-0.17	-0.20
	7.98	8.07	-0.09	+0.01	2.30	2.44	-0.14	+0.02
	7.57	7.72	-0.15	+0.03	1.97	2.06	-0.09	-0.10
	7.44	7.58	-0.14	-0.03	1.81	1.94	-0.13	-0.09
	7.81	7.98	-0.17	+0.02	2.33	2.38	-0.05	+0.01
650 μ	9.12	9.38	-0.26	+0.06	3.72	3.58	+0.24	-0.48
	9.06	9.29	-0.23	-0.16	3.15	3.79	-0.64	-0.09
	8.65	8.91	-0.16	-0.08	3.05	3.06	-0.01	-0.05
	8.19	8.40	-0.21	-0.05	2.52	2.60	-0.08	+0.02
	7.78	7.98	-0.20	-0.03	2.13	2.29	-0.16	-0.06
	7.56	7.69	-0.13	-0.03	1.90	1.99	-0.09	-0.04
	7.80	7.98	-0.18	-0.01	2.32	2.32	0.00	-0.02

The systematic error does not affect the results in any way, as the curve plotted is the difference of all the values from a mean value. In plotting, as described below, the probable error is reduced with the values of ΔV and ΔE by $\frac{100}{E. F. L.}$. Thus, in the case of a lens of 200 mm E. F. L., the probable errors are reduced by one-half. Very few of the lenses plotted were less than 175 mm E. F. L., while one was as high as 400 mm.

4. METHOD OF PLOTTING.—The method of plotting the results has been described in the discussion of spherical and sine condition aberrations. The mean values for V and E were so chosen as to show the places of best correction of the lenses in question. Thus, the values of the first two central zones in all cases were disregarded (in taking the mean), as these values are unreliable. In photographic lenses, where no provision is made for the correction of the red rays, the mean of the yellow-green, blue, and violet was taken. On the other hand, in projection and telescope lenses, the violet was disregarded, and the mean of the red, yellow-green, and blue was used. The mean values of V and E were then subtracted from each of the values of V and E , respectively, and the resulting differences reduced to 100 mm focal length. This procedure makes the vertical axis the mean value for both E and V . A deviation from the axis means an aberration. The dotted line shows the variation of the E. F. L., and the solid line shows the spherical aberration. The distance between the curves ΔE and ΔV is a measure of the coma.

The curves for each wave length are plotted separately, so as not to create confusion, but the vertical axis represents the same line and, for comparison, the figures should be superposed with the axes coinciding, or else the distances from $X' Y$ should be compared.

The vertical distances h have been magnified 4 times and the horizontal distances 20 times, in accordance with convention used by Von Rohr and others.

The wave lengths used were:

Limits of band	Mean	Color
	$\mu\mu$	
423-427 $\mu\mu$ (about).....	425	Deep violet.
475-479 $\mu\mu$ (about).....	475	Blue.
547-558 $\mu\mu$ (about).....	550	Yellow-green.
647-663 $\mu\mu$ (about).....	650	Red.

The "limits of band" gives the width of the band of spectrum used for each color.

5. APPLICABILITY OF METHOD.—The method can be successfully used on all optical systems, of relatively large aperture, and short enough focus to permit the plate to be placed sufficiently near the focus to obtain strong light, and still separate the disks a measurable distance. The diameter must be large enough to obtain several zones without crowding the holes in the screen too close together. Such lens systems are photographic lenses, short-focus telescope objectives, etc.

The method has been used on complete instruments, such as telescopes. In a complete telescope, the consideration for a theoretically perfect image requires the rays to be parallel on coming out of the eyepiece. In general, they will not all be

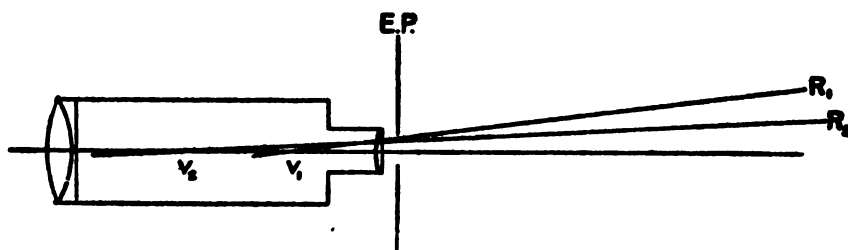


FIG. 10.—Method applied to complete telescope

parallel, but some will intersect the axis at some point behind the exit pupil. If we take two shadowgraphs, one near the exit pupil and one some distance behind it, we can obtain the intersections of the rays with the axis.

Let R_1 and R_2 (Fig. 10) be two rays intersecting the axis at distances V_1 and V_2 from the rear principal plane of the instrument. Then U_1 and U_2 will be such distances that

$$\frac{1}{U_1} = \frac{1}{f} - \frac{1}{V_1}$$

$$\frac{1}{U_2} = \frac{1}{f} - \frac{1}{V_2}$$

where f is the accommodation used by the observer

$$\frac{1}{U_1} - \frac{1}{U_2} = -\frac{1}{V_1} + \frac{1}{V_2} = -\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$

The quantity $-\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$ represents the range of accommodation the eye has to assume simultaneously in focussing the rays from the instrument on the retina. Since this is independent of f , we see that it does not depend on the observer's eye, nor on the amount of accommodation he requires in using the telescope, nor whether the telescope is focussed on a near object or a distant object, but depends only on the difference in power of the various zones.

6. ILLUSTRATIVE DESCRIPTION OF A SET OF CURVES.—To illustrate the meaning of a set of curves let us take the lens represented by Fig. 25. This is a foreign-made projection lens of the Petzval type for motion-picture projection work. A glance at the curves shows that the lens is best corrected for that region of the spectrum between $475\mu\mu$ and $550\mu\mu$, or between the yellow-green and the blue. Here the dotted curve and the full almost coincide, showing freedom from coma near the axis. At about 13 mm out from the lens center the curves show appreciable separation. This corresponds to an aperture $1:3.9$ $\left(\frac{100}{2 \times 13} = \frac{1}{3.9}\right)$.

At this aperture the maximum value of $\Delta E - \Delta V$ is 0.09 mm for wave length $550\mu\mu$ and 0.05 mm for wave length $475\mu\mu$ (disregarding the center zone of the lens); for wave length $425\mu\mu$, or the violet, the spherical is practically zero, but the two curves do not fit so closely, $\Delta E - \Delta V$ becoming equal to 0.15 mm. This is quite small, but it is three times that of wave length $475\mu\mu$ and twice that of wave length $550\mu\mu$.

This lens being designed for projection work, the color correction should be between the red and the blue; that is, the red and the blue would focus in the same point and the others not far away, with no attention paid to the extreme violet. The distance of the full line from the axis (the zero line) gives the amount of color aberration for the particular color. The average distances of the four colors from the axis, using the mean of the red, yellow-green, and the blue as zero, are as follows: Violet, -0.18 mm; blue, -0.13 mm; yellow-green, 0.00 mm; and red, $+0.22$ mm. This shows the red to be far out of achromatism, as it focusses 0.35 mm and 0.40 mm behind the blue and violet, respectively. The progression of the curves in the same direction (-0.18 mm to $+0.22$ mm) shows that no two colors focus at the same point, hence there is no color correction.

The distances of the dotted curves from the axis shows the variation of the E. F. L. with the color, hence the variation of the size of the image with the color. This is the oblique achromatism. These average distances for each color are as follows: Violet, -0.18 mm; blue, -0.13 mm; yellow-green, $+0.06$ mm; and red, $+0.25$ mm; that is, the red image is 0.4 per cent bigger than the violet, with the sizes of the other images in between. Hence the image projected by this lens shows red fringes on the outside and violet on the inside. This lens could be better used to advantage with a color screen, as each color is well corrected for itself; but the only color for which the variation is small is the blue, and there is no real achromatism as defined earlier in this paper.

The other sets of curves furnish similar information regarding the other lenses, allowing us to draw conclusions as to the best use of the lens as far as the center of the field is concerned.

V. CLASSIFICATION OF CURVES

The following curves divide themselves into groups of lenses of several classes. Figs. 11 to 18 represent various types of photographic lenses. The curves here run very smoothly, showing good work in surfacing, as most of these lenses are of a high-grade make. Figs. 14 and 15 are not so regular, but the residuals are so small as to be negligible in any case. Fig. 16 is of an old-type, rectilinear lens; the rest are modern anastigmats. It will be noticed how the spherical aberration in Fig. 16 does not begin to return to the axis, whereas in the case of the others they either cut the axis again or approach it.

Figs. 19 to 26 are projection lenses of the Petzval type.³ The curves are not smooth and regular, showing less care in surfacing than in the photographic objectives. The lenses represented by Figs. 19 to 23 were made in this country, the others being imported. It will be seen that our domestic lenses compare favorably with foreign ones. Figs. 24 and 26 show considerable coma. All the lenses of this group are of rather large aperture, the largest being $f:2.7$, approximately.

Fig. 27 is a telescope objective (aperture $f:3.3$) made by Zeiss. The lens is exceedingly well corrected for wave length $550\mu\mu$, which is the brightest yellow region of the spectrum, with the red and the blue not far away. The violet has been entirely

³ First brought out by J. Petzval in 1840. It is described in all books on optical instruments.

neglected. This is also true of the color correction of the projection lenses, as they all have to be used for visual work.

VI. SUMMARY

The errors which affect the definition of a lens are discussed and methods of representing graphically the central errors described. The condition for freedom from coma near the axis is arrived at. The relative importance of the errors in the different types of lenses is discussed. Hartmann's method is extended, permitting one set of measurements to give all the important central errors—spherical aberration, zonal variation of the equivalent focal length, axial and oblique achromatism. The apparatus and the procedure are described, and the accuracy of the adjustments and measurements discussed. The method is applicable to all systems of relatively short focus and large aperture, such as photographic lenses, projection lenses, and telescope objectives, and also to complete optical systems. The results of the method as applied to a complete telescope are discussed, and are shown to be independent of the accommodation of the observer. Seventeen sets of curves for as many different lenses are given, and an illustrative discussion of one set of curves, together with a general description of the types of lenses represented by each group of curves.

WASHINGTON, September 19, 1916.

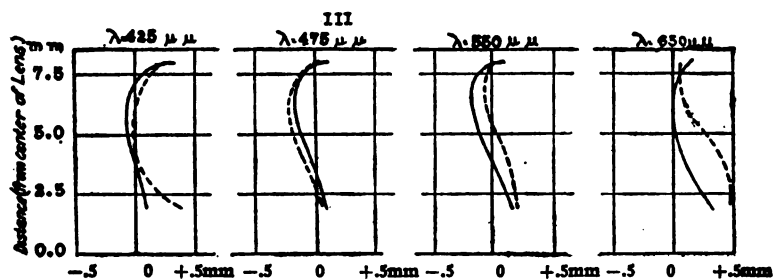


FIG. 11.—Photographic anastigmat lens

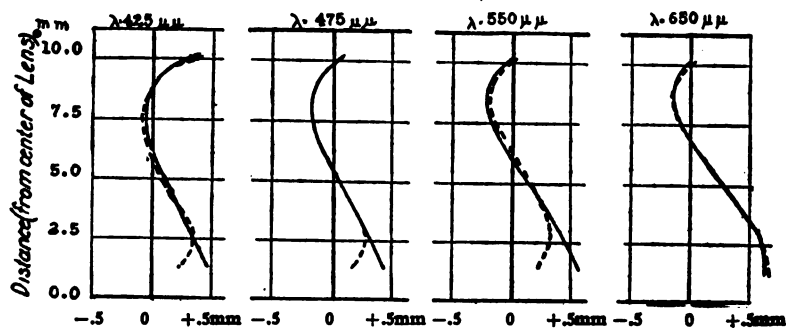


FIG. 12.—Photographic anastigmat lens

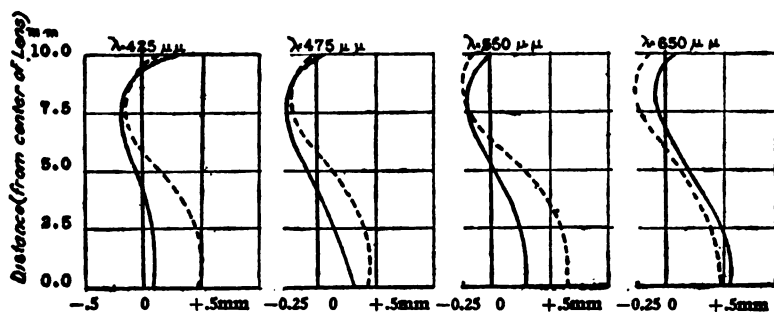


FIG. 13.—Photographic anastigmat lens

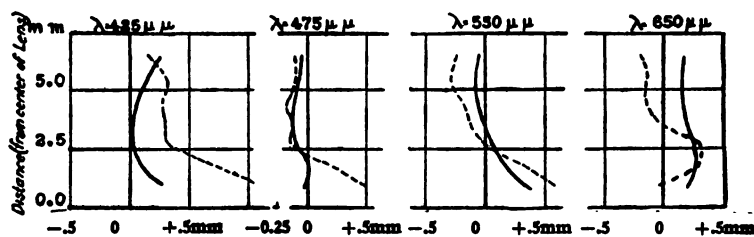


FIG. 14.—Photographic anastigmat lens

Spherical aberration —————
Sine condition

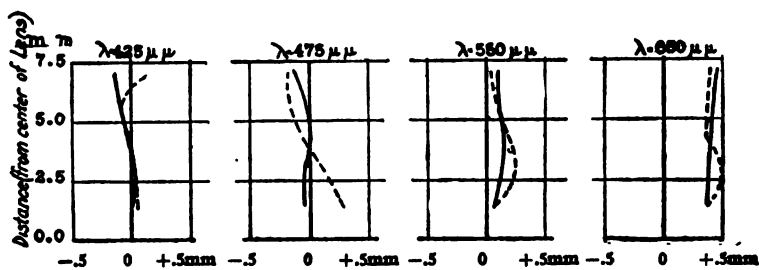


FIG. 15.—Photographic anastigmat lens

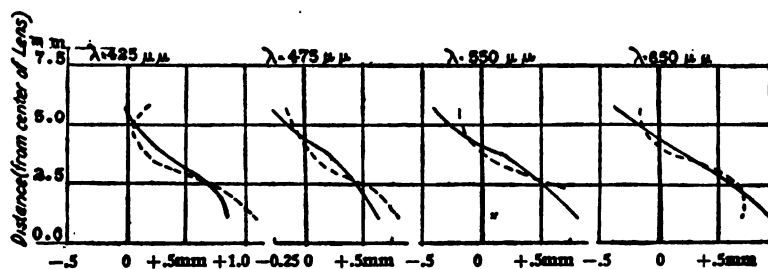


FIG. 16.—Old type Rectilinear lens

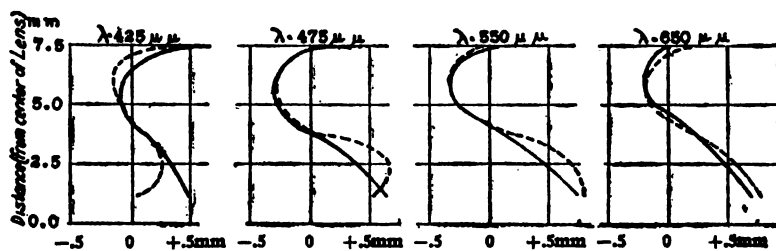


FIG. 17.—Photographic anastigmat lens

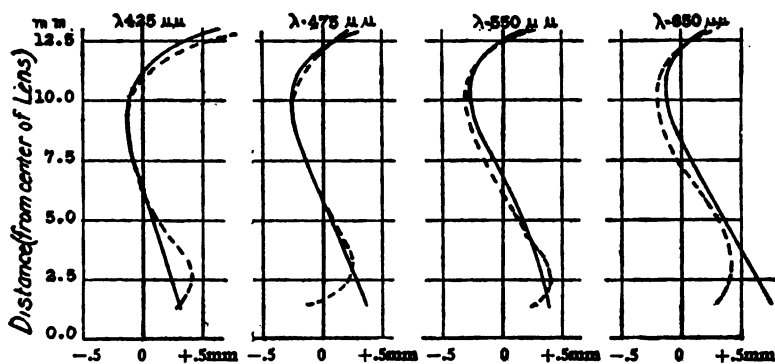


FIG. 18.—Photographic anastigmat lens

Spherical aberration —————
Sine condition —————

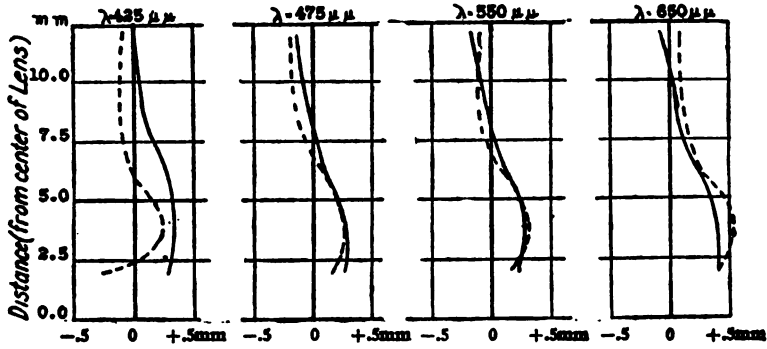


FIG. 19.—Projection lens

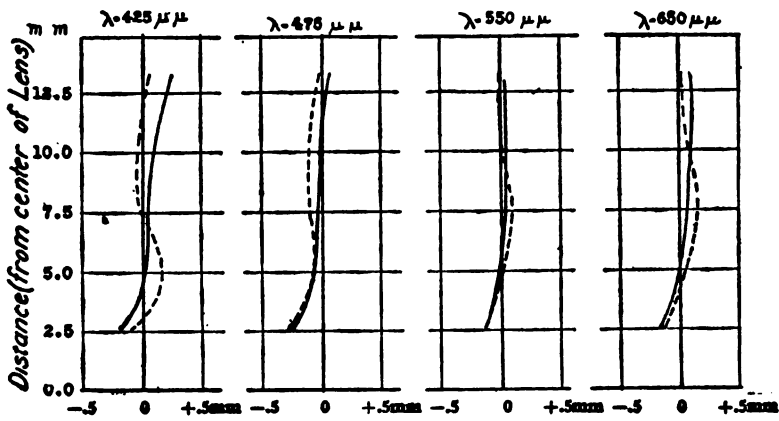


FIG. 20.—Projection lens

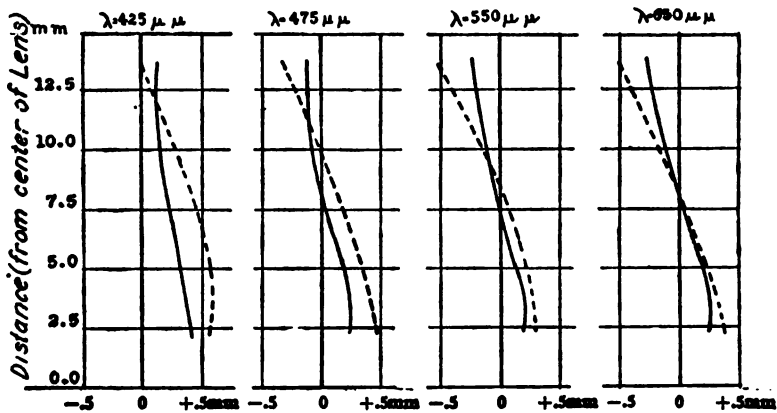


FIG. 21.—Projection lens

Spherical aberration —————
Sine condition

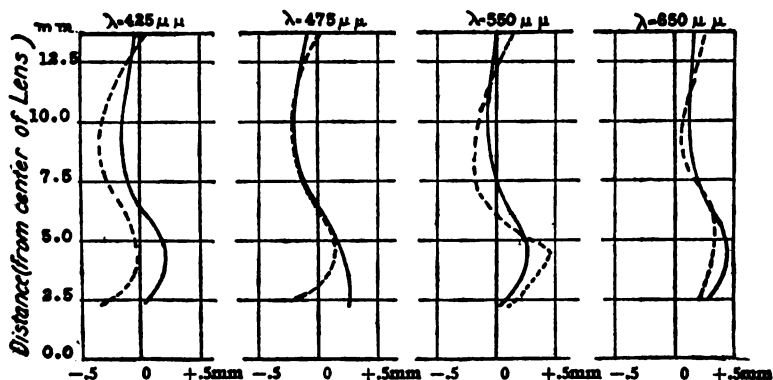


FIG. 22.—Projection lens

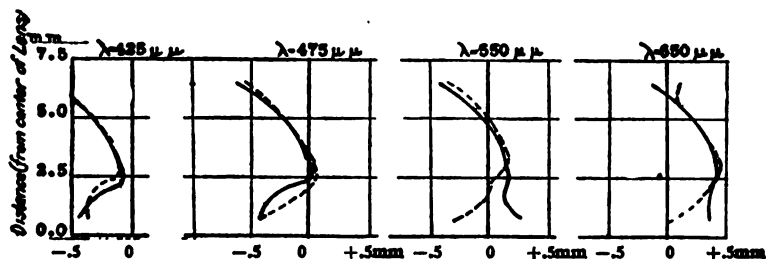


FIG. 23.—Projection lens

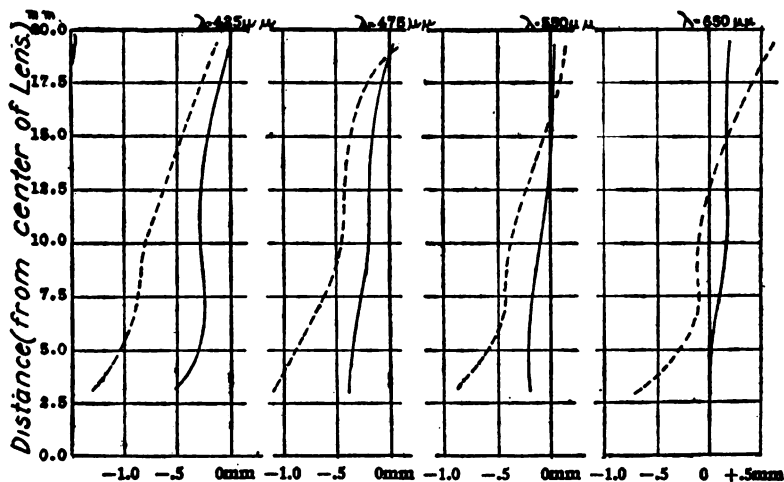


FIG. 24.—Projection lens

Spherical aberration —————
 Sine condition

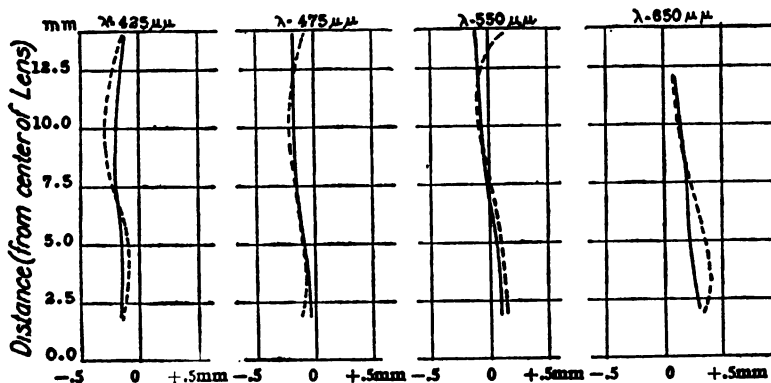


FIG. 25.—Projection lens

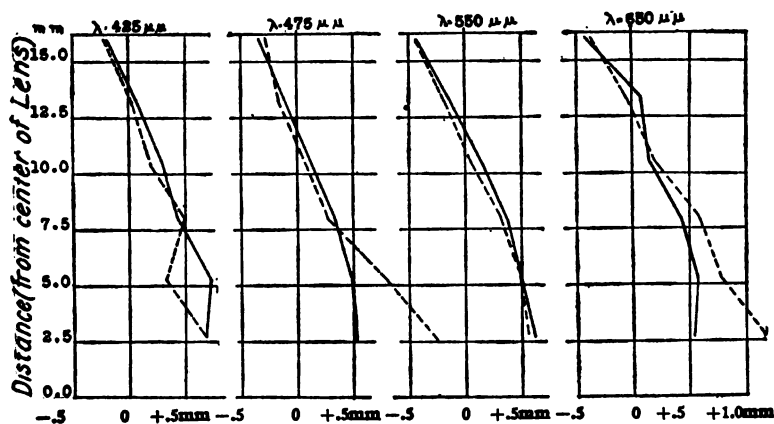


FIG. 26.—Projection lens

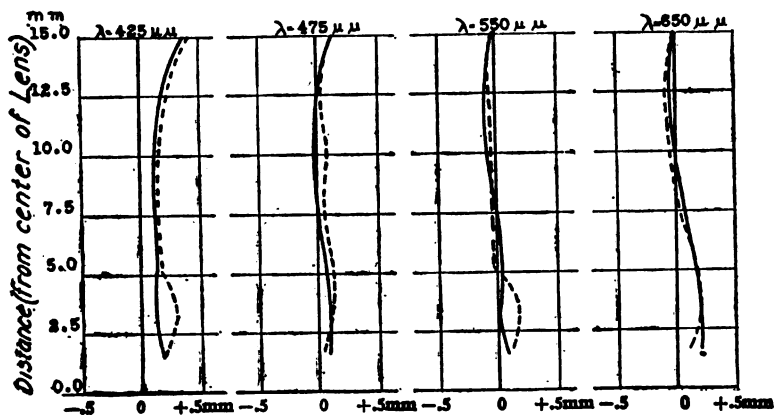


FIG. 27.—Zeiss telescope lens

Spherical aberration —————
Sine condition

WAVE-LENGTH MEASUREMENTS IN SPECTRA FROM 5600 Å TO 9600 Å

By W. F. Meggers

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I. INTRODUCTION

During the past 30 years the spectra of the chemical elements have been quite thoroughly investigated in the wave-length region to which ordinary photographic plates are most sensitive—that is, from 2000 Å to 6000 Å (Å = angstrom = 0.0000001 mm). This was made possible by Prof. Rowland's invention of the concave grating and the establishment of his system of standard wave lengths. Since 1904 a new system of wave-length standards—the international system—has been established and the wave lengths from 2000 Å to 6000 Å of many spectra have been re-measured in international Angstroms.

The long waves have never been so extensively or carefully investigated, because of the great difficulty in photographing them. Measurements of the wave lengths of some of the strong lines in

the red and adjacent infra-red regions of the spectrum have been made by using radiometers or phosphoro-photography to detect the waves. These methods, however, are difficult and yield results which are not in very good agreement. It is generally conceded that nothing can compete with direct photography for the accurate determination of the structure and wave lengths of spectral lines.

The photographic work on long wave lengths has been done principally with ordinary photographic plates which have been specially treated with dyes to make them sensitive to these long waves. Such dyes as alizarin, nigrosin, cyanin, and dicyanin have been used for the purpose. Up to the present, however, comparatively few spectra have been investigated by this method. In most cases photography with stained plates has not registered waves much longer than 8000 Å, although it is possible to reach much longer waves by this method. Furthermore, such work has been done chiefly with the low dispersion of prisms or concave gratings with small radius of curvature, and very few long wave lengths have been measured in international angstroms. By using the photographic method with interferometers or with larger gratings, accurate information concerning the spectra of the elements can be extended to regions beyond 9000 Å without much difficulty.

II. PURPOSE

Some work on spectroscopic analysis at the Bureau of Standards led to a photographic investigation of the spectra of some of the elements in the region of longer wave lengths. The photographic sensitizers dicyanin and dicyanin A were used and found to be of great value in photographing spectra between the wave-length limits, 5600 Å to 9600 Å. This work was begun with the plan of studying the longer waves of the spectra of elements commonly found in iron as impurities, such as nickel, cobalt, chromium, manganese, copper, titanium, vanadium, silicon, calcium, and carbon. The success in photographing these led to the photography of the spectra of the following elements in addition to those mentioned: Lithium, sodium, potassium, rubidium, caesium, beryllium, strontium, barium, and magnesium. Thus the arc spectra of 20 of the chemical elements were photographed from 5600 Å to 9000 Å or beyond.

Accurate measurements of wave lengths to 8824 Å in the arc spectrum of iron have been made with interferometers by Burns.¹

¹ Burns, *Journal de Physique* (5), 8, p. 457; 1913.

Similar measurements have been made to 8210 Å in the barium-arc spectrum by Werner.² These spectra do not contain a sufficient number of evenly distributed and sharp lines in this region to recommend them as entirely satisfactory for standards. A photographic survey of the spectra of the elements may disclose a more satisfactory source for long-wave standards.

Some of the spectra which were photographed, notably those of cobalt, nickel, titanium, vanadium, manganese, and chromium, were found to have sharp lines whose wave lengths can be more accurately obtained with interferometers than from the grating photographs made in this work. Furthermore, the number, distribution, and intensity of lines in this part of the cobalt-arc spectrum were found to be more satisfactory than in the iron-arc spectrum.

If the sharpness of these cobalt lines be examined with the interferometer, the cobalt arc may be found superior to the iron arc as a source of long-wave standards. The wave-length measurements in these sharp-line spectra will, therefore, be postponed until the interferometer is applied.

Many elements, especially the alkali metals, have spectra the greater part of whose lines are broad, diffuse, reversed, or unsymmetrical. A careful study of the long-wave spectra of some of these elements has been made.

The wave lengths have been measured in international angstrom units, and the results are probably as accurate as the structure of the lines will permit. These results are of special interest because of the regularities and series relationships which exist in the spectra of the II and III groups of elements in the periodic system. The apparatus and method used in photographing and measuring these spectra are described in this paper, and the results are given for lithium, sodium, potassium, rubidium, caesium, copper, beryllium, calcium, strontium, barium, and magnesium.

III. APPARATUS

The spectra were photographed in the first-order spectrum of a concave grating ruled by Dr. J. A. Anderson at the Johns Hopkins University. The grating has a radius of curvature of 640 cm and the ruled surface is 7.5 cm by 13.3 cm with 299 lines per millimeter, or 39 800 lines in all. The mounting is shown in Fig. 1.

²Werner, *Ann. d. Physik*, 44, p. 289; 1914.

An image of the arc *A* was magnified about three diameters and focused by the lens *L* on the slit *S* of the spectograph. The light passing through the slit *S* filled a concave mirror *M*, which sent out a parallel beam to the grating *G*, which was placed close beside the slit *S*. The grating *G* focused the spectra on the photographic plate *P*. The grating was fixed at one end of a webbed steel beam, and the camera was movable along the other end of the beam. The camera could be rotated by sliding this end of the beam along a double track, rotation taking place about a vertical axis through the center of the grating. The grating focused the spectra in a circle with its center at the grating and radius equal to the focal length of the grating. This spectrograph gave a dispersion of 10 angstroms per mm in the first order, so that a spectrum length of 2000 Å could be photographed on a 20 cm plate. The focal surface was practically plane for this distance.

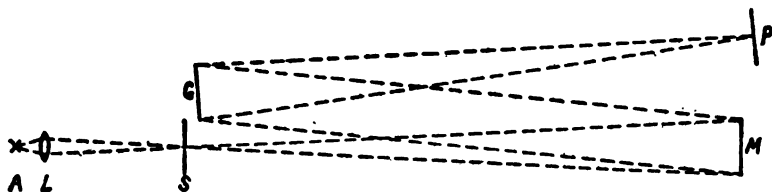


FIG. 1.—Plan of apparatus

After the camera was focused for a particular spectral region it was clamped to the double track. The whole apparatus was clamped to massive brick piers cemented to the thick concrete floor in a basement room where mechanical disturbances and temperature variations were small. A wooden house surrounded the entire apparatus and protected the camera from stray light.

The advantages of the grating mounted in parallel light, in addition to compactness and stability, are intensity of spectra and freedom from astigmatism. The intensity given by the so-called Rowland mounting is quadrupled by this mounting in parallel light. This is of great importance for the photographing of faint spectra and in regions where specially prepared plates must be used. The stigmatism of the slit images when the grating is mounted in parallel light allows the comparison spectrum, containing standards, to be photographed directly beside any other spectrum. This was accomplished by placing suitable diaphragms in front of the slit. An aperture was used which allowed light from an arc to cover 2 mm of the slit, and then this portion was

covered and 5 mm on either side was exposed to the iron arc. The slit width was usually about 0.02 mm.

IV. SOURCES

Arcs were made of metallic electrodes in all cases where it was possible. In the other cases the following salts were used in Acheson graphite electrodes: LiCO_3 , NaCl , KCl , RbCl , CsCl , BeCO_3 , SrCl_2 , and BaCl_2 . For calcium and magnesium arcs the lower electrode was of the metal and the upper one of graphite.

Direct current was supplied at a potential difference of 220 volts. The current strength was made to correspond roughly to the wave lengths to be photographed, i. e., 6 amperes were used to photograph from 6000 Å to 7000 Å, 7 amperes from 7000 Å to 8000 Å, etc. An exception was made in the case of magnesium. The metallic electrode would ignite with large currents, but burned quite satisfactorily with 3 amperes and 110 volts. Electrodes of Norway iron were used for producing the light of the comparison spectrum, and the arc was operated under standard conditions.⁸

In every case only light from the center of the arc was photographed, the light from the electrodes being screened from the slit by the diaphragm.

There is an important difference between so-called chemically pure substances and substances which are spectroscopically pure. In spite of the great skill and care taken in preparing the material, chemists seem unable to produce elements in an absolutely pure state. The use of impure materials for sources has naturally led to frequent mistakes in assigning spectrum lines to the proper element. Some of the long waves which have been wrongly identified by others will be pointed out in the results of this work. It is possible that some errors of this kind still remain.

Spectroscopic analysis of the Acheson graphite showed the presence of sodium, calcium, and barium. All the lines photographed were measured, but the wave lengths due to impurities were separated from the others and will be found in Table I I. This table shows the impurities in the "chemically pure" salts used in this work.

When large quantities of salts are used in graphite electrodes the spectrum of carbon is quite effectively suppressed. If only small quantities are used the thousands of lines due to carbon become very troublesome. Electrodes of copper, cored and filled

⁸ *Astroph. J.*, **89**, p. 93; 1914.

with salt, may be recommended for work in the region of long wave lengths, for the copper spectrum has very few strong lines in this region.

V. PHOTOGRAPHY OF RED AND INFRA-RED SPECTRA

Among the various photographic sensitizers which have been used to photograph red and infra-red spectra, the most efficient and convenient is probably dicyanin. Eder,⁴ Geiger,⁵ Burns,⁶ and others have used it successfully, and it is regrettable that its use has not become more universal.

Photography with stained plates is generally thought to be difficult, troublesome, and uncertain. Perhaps some have tried dicyanin without success because their dye was inferior or worthless. This dye is quite easily decomposed by the action of light or heat, after which its value as a photographic sensitizer is lost. If stored in a cool, dark place it may be kept months without decomposing. The process of staining plates is probably thought to require special apparatus and technique. It is possible, however, to obtain satisfactory results with a very simple procedure.

The most efficient staining bath was found to be the one recommended by Burns.⁶ It consisted of a mixture of water, alcohol, ammonia, and dicyanin in the following proportions: About 4 cc of a stock solution of 1 part dicyanin to 1000 parts alcohol were added to 50 cc distilled water, 50 cc ethyl alcohol, and 5 cc of strong ammonia. Ordinary photographic plates, like Seed 27 or Graflex, were soaked in such a bath from 3 to 5 minutes, rinsed in alcohol 30 seconds, and dried by a current of air from an electric fan. Plates treated in this manner were found to be quite sensitive to wave lengths between 6000 Å and 9000 Å. The strong lines between 7500 Å and 8500 Å in the spectra of the alkali metals were photographed with exposures of 1 minute or less. In the barium-arc spectrum a line of wave length 9370 Å was registered with an exposure of 20 minutes, while an exposure of 60 minutes on a plate stained with dicyanin A showed the line of wave length 9608 Å quite strongly. (See Fig. 7.) The exposures were usually limited to 30 minutes in length, so as to be reasonably sure that no displacements had taken place in the spectrograph. With these half-hour exposures the spectrum of the iron arc has been photographed to 9118 Å, the chromium arc to 9290 Å, the vana-

⁴ Eder, *Kaiser. Akad. d. Wiss.*, 193, p. 2289; 1914.

⁵ Geiger, *Ann. d. Phys.*, 39, p. 752; 1912.

⁶ Burns, *Journal de Physique*, (5), 8, p. 457; 1913.

dium arc to 9087 Å, the nickel arc to 8968 Å, the cobalt arc to 8926 Å, and the titanium arc to 8734 Å. In the region 5600 Å to 7600 Å the exposure times ranged from 1 to 15 minutes.

A few typical spectrum photographs as made with the dicyanin-stained plates are reproduced in Figs. 3, 4, 5, 6, and 7. These photographs are enlargements of three diameters made from copies of the original spectrum plates. They show the first-order spectrum of some element bounded on either side by the arc spectrum of iron as photographed in the second order. An exception to this arrangement is found in the last two spectra on Fig. 3. In these cases the first-order spectrum of iron is shown below that of Caesium. The wave lengths of the strong lines in the middle spectrum are indicated by the accompanying numbers.

VI. WAVE-LENGTH MEASUREMENTS

The plates were measured with the excellent measuring machine at the Johns Hopkins University. This machine has a screw which is nearly perfect throughout its entire length of more than 50 cm. The pitch of the screw is 1 mm. The head has a diameter of 25 cm and the circumference is divided into 1000 parts. One division thus corresponds to 1μ along the axis of the screw or to 0.01 Å on a spectrum photograph with 10 Å per mm.

Turning the head moved the microscope along ways which were parallel to the screw. The spectrum plate was clamped firmly to a shelf fastened to the ways. A five-fold magnified image of a part of the plate was then visible in the field of the microscope with the cross-hair images superposed. The microscope contains two pairs of cross hairs, as shown in Fig. 2. Readings were made when the cross-hair intersection b_1 was on a spectrum line and again when b_2 was on the same line. The mean of these two readings was used for the position of the line on the plate. The positions of the standards in the comparison spectrum of iron were obtained from readings when the cross-hair intersections a_1 and a_2 were on the lines. After the two spectra were completely measured from short to long wave lengths, the plate was reversed and remeasured from long to short wave lengths. Thus each determination of

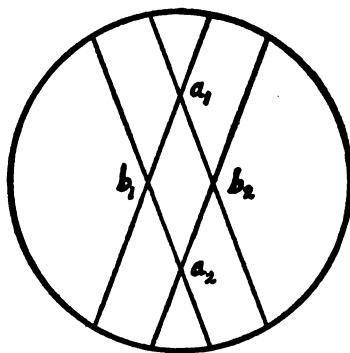


FIG. 2.—Cross hairs

wave length is based upon four independent observations of the position of a line with respect to the neighboring standards.

The scale of the plate, assuming normal dispersion, was obtained by dividing the difference in wave length by the measured distance between the first and last standards measured on the plate. All the other measurements were then reduced to wave lengths on this basis. The deviations from standard values of the values thus obtained for the iron lines were plotted as a function of wave length and a curve was drawn through these points. From this curve the correction required to express any other wave length on the international scale was obtained.

Interference measurements of wave lengths in the iron spectrum by Burns, Meggers, and Merrill⁷ were used as standards. These standards contain the 85 international secondary standards from 3233 Å to 6750 Å, together with 318 other iron lines compared with the international standards by the interferometer method. This set of standards contains chiefly the sharpest iron lines, of different intensities, consistent with regularity of distribution throughout the spectrum. The average distance between these standards is less than 9 Å. Eighty-six per cent of the 85 international secondary standards are lines of intensity 4 to 6, inclusive, while only 47 per cent of the additional 318 lines have these intensities, the remainder being distributed among fainter and stronger lines. These facts make this set of standards very convenient for the measurement of wave lengths.

The comparison iron spectrum from 3300 Å to 4800 Å was usually photographed in the second order of the grating, when long waves, from 6600 Å to 9600 Å of other spectra were photographed in the first order. Orders overlapping on the first order spectrum were removed by placing a screen of Jena red glass in front of the slit. The first-order spectrum of iron was photographed beside the shorter waves of the other spectra. On a few plates the first-order spectrum of iron was used to 8600 Å and Burns's values were then used for the standards longer than 6750 Å.

VII. RESULTS

The results are contained in the following tables. The first column of each table gives the wave lengths as measured in international angstroms. The second column indicates the intensity and character of each line. The strongest lines are called intensity

⁷ This Bulletin, 18, p. 245; 1916.

10. Faint but measurable lines are called intensity 1. The character or appearance of a line is represented by letters which have the following meanings:

b=broad.
 d=perhaps double.
 h=hazy.
 H=very hazy.
 I=shaded to red.
 L=much shaded to red.
 R=broadly reversed.
 rs=narrow reversal, red component stronger.
 Sb=very broad.

The third column indicates the probable error of the measurement of each wave length. The letters have the following significance:

A=probable error 0.000 Å to 0.010 Å.
 B=probable error 0.010 Å to 0.020 Å.
 C=probable error 0.020 Å to 0.030 Å.
 D=probable error >0.030 Å.
 E=only one determination.

A and B also indicate that the line was measured more than twice. In fact most of the stronger lines were measured on three to six plates. The lines marked "E" are generally faint and show only on the longest exposures. Their wave lengths are not very accurate, but they are all included in the table to show how many of them have been recorded by comparatively short exposures of the dicyanin-stained plates. In the following tables I. A. indicates that the wave lengths are given in international angstroms. The others are based on Rowland's system of standards.

The tables contain also the results of others who have photographed in this region with stained plates. The only investigations of this kind embracing more than one spectrum are those of Lehmann,⁸ Hermann,⁹ Saunders,¹⁰ Lorensen,¹¹ and Eder.¹² Among these, Eder's results are the only ones expressed in International Angstrom units.¹³ Lorensen used the Hartmann¹⁴ values for the iron lines as standards. For purposes of comparison, the

⁸ Lehmann, *Ann d. Phys.*, (4), 5, p. 633, strong line 1, weak line 4; 1901.

⁹ Hermann, *Ann d. Phys.*, (4) 16, p. 684; 1905.

¹⁰ Saunders, *Astrophys.*, JI., 20, p. 188; 1904.

¹¹ Lorensen, *Dissertation*, Tübingen; 1913.

¹² Eder, *Sitzungsber. d. k. Akad. d. Wissensch., Wien*, 123, p. 2289, strong line 100; 1914.

¹³ Science Abstracts, 20, Abs. 223, 1912, briefly describes "Researches and Measures of Wave Lengths in the Red and Infra-red Regions of the Spectrum," by K. W. Meissner, whose work is published in *Ann. d. Physik*, 60, p. 713, 1916. Dicyanin-stained plates were used to investigate the spectra of Cs, Fe, Na, K, Rb, Al, Ca, Ag, Cu, Cr, U, S, O. It has been impossible to obtain a copy of Meissner's publication. This explains the absence of his results in the following tables.

¹⁴ Hartmann, *Physik. Zeitschr.*, 10, p. 123; 1909.

results based upon Rowland's standards may be changed to the international scale by subtracting the following quantities:

0.22 A from 5500 to 6050
.21 A from 6050 to 6500
.22 A from 6500 to 6570
.23 A from 6570 to 6750
.24 A from 6750 to 6850
.25 A from 6850 to 7000
.26 A from 7000 to 7200
.27 A from 7200 to 7400
.28 A from 7400 to 7700
.29 A from 7700 to 8000
.30 A from 8000 to 8200
.31 A from 8200 to 8300
about .35 A from 8300 to 8800

For the proper discussion of series relationships among the lines in these spectra, the wave lengths should be reduced to vacuum. This can not be done accurately at the present time because the dispersion of air has never been determined for waves longer than 6800 Å. For this reason an investigation on atmospheric dispersion in the region of long wave lengths has been begun. The frequency differences of pairs of lines which appear in doublet series in some of the spectra will not be affected appreciably by the dispersion of the atmosphere. Such frequency differences, as determined from my wave-length measurements, will be given below. The oscillation frequency used here is the reciprocal of the wave length or the number of waves per centimeter. This number is proportional to the absolute frequency or number of waves per second and is more convenient and in wider use than the latter.

TABLE 1.—Lithium

Meggers			Eder		Saunders	Lehmann		Kayser and Runge ¹⁵	
λ I. A.	Notes	p. e.	λ I. A.	Notes	λ	λ	Notes	λ	Notes
6103.53	10b	E			6240.3			6103.77	10R
6707.85	10R	B	6707.85	100R				6708.2	10R
8126.52	10	B	8126.27	10b	8127.0	8127.34	1		

The arc was produced between graphite electrodes containing lithium carbonate. The impurities present were sodium, potassium, calcium, and barium.

The spectra of the alkali metals have attracted much attention because of their similarity. They all contain definite sequences

¹⁵ Kayser and Runge, Wied. Ann., 41, p. 302; 1890.

or series of lines, and the members of each series possess the same spectral character. In general three types of series exist in each of these spectra. The intense and easily reversed lines constitute the so-called principal series. Most of the other lines are broad and diffuse. They may be arranged in two subordinate series, the lines of the first subordinate series being broadened by shading toward the red. In the spectra of sodium, potassium, rubidium, and caesium all the series consist of doublets, but the lines in the lithium spectrum have generally been measured as single. The lines are, no doubt, very narrow pairs, and King ¹⁶ has recently found the line 6707 double under certain conditions. Two components for this line have also been observed by others. The separation is about 0.15 Å.

TABLE 2.—Sodium

Meggers			Eder		Kayser and Runge ¹⁷		Saunders	Hermann	
λ I. A.	Notes	p. e.	λ I. A.	Notes	λ	Notes	λ	λ	Notes
5882.97	8	E			5882.90	7ur			
88.61	10	E			88.26	8ur			
5895.97	10R	E			5895.19	10R			
95.94	8R	E			96.16	9R			
6154.40	4	E			6154.62	8ur			
60.96	5	E			61.15	8ur			
							7369.4		
							77.4		
							7410.0		
					Lehmann		18.3		
8183.30	8ur	C	8183.35	100R	8184.33	1	8184.5	8183.74	2
								8188.17	0
8194.82	10ur	C	8194.92	100R	8194.76	1	8196.1	8195.33	3

Sodium chloride was used in graphite electrodes for the arc. The impurities present were potassium, strontium, lithium, calcium, and barium. When a large amount of sodium chloride was used in the arc the lines 8183 and 8194 showed narrow reversals displaced from the centers of the lines toward shorter wave lengths. The wave lengths given are from measurements on these reversals. When unreversed the lines are shaded to the red, and the measured wave lengths are then usually from 0.1 Å to 0.2 Å larger. These two lines are the first pair of the first subordinate

¹⁶ King, *Astrophys. J.*, 44, p. 169; 1916.

¹⁷ Kayser and Runge. See note 15.

series and their frequency difference is 17.2, which is identically the frequency difference of the first pair of the principal series, 5889 and 5895. The lines 6154 and 6160 are the first pair of the second subordinate series, and their frequency difference from these measurements is 17.3. The frequency difference of the second pair, 5682 and 5688, in the first subordinate series, is 17.4. The range in frequency differences for these four pairs of lines is little more than 1 part in 100 000 in the actual number of waves per centimeter, and this is of the same order of magnitude as the errors in the wave-length measurements.

TABLE 3.—Potassium

Meggers			Eder		Kayser and Runge ¹⁵		Saunders	Hermann		Lehmann	
λ L. A.	Notes	p. e.	λ L. A.	Notes	λ	Notes	λ	λ	Notes	λ	Notes
5782.60	2h	E			5782.67	5u R					
5801.96	4h	E			5802.01	6u R					
12.52	2h	E			12.54	6u R					
5832.09	2h	E			5832.23	7u R					
6911.30	10	A	6911.31	5	6911.2	7	6911.8				
38.98	10	A	39.07	6	38.8	8	39.5				
							6966.3				
7664.94	10R	A	7664.95	150R	7665.6	10u R	7664.91	7665.29	8	7668.54	1
99.01	10R	A	99.02	100R	99.3	10u R	99.08	99.32	6	7701.92	1
							7931.8				

Potassium chloride was used in graphite electrodes. The impurities in the spectrum were sodium, rubidium, calcium, barium, and lithium.

The lines 7699 and 7664 are the first pair of the principal series and their frequency difference from these measurements is 57.74. The next pair, 6938 and 6911, are the first lines of the first subordinate series. Their frequency difference is 57.72, or identically that of the first pair of the principal series. The lines 5801 and 5782 are the second pair in the first subordinate series and their frequency difference is 57.71. The pair 5832 and 5812 belong to the second subordinate series. The difference in waves per centimeter for this pair is 57.73. The above measurements thus give exactly the same frequency difference for these four pairs.

¹⁵ Kayser and Runge. See note 15.

TABLE 4.—Rubidium

Meggers			Eder		Eder and Valenta ¹⁹		Saunders	Lehmann	
λ I. A.	Notes	p. c.	λ I. A.	Notes	λ	Notes	λ	λ	Notes
6070.95	2l	E			6071.30	3	6071.1		
6159.84	5l	E			6160.20	4	6160.0		
6206.48	8l	E			6206.74	6	6206.7		
98.50	10l	E			98.85	8	98.8		
					7060.09	1			
7280.22	10l	C			7280.53	1	7280.3	7277.01	4
7408.37	10l	C	7408.49	1l	7408.71	1	7408.5	7406.19	4
7619.12	8l	C	7619.17	2b	7619.7	1	7619.2	7626.66	3
7757.80	8l	B	7757.70	2b			7757.9	7753.58	3
59.58	1	C			7759.6	1	59.5		
7800.29	10R	A	7800.30	100R	7800.3	4R	7800.2	7805.98	1
7947.64	8R	A	7947.63	50R	7947.7	2R	7947.6	7950.46	2
			8521.21	3a				8513.26	4

Rubidium chloride was used in graphite electrodes. The arc spectrum showed sodium, potassium, caesium, calcium, barium, and lithium as impurities.

The line 7060 observed by Eder and Valenta is due to barium. The line measured as 8521 by Eder and 8513 by Lehmann belongs to caesium.

The lines 7947 and 7800 are the first pair of the principal series. Their frequency difference is 237.7. Lines 7757, 7619, and 6298, 6206 are the first two pairs of the first subordinate series. Their frequency differences are 234.5 and 235.4. Lines 7408, 7280, and 6159, 6070 are the first two pairs of the second subordinate series. Their frequency differences are 237.5 and 237.7. Thus it appears from these measurements that the frequency differences of pairs in the second subordinate series are identical with that of the first pair of the principal series, while those of the first subordinate series are about 1 per cent smaller. This is also true for the succeeding pairs.

¹⁹ Eder and Valenta, *Atlas Typischer Spektren*, Wien, 1911.

TABLE 5.—Caesium

Meggers			Eder		Lehmann		Saunders
λ I. A.	Notes	p. e.	λ I. A.	Notes	λ	Notes	λ
5663.8	H	D					
5844.7	H	D					
6010.33	7b	C					
6212.87	8b	C					6034.8
6217.27	1b	E					6213.1
6354.98	4b	C					6217.6
							6325
							6355.3
							6359
							6434
							6475
							6587.3
6586.94	5i	D	6587.11	6i			6588.0
							6630.5
6723.18	10b	A	6723.28	20			6723.7
							6826.9
							6872.6
6973.17	10b	A	6973.35	20R			6973.1
83.37	5b	A	83.39	3b			6983.8
7228.5	L	E			7227.46		7228.8
7279.7	L	E					7280.5
7609.13	5i	B	7609.28	2i	7616.58	3	7609.7
			7800.11	2b			
			7941.1.	1u			7944.7
7944.11	6b	C					
8016.2	} 10L	D			8019.62	3	8007.7
8016.9							
8079.1	} 10L	D			8082.02	3	
8079.8							
8521.12	10R	A			8527.72	1	
8761.35	5b	B			8766.10	2	
8943.46	8R	A			8949.92	2	
9172.23	2b	A			9171.38	3	
9208.40	1b	A			9211.86	3	

Caesium chloride was used in graphite electrodes. The arc spectrum showed sodium, potassium, rubidium, lithium, calcium, and barium as impurities. The line 7800 measured by Eder is due to rubidium. The line 8079 and the line 8016 are very unsymmetrical. The photographic maximum is therefore dependent on the duration of the exposure for its position, and the measurements on this maximum vary between the limits indicated.

The following lines belong to series:

		Meggers	Kayser
		$\Delta\nu$	$\Delta\nu$
First pair of principal series.....	8943.46 8521.12	554.1	546.9
First pair of first subordinate series.....	9208.40 8761.35	554.2	552.0
Second pair of first subordinate series.....	6973.17 6923.18	533.2	533.8
Third pair of first subordinate series.....	6212.87 6010.33	542.2	543.0
Fourth pair of first subordinate series.....	5844.7 5663.8	546.5	547.0

Kayser²⁰ suspected that accurate measurements would show these frequency differences to be constant. Although my measurements show the frequency differences to be identical for the first pairs of the principal and subordinate series, the frequency differences for the next three pairs of the first subordinate series show the same deviations from constancy as the older measurements.

TABLE 6.—Copper

Meggers			Aretz ²¹		Hasbach ²²	
λ L. A.	Notes	p. e.	λ L. A.	Notes	λ L. A.	Notes
6415.18	1	E	6415.155	1		
27.57	1	A	27.564	2		
			52.287	0		
			56.672	1u		
74.20	3	A	74.176	5u	6474.2	1u
85.18	2	D	85.142	5u		
			6504.051	0	6485.15	1
			06.142	2u		
			31.437	1		
6544.51	1h	D	44.427	2u		
50.98	1	E	50.977	3		
65.54	3h	C	65.555	2u		
			83.542	2u		
			99.681	5		
6621.61	4	A	6621.623	1u	6621.59	1
29.67	1	C	29.730	1u		
72.23	3	A	72.234	5	72.2	1u
6741.42	7	A	6741.418	6	6741.4	1u
49.29	2b	E				
75.64	2b	C				

²⁰ Kayser, *Handbuch der Spectroscopie*, 2, p. 539.

²¹ Aretz, *Zeitschr. f. wiss. Phot.*, 9, p. 196; 1910.

²² Hasbach, *Zeitschr. f. wiss. Phot.*, 13, p. 399; 1914.

TABLE 6.—Copper—Continued

Meggers			Aertz		Hasbach	
λ I. A.	Notes	p. e.	λ I. A.	Notes	λ I. A.	Notes
6821.86	1h	C	81.869	0u		
35.46	1	E				
40.99	1h	A				
81.94	2	A				
89.92	2	C				
90.90	2	A				
6905.90	6	A	6905.937	3u		
20.09	4h	A	20.287	1u		
35.80	2h	C				
68.36	1	C				
7000.02	1h	E				
39.34	3	B				
7124.66	1	E				
54.29	1	E				
93.56	2h	A				
7427.26	1	E				
7570.09	5	A				
7771.98	4	A				
74.18	3	A				
75.41	2	A				
7848.55	1	C				
7911.95	1	C				
33.20	10	A				
54.23	2	B				
8006.27	2	B				
17.78	2	B				
71.06	2	C				
92.74	10	B				
8114.17	2	B				
78.96	2	B				
87.90	1	E				
8216.22	1	C				
23.13	1	E				
42.27	1	E				
73.45	1	E				
8408.00	1h	E				
46.40	2	C				
8680.08	1	E				
83.17	1	E				

Copper rods 6 mm in diameter were used as arc electrodes. The D lines of sodium appeared in the spectrum. The work of Aertz and of Hasbach should be consulted for wave lengths shorter than 6621 Å. Aertz made exposures on Wratten & Wainwright plates for six hours and obtained lines below 6621 Å which were not recorded by the 15-minute exposures on my plates. The line 6599 which Aertz marked intensity 5 did not appear on my plates. It may represent an impurity.

The copper-arc spectrum also has a doublet system of series. The first pair of the principal series, 3247.550 Å and 3273.967 Å, according to Hasbach, has a frequency difference of 248.45. Lines 7933.19 Å and 8092.76 Å are the first lines of the second subordinate series and have a frequency difference of 248.55. The lines 5133.261 Å and 5220.083 Å have a frequency difference of 248.40. This pair belongs to the first subordinate series. The frequency differences of these three pairs representing three different series differ from each other by less than one part in 100 000 in the wave number.

Beryllium

Investigations in the arc spectrum of beryllium have thus far disclosed only 12 lines with certainty. The longest of these has a wave length of 4572 Å, which is in the blue part of the spectrum. Exposures of a half hour on my stained plates showed no traces of lines between 4572 Å and 9000 Å. Beryllium carbonate was used in graphite electrodes to produce the arc spectrum.

TABLE 7.—Strontium

Meggers			Eder		Hampe ²³	Lohmann		Eder ²⁴ and Valenta	Lorenser	
λ I. A.	Notes	p. e.	λ I. A.	Notes	λ I. A.	λ	Notes	λ	λ	Notes
6363.98	4	C			6363.932			6364.21	6364.19	5
70.00	5	C			69.939			70.20	70.18	6u
80.77	5	C			80.740			80.95	80.94	10u
86.57	7	C			86.507			86.76	86.76	9u
88.32	6	D			88.245			88.50	88.48	8u
6408.49	9	C			6408.465			6408.70	6408.69	15
46.70	1	C			46.676				46.91	3
65.78	1	C			65.788			66.08	66.10	2
6504.02	6	C	6504.01	6	6503.990	6504.07	1	6504.26	6504.26	12u
									16.07	1
21.29	1								21.53	1
46.82	4	B	46.79	4	46.785	46.27	4	47.09	47.06	8u
50.28	5	B	50.29	6	50.253	50.19	3	50.53	50.51	12u
6617.28	5	B	6617.27	4	6617.268	6616.92	3	6617.50	6617.54	10u
43.58	4	A	43.52	3	43.545	44.05	3	+ 43.78	43.80	6u
						6708.10	4			
						54.21	4			
								6769.59		1
6791.08	5	A	6791.06	6	6791.046	6792.19	1	6791.30	91.35	10
								6803.55		1
6878.36	10	A	6878.36	7	6878.347	6880.69	1	6878.63	78.65	15v
92.62	6	A	92.62	4	92.598	93.37	3	92.83	92.86	8u
7070.15	10	A	7070.12	20R	7070.102	7070.34	2	7070.45	7070.53	20v
								88.90		1

²³ Hampe, Zeitschr. f. wiss. Phot., 18, p. 348; 1914.

²⁴ Eder and Valenta, Atlas Typischer Spektren, Wien; 1911.

TABLE 7.—Strontium—Continued

Meggers			Eder		Hampé	Lehmann		Eder and Valenta	Lorenser	
λ I. A.	Notes	p. e.	λ I. A.	Notes	λ I. A.	λ	Notes	λ	λ	Notes
7153.08	4	A	7153.07	2				7153.24	7153.43	4a
67.24	6	A	67.29	5		7168.02	3	67.49	67.66	10a
7232.24	6	A	7232.20	4		7232.10	3	7232.53	7232.56	8a
						54.44	3			
87.44	1h	E	87.57	1					87.75	4
			89.25	1					89.19	2
7309.47	6	A	7309.46	10				7309.65	7309.70	12a
									48.72	1
									62.89	1
									7406.07	2
			7408.13	1					68.32	3
									38.53	2
7621.54	5	A	7621.55	3					7621.76	8
73.11	6	A	73.07	6					73.38	10
			8183.98	4						
			95.14	4b						

Strontium chloride in graphite electrodes was used for the arc. Lithium, sodium, calcium, and barium were found as impurities. The lines 8183 and 8195 observed by Eder are due to sodium. The unsymmetrical character of these two lines makes their measured wave lengths several tenths of an angstrom unit larger than when measurements are made on the reversals. The line 6708 measured by Lehmann is probably a lithium line. The line 7408 measured by Eder and by Lorenser may belong to rubidium.

TABLE 8.—Calcium

Meggers			Eder		Crew and McCauley ²⁰		Holtz ²¹	Lehmann		Lorenser	
λ I. A.	Notes	p. e.	λ I. A.	Notes	λ I. A.	Notes	λ I. A.	λ	Notes	λ	Notes
6439.13	10	E			6439.086	9	6439.061			6439.31	20v
49.83	5	E			49.811	7	49.794			50.05	10
55.60	3	E			55.606	3	55.560			55.82	8a
62.62	10	E	6462.55	8	6462.576	9	62.550			62.82	20v
										64.93	2a
71.69	5	E	71.64	5	71.659	5	71.644			71.92	10
93.83	7	E	93.77	7	93.789	8	93.762			94.04	20v
99.67	5	E	99.64	5	99.648	4	99.624			99.94	10
					6508.742	1		6508.02	3		
										6509.10	4a
6572.78	2	E	6572.78	4	6572.783	3	6572.71	71.98	2	73.03	6a
										6656.1	1a

²⁰ Crew and McCauley, *Astrophys. J.*, 89, p. 89; 1914.

²¹ Holtz, *Zeitschr. f. wiss. Phot.*, 19, p. 101; 1913.

TABLE 8.—Calcium—Continued

Muggen			Eder		Crew and McCauley		Holtz	Lehmann		Lorenser		
λ L. A.	Notes	p. a.	λ L. A.	Notes	λ L. A.	Notes	λ L. A.	λ	Notes	λ	Notes	
6717.78	8	C						6666.98	2	65.6	1u	
			6707.88	2	6707.866	1	6707.81	-----	-----	6708.06	6u	
										10.12	1	
			17.69	8	17.688	5	17.70	6714.47	3	18.01	15v	
										44.0	1u	
										47.2	1u	
								67.02	1	-----	-----	
								82.85	4	-----	-----	
										84.13	4u	
										89.38	1u	
7148.15	10	B						6833.50	4	-----	-----	
			7148.14	6	7148.123	3	-----	7146.45	3	6866.80	1u	
			7202.15	4	7202.161	2	-----	99.83	1	84.29	2	
			7326.10	5	7326.099	1	-----	7322.95	2	7148.49	15v	
										7202.51	15v	
										7326.43	20v	
										7468.69	3	
										7521.22	1u	
										34.75	2u	
										66.08	3u	
8153.13	1	E								82.39	1uR	
										87.79	2uR	
										98.40	3uR	
								7600.74	4u	-----	-----	
										7602.78	4uR	
										10.66	6uR	
								12.34	4u	45.25	1u	
										7984.25	1u	
										95.31	2u	
										8153.13	2	
Hermann												
						λ L. A.	Notes					
8497.98	8	B	8498.11	4	-----	-----	8498.32	00	8499.20	3	8498.35	3
8542.06	10	B	8542.25	1u	-----	-----	8542.48	1	8543.08	1	8542.47	5
8662.10	9	B	-----	-----	-----	-----	8662.42	0	8662.10	2	8662.50	3

An electrode of metallic calcium was used below and a graphite electrode above to produce the arc. Sodium, lithium, potassium, and strontium contaminated the spectrum. The line 6707 measured by Eder, Crew and McCauley, Holtz and Lorenser is due to lithium, although Eder insists that it is a calcium line. Woodward²⁷ has shown that this line is absent in the arc spectrum of pure calcium. By means of a 25-ampere arc and longer exposures

²⁷ Woodward, *Astrophys. J.*, 41, p. 169; 1915.

²⁰ George, *Zeitschr. f. wiss. Phot.*, 12, p. 237; 1913.

Meggert			Lorenser		Eder		Burns	Werner	Lehmann		Hermann
λ L. A.	Notes	p. e.	λ	Notes	λ L. A.	Notes	λ L. A.	λ L. A.	λ	Notes	λ
			8329.17	3u							
50.64	1	E									
90.38	1	E									
8414.52	1h	E									
8514.23	2h	C	8514.50	1u					8518.24	4	
21.91	1	C									
			42.72	1u							
8559.90	10	B	8560.21	6u	8559.98	5			8563.92	1	8560.20
67.53	3h	B	68.08	3u R							
69.11	1	E	69.60	2u R							
			70.34	1u R							
82.04	4	B	82.66	3u					85.70	4	
93.40	1	E									
8654.02	4	B	8654.40	3					8659.38	4	8654.33
			59.65	1							
8710.74	1	A									
37.71	1	C									
99.70	2h	C							8806.22	4	8799.86
8860.96	4	B	8861.32	2					8868.50	4	8861.40
8914.96	4	A	8915.40	2					8921.04	3	8915.19
27.36	1	E									
37.85	1	E									
9091.15	1	E									
9133.70	1	E									
89.43	2	E									
9219.65	2	B									
9308.09	1	E									
24.53	1	E									
70.05	3	B									
9455.94	1	E									
9608.83	1	E									

The arc was operated with barium chloride in graphite electrodes. Lines belonging to sodium, lithium, potassium, strontium, and calcium appeared in the spectrum. The line 9370 was registered by the phosphoro-photographic method by Lehmann,³¹ who measured the wave length as 9367 Å. The line 9608 was detected by Randall³² with a thermopile, and this wave length was given as 9610.7 Å. Randall measured a line of wave length 9527.3 Å, which did not appear on my plates.

It is unfortunate that the line structures and wave lengths in the barium spectrum are so sensitive to the conditions under which the arc is produced. Werner used the arc in a vacuum in order to obtain sharp lines. Under atmospheric pressure many of

³¹ Lehmann, *Ann. d. Phys.*, **89**, p. 53; 1912.

³² Randall, *Astrophys. J.*, **34**, p. 1; 1911.

the lines begin to broaden. Lorenser used large quantities of barium chloride in a carbon arc fed by 30 amperes. Under these conditions nearly all of the lines are broad or unsymmetrical.

TABLE 10.—Magnesium

Meggens			Nacken ²²		Lorenser		Hermann	
λ I. A.	Notes	p. c.	λ I. A.	Notes	λ	Notes	λ	Notes
5528.49	10	C	5528.466	5				
5711.14	5	C	5711.127	1				
82.10	1	C						
6021.70	1	C						
6318.5	2d	D			6318.55	2u R		
					18.82	2u		
					19.08	1u		
					32.48	2u		
					47.27	4u		
					6457.03	1		
					64.51	2		
					6505.62	2		
					45.66	2u V		
					46.77	5u R		
7657.5	1h	D			7658.46	3		
8806.75	5	B			8807.20	5	8806.96	8
							8929.35	2
							9224.44	00

Metallic magnesium was used as a lower electrode and graphite as an upper electrode in the arc. It is difficult to burn such an arc in air because of the tendency of the magnesium to ignite and oxidize. Lorenser burned the arc in a vacuum.

Comparison with the tables of solar wave lengths, published by Capt. (Sir) W. de W. Abney,²⁴ brings out certain interesting facts. Aside from the telluric lines, a large percentage of the rays in the infra-red solar spectrum are due to iron, as would be expected from the character of the solar spectrum in the visible and ultra-violet regions. The strongest solar lines in the infra-red are due to calcium and magnesium. These are the lines X_I, X_{II}, X_{III}, and X_{IV} of Abney, the last and weakest being due to magnesium. Some of the strong lines of barium seem to show in the solar spectrum, as well as the strong lines of potassium (7664 and 7699), sodium (8183 and 8194), lithium (8126), and copper (7933).

²² Nacken, *Zeitschr. f. wiss. Phot.*, 12, p. 54; 1913.

²⁴ Abney, *Phil. Trans.*, 177, p. 457; 1886.

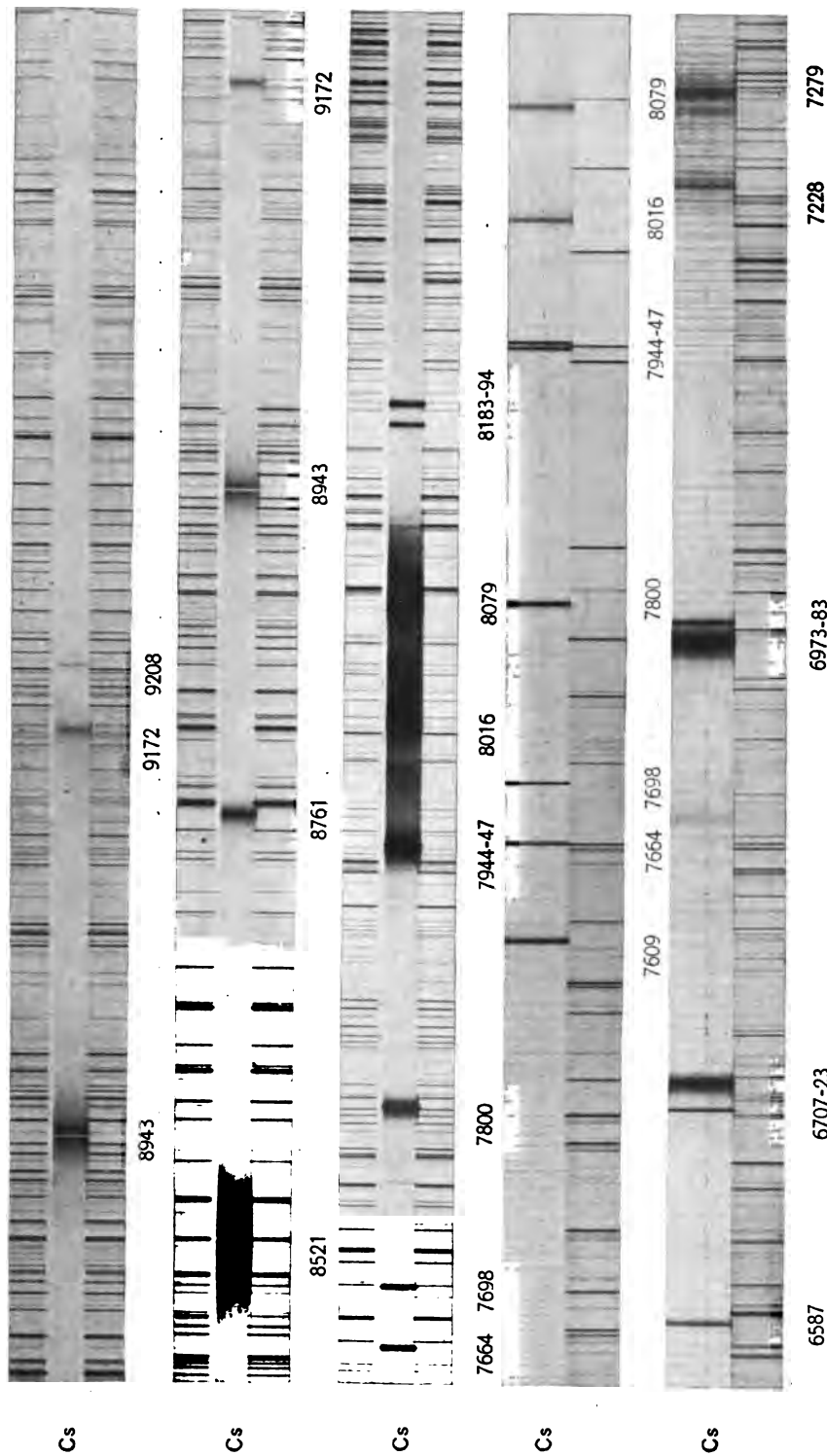


FIG. 3

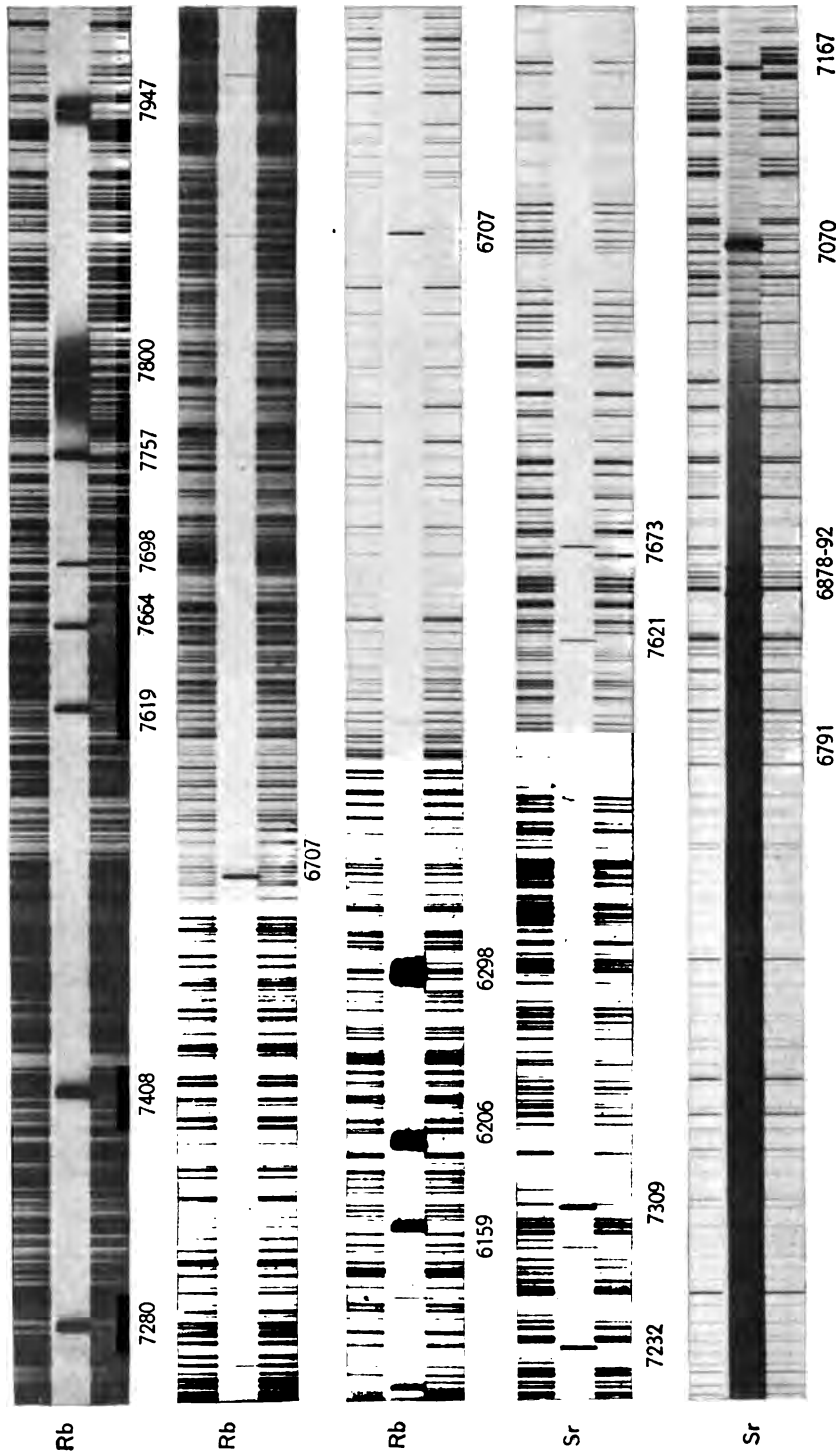


FIG. 4

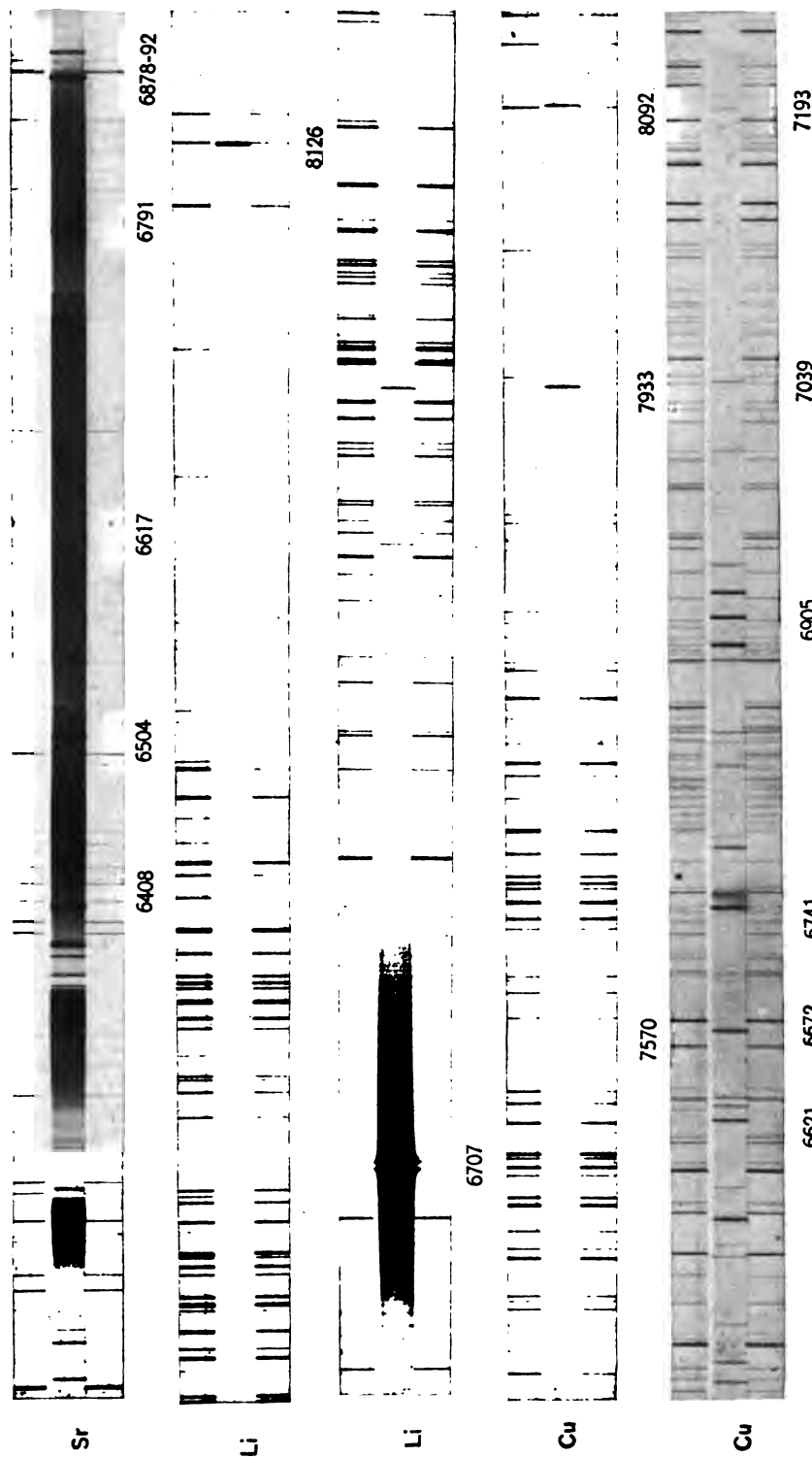


FIG. 5

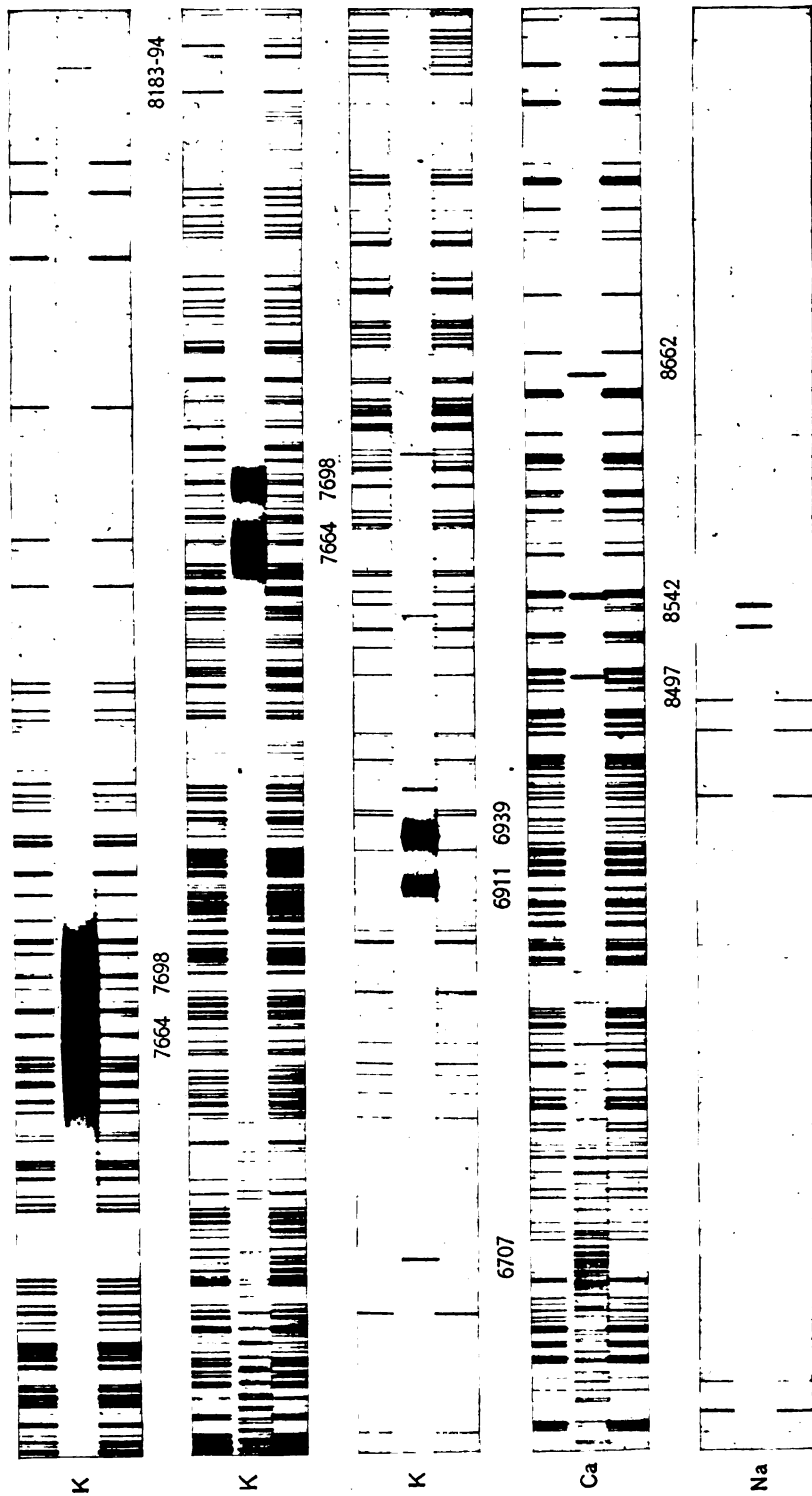


FIG. 6

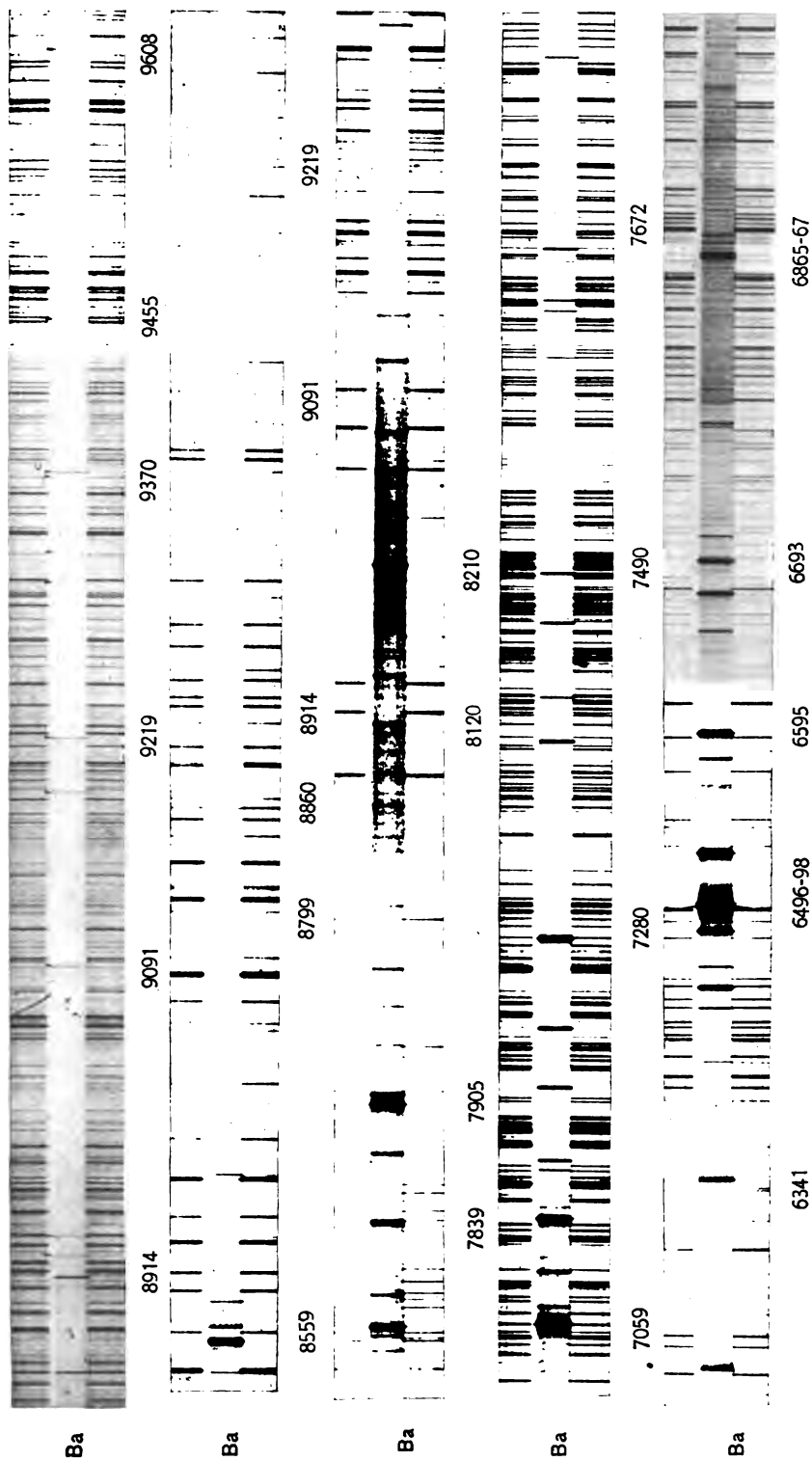


FIG. 7

TABLE 11.—Wave-Length Measurements of Impurity Lines in Spectra

	Li	Na	K	Rb	Cs	Cu	Ca	Sc	Ba	Mg
Li	6707.65 10R									
	5889.97 10R	0.85 2A	0.81 4A	0.80 8E	0.84 8A	0.96 2C	0.82 4A	0.83 8	0.84 2A	0.75 1A
	95.94 8R		.96 10E	.96 10E	.97 10C	.93 1C	90.00 8E	90.01 4E	.97 2C	.97 3C
Na	8183.30 8R		.93 8E	.96 8E	.96 8C		.97 6E	.94 3E	.94 2C	.93 2C
			.49 4D	.26	.40 1A		.33 1E		.2 1E	.34 1E
K	94.82 10R		95.06 5D	94.95	.98 1B		.91 1E		.9 2E	.90 1E
	7664.94 10R	92.5B		.96 10A	.94 8A		.92 1C		.92 2A	.9 1D
	98.01 10R	.98 4B		.96 8A	.99 6A		.98 1C		.99 2A	.9 1D
Rb	7800.29 10R		.82 2B		.84 10A					
	7947.64 8R		.64 1B		.67 4A					
	8521.12 10R			.09 3E						
Ca	6102.78 6	.76 1E	.76 1E	.76 2E				.77 1E	.76 1E	
	22.30 8	.26 2E	.26 3E	.26 3E	.23 2C			.28 3E	.26 1C	
	62.20 10	.14 2E	.18 3E	.21 4E	.16 3D			.23 4E	.20 2A	
Sc	6439.13 10	.06 2E	.10 2E	.11 3E	.13 1E			.10 1C	.07 2C	
	62.62 10	.58 2E	.55 1E	.59 3E	.57 1E				.57 2C	
	93.87 7	.78 1E	.80 1E	.82 1E				.84 1E		
Cu	7148.15 10	.16 2E	.17 1C	.14 1A	.10 1E			.23 1E	.16 2B	
	7302.21 8	.16 1E	.23 1E	.15 1E				.22 1E		
	7326.12 8	.12 2E								
Ba	8497.98 8	.90 1E							.95 1C	
	8542.06 10	.09 2E	.07 1E	.01 1E	.06 1E				.07 4B	.03 1E
	8662.10 9	.15 1E	.06 1E	.07 1E	.13 1E				.12 2A	.11 1E
Sr	6408.49 9									
	6878.36 10						.40 2E		.39 2E	
	7070.16 10	.12 2E					.17 1E		.16 2C	
Ba	6141.75 10	.68 1E	.70 1E	.73 2E	.71 1E			.77 2E		
	6496.93 10		.99 1E	.92 1E				.92 1C		
	7059.94 10	.94 1E	.94 1E	.92 1A						
	7280.20 10	.80 1E								

Many of Abney's wave lengths are in error by several tenths of an angstrom, and further identification of these Fraunhofer lines with elements in the sun is difficult and uncertain. The solar spectrum from 6800 Å to nearly 10 000 Å has been photographed recently on dicyanin-stained plates, and it is hoped that the wave-length measurements will make more identifications possible.

VIII. SUMMARY.

Accurate measurements of wave lengths and determinations of the characteristics of the emission lines in the spectra of the elements are of importance in spectroscopic analysis and for the discussion of regularities in spectra. Securing such data about the long waves has been delayed chiefly by the insentiveness of ordinary photographic plates to the red and adjacent infra-red spectral regions. More extensive use of photographic dyes is important for these spectral investigations.

Dicyanin is especially valuable and efficient as a photographic sensitizer for the long waves. The simple procedure of staining ordinary photographic plates in a mixture of dicyanin, water, alcohol, and ammonia renders the plates quite sensitive to wave lengths from 6000 Å to 9000 Å. Such plates were used to photograph the arc spectra of 20 of the chemical elements, including the alkali metals, the alkaline earths, and elements commonly found in iron as impurities.

The photographs were made in the first-order spectrum of a concave grating of 640 cm radius, the grating being mounted in parallel light. Exposures were usually limited to 30 minutes, and these sufficed to record waves longer than 9000 Å in many of the spectra. The second-order spectrum of the iron arc was photographed on either side of the first order and the long wave lengths were obtained from the standards in the iron spectrum. The wave-length measurements are given on the international scale for the arc spectra of the following elements: Lithium, sodium, potassium, rubidium, caesium, copper, calcium, strontium, barium, and magnesium. The wave lengths range from 5600 Å to 9600 Å, and the probable error is less than 0.02 Å for all lines measured more than twice. The broad and unsymmetrical character of some of the lines imposes a limit on the accuracy obtainable in the measurements.

Frequency differences of doublets in the spectra of sodium, potassium, rubidium, caesium, and copper are shown by these

wave-length determinations to be constant in most cases to one part in 100 000 in the number of waves per centimeter.

Comparison of the spectra made it possible to detect many impurities in the elements used for light sources. Still more extensive spectral investigations are required in the region of long wave lengths to identify all the lines with certainty.

In conclusion, I wish to express my thanks to Prof. J. S. Ames and to Dr. K. Burns for their interest and encouragement in this work.

WASHINGTON, March 20, 1917.

SPECIFIC HEAT OF LIQUID AMMONIA

By Nathan S. Osborne and Milton S. Van Dusen

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I. INTRODUCTION

In reviewing the existing data on the thermodynamic properties of ammonia, the dearth of calorimetric measurements is at once apparent. Only when it is remembered that calorimetric data are of primary importance in the computation of tables adapted to the needs of the engineer can the difficulties of the computer be appreciated. With regard to the specific heat of liquid ammonia it appears that the experimental difficulty of measurement and the absence of urgent need for accurate values have deterred all but a few from the attempt. More recently the progress in the production of artificial refrigeration has led to a need for more accurate tables than those existing, and the measurement of the specific heat of liquid ammonia, together with other thermodynamic properties, has been undertaken in response to the expressed wish of the refrigeration industries.

² Phil. Mag., 25, p. 393, 1893; Am. Jour. Science, Series (3), 45, p. 200, 1893.

it can be estimated that the heat capacity between 0° and 24° was approximately 15 per cent lower than it would have been under saturation conditions and that between 26° and 46° it was about 21 per cent lower. Both the published values and the values recomputed as indicated are shown in Fig. 1.

Elleau and Ennis³ (1898) also used the method of mixtures between 0° and $+25^{\circ}$. Nine grams of ammonia in a capsule of 20 cm³ capacity were used.

Dieterici and Drewes⁴ (1904) used the Bunsen ice calorimeter. The specimen of ammonia was inclosed in a sealed glass capsule of 1.2 cm³ capacity.

A. J. Wood⁵ (1912) used the method of mixtures with a sample of about 60 g of ammonia. Six experiments were made between 16° and 20° C.

Keyes and Brownlee⁶ (1916) have given an equation based upon unpublished experimental work of Babcock.

III. GENERAL DESCRIPTION OF APPARATUS AND METHOD

In carrying out the measurements of specific heat of the liquid two independent methods were used. In one method the heat added to a fixed amount confined in the calorimeter and the resulting change in temperature are measured under saturation conditions. By using independent data for the specific volumes of the two phases and for the latent heat of vaporization the corrections for the heat used to change the temperature and amount of vapor present may be determined and the specific heat of the saturated liquid obtained.

In the other method the calorimeter is kept full of liquid at a constant pressure above saturation, the heat added to the variable quantity, and the resulting change in temperature measured, due consideration being paid to the amount and temperature of the expelled portion. The result of this method of measurement is to give the specific heat of the liquid at a constant pressure.

By using the independent data for the latent heat of pressure variation of the liquid, the specific heat of the saturated liquid may be computed by general thermodynamic formulas, thus giving an independent check on the two methods of measurement.

³ Jour. Franklin Inst., 145, pp. 189, 280, 1898.

⁴ Zeit. für die ges. Kalte Ind., 11, pp. 21, 47, 1904.

⁵ Ice and Refrigeration, April, 1912.

⁶ Thermodynamic Properties of Ammonia, John Wiley & Son, 1926. (Since the foregoing went to press a description of the experimental determinations has appeared in Jour. Am. Chem. Soc., XXXIX, p. 1524; August, 1917.)

The calorimeter used in making the experimental determinations having been previously described in detail elsewhere,⁷ only a brief description is here given. The instrument is of the aneroid type and was specially designed to meet the requirements of this investigation. A metal shell of sufficient strength is made in the form of a cylinder having a reentrant central tube. The interior annular space contains the material to be investigated. An electric heating coil and a resistance thermometer are located in the central tube. Heat developed in the coil is transmitted to the surrounding liquid, the distribution being favored by radial metal vanes. The interior of the shell is tinned and the outside nickered. For preventing the transfer of unmeasured heat between this calorimeter and its environment it is suspended within a shield consisting of a thermally controlled jacket, with an air space between for thermal insulation. For evaluating the thermal leakage—that is, the heat exchanged with the surroundings by reason of temperature differences—multiple thermocouples with junctions distributed on the surfaces indicate temperature differences between calorimeter and jacket. Leakage is usually annulled by keeping the average jacket and calorimeter surface temperatures equal. The heat supplied to the calorimeter for producing temperature changes is developed at a nearly constant rate in the heating coil by current from a storage battery. Temperature changes are measured either by the resistance thermometer in the calorimeter or by an auxiliary resistance thermometer in the envelope, using thermocouples for the transfer. The energy supplied is determined by potentiometer measurements of current and potential drop and by the duration of the heating current, which is measured by the standard clock.

The material which is the subject of the measurement is introduced into the previously evacuated calorimeter through the tube provided for this purpose. The sample is confined in a special steel reservoir, which is closed by means of a valve. After being weighed the reservoir is suspended in an auxiliary thermoregulated bath and connected to the calorimeter. When the valves in this connection are open, the flow into the calorimeter is induced by vapor pressure difference caused by heating the reservoir above the temperature of the calorimeter. If the connecting tube is made to extend down to the bottom of the reservoir the transfer occurs by liquid flow, while if this tube terminates at the top of

⁷ This Bulletin, 14, p. 133; 1917.

the reservoir the transfer occurs by distillation. The removal of the material from the calorimeter, except in the case of actual overflow by expansion when already full of liquid, is necessarily by distillation, since the orifices are at the top. The large amount of heat transferred when distilling to or from the calorimeter can be readily extracted or added by use of the cooling device or the heating coil as needed.

IV. THEORY OF METHODS

1. NOTATION

M = mass, in grams, of ammonia in the calorimeter.

θ = temperature, in centigrade degrees, of the thermodynamic scale.

t = time.

I = heating current in amperes (mean value).

E = potential difference, in volts, across heating coil (mean value).

Q = heat added in joules.

N = heat capacity in joules per degree of calorimeter.

V = volumetric capacity in cm^3 of calorimeter.

t' = duration of heating current.

t_2 = duration of entire experiment.

h = mean thermocouple indication, in millimeters of the galvanometer scale, during entire experiment.

B = coefficient of thermal leakage in joules per minute per millimeter galvanometer deflection.

θ_1, θ_2 = initial and final temperatures of calorimeter and contents when in equilibrium.

$\Delta\theta = \theta_2 - \theta_1$.

$\theta_m = \frac{\theta_2 + \theta_1}{2}$. The subscript m will be used to refer to temperature θ_m .

C_p = specific heat at constant pressure p and temperature θ .

σ = specific heat, in joules per gram per degree, of the saturated liquid—that is, the liquid in equilibrium with the vapor.

σ' = specific heat, in joules per gram per degree, of the saturated vapor.

u = specific volume in cm^3/g of the saturated liquid.

u' = specific volume in cm^3/g of the saturated vapor.

L = latent heat of vaporization, in joules per gram, under saturation conditions.

x = mass of vapor per gram total contents; i. e., dryness factor.

$1 - x$ = mass of liquid per gram total contents.

p = pressure.

π = saturation pressure at temperature θ .

l = latent heat of pressure variation of liquid.

2. DETERMINATION OF HEAT ADDED

The heat supplied to the system composed of the calorimeter and the material which is the subject of measurement consists of two parts, namely, the heat developed in the electric heating coil and the heat transferred by thermal leakage. The latter is usually zero or small. This heat added is distributed in two parts, namely, that which is absorbed by the calorimeter and that absorbed by the contents. Using the symbols adopted, the above statement is expressed by the equation:

$$\Delta Q + N\Delta\theta = I E t' + B h t, \quad (1)$$

where ΔQ denotes the heat absorbed by the contents.

3. DETERMINATION OF SPECIFIC HEAT OF SATURATED LIQUID

The formula expressing the specific heat, σ , of the saturated liquid in terms of the measured quantity of heat added, ΔQ ; the observed change in temperature, $\Delta\theta$, from the initial to the final equilibrium condition; the total mass, M , present; the latent heat of vaporization, L ; and the dryness factor, x , is obtained by the use of two thermodynamic relations for saturation conditions.

Disregarding as insignificant the thermal expansion of the calorimeter and consequent change in total volume of the contents, the amount of heat, dQ , required to produce a temperature change, $d\theta$, under saturation conditions, no external work being done, is given by the equation:

$$\frac{dQ}{M} = (1 - x)\sigma d\theta + x\sigma' d\theta + L \frac{dx}{d\theta} d\theta \quad (2)$$

Since $\sigma' - \sigma = \frac{dL}{d\theta} - \frac{L}{\theta}$, by substituting and combining terms:

$$\frac{dQ}{M} = \sigma d\theta - \frac{Lx}{\theta} d\theta + \frac{d}{d\theta}(Lx) d\theta \quad (3)$$

Integrating between θ_1 and θ_2 and dividing by $\Delta\theta$

$$\frac{1}{M} \frac{\Delta Q}{\Delta\theta} = \frac{1}{\Delta\theta} \int_{\theta_1}^{\theta_2} \sigma d\theta - \frac{1}{\Delta\theta} \int_{\theta_1}^{\theta_2} \frac{Lx}{\theta} d\theta + \frac{L_2x_2 - L_1x_1}{\Delta\theta} \quad (4)$$

For numerical computation of σ this equation may be simplified if the variations of σ and $\frac{Lx}{\theta}$ with θ are such that over the temperature interval $\theta_2 - \theta_1$ the following approximations may be made:

$$\frac{1}{\Delta\theta} \int_{\theta_1}^{\theta_2} \sigma d\theta = \sigma_m, \quad \frac{1}{\Delta\theta} \int_{\theta_1}^{\theta_2} \frac{Lx}{\theta} d\theta = \left(\frac{Lx}{\theta}\right)_m$$

where σ_m denotes the value of σ at the arithmetical mean temperature θ_m and $\left(\frac{Lx}{\theta}\right)_m$ denotes the value of $\frac{Lx}{\theta}$ at that same temperature.

Making these substitutions and solving for σ_m

$$\sigma_m = \frac{\Delta Q}{\Delta\theta M} - \frac{L_2x_2 - L_1x_1}{\Delta\theta} + \left(\frac{Lx}{\theta}\right)_m \quad (5)$$

The dryness factor, x , is given by the formula

$$x = \frac{\frac{V}{M} - u}{u' - u} \quad (6)$$

4. DETERMINATION OF SPECIFIC HEAT OF LIQUID AT CONSTANT PRESSURE

In this method of measurement of specific heat, all the heat, ΔQ , added to the contents of the calorimeter is used to produce a change in temperature of the liquid phase, but not all of the contents undergo the same change in temperature. Only that part which still remains in the calorimeter at the final equilibrium temperature passes through the entire temperature increment, that which flows out having a certain temperature, θ' , for each infinitesimal portion at the time when it emerges. This variable temperature, θ' , is observable by means provided in the instrument, and although not actually observed in each experiment its relation to the other variables can be established once for all.

The quantity of heat, dQ , added to the liquid in consequence of the change of temperature, $d\theta$, can be divided into two parts, the first the quantity required to produce the change, $d\theta$, in the entire

amount, M , contained at any temperature, θ , and second the quantity required to heat from θ to θ' the amount, dM , expelled during the change, $d\theta$, thus

$$dQ = MC_p d\theta + dM \int_{\theta}^{\theta'} C_p d\theta \quad (7)$$

where θ denotes the uniform temperature which the contents would ultimately attain if thermally isolated at constant pressure. This temperature may be called the effective mean temperature.

In order to get an expression for the total amount of heat, ΔQ , added to the liquid during an experiment, M , C_p and θ' must be expressed in terms of θ , so that equation (7) may be integrated.

Now the experiments show that M and C_p vary with the temperature in such manner that no significant error will be incurred within the range of temperature of any single experiment if they be expressed as linear functions of θ , thus

$$M = (M)_m [1 + b(\theta - \theta_m)]$$

$$C_p = (C_p)_m [1 + a(\theta - \theta_m)]$$

where $(M)_m$ and $(C_p)_m$ are the values of M and C_p , respectively, at the mean temperature θ_m , and b and a are constants. From the first of these equations it follows that

$$dM = (M)_m b d\theta$$

and from the second

$$\int_{\theta}^{\theta'} C_p d\theta = (C_p)_m (\theta' - \theta) \left[1 - a\theta_m + \frac{a}{2} (\theta' + \theta) \right]$$

Making the foregoing approximations in equation (7) by substituting the four above relations, the equation becomes

$$dQ = (M)_m (C_p)_m \left\{ \left[1 + a(\theta - \theta_m) \right] \left[1 + b(\theta - \theta_m) \right] + b(\theta' - \theta) \left[1 - a\theta_m + \frac{a}{2} (\theta' + \theta) \right] \right\} \quad (8)$$

For convenience of integration the equation may be rearranged and written in the form

$$dQ = (M)_m (C_p)_m \left[(1 - a\theta_m)(1 - b\theta_m) + a(1 - b\theta_m)\theta + \frac{ab}{2}\theta^2 + b(1 - a\theta_m)\theta' + \frac{ab}{2}\theta'^2 \right] d\theta \quad (9)$$

in which $(M)_m$, $(C_p)_m$, a , b , and θ_m are constants, while θ' is yet to be expressed in terms of θ .

The relation between the temperature, θ' , at which the overflow leaves the calorimeter and the instantaneous effective mean temperature, θ , of the liquid remaining behind was investigated by means of supplementary experiments, the details of which are described in another section. At the start of an experiment θ and θ' are both equal to the initial temperature θ_1 . As the experiment proceeds and θ rises, θ' also rises but less rapidly at first, so that it lags behind θ . After a certain time, however, this lag becomes nearly constant, and during the remainder of the period of outflow θ' is below θ by a nearly constant amount, which may be denoted by λ . Since the initial period is short and λ is small, it is sufficiently accurate to write

$$\begin{aligned}\theta' &= \theta_1, \text{ for } \theta_1 \leq \theta \leq \theta_1 + \lambda \\ \theta' &= \theta - \lambda, \text{ for } \theta_1 + \lambda \leq \theta \leq \theta_2\end{aligned}$$

In integrating equation (9) from θ_1 to θ_2 it is therefore necessary to proceed with the terms which contain θ' in two steps, the first from θ_1 to $\theta_1 + \lambda$ with $\theta' = \theta_1$, and the second from $\theta_1 + \lambda$ to θ_2 with $\theta' = \theta - \lambda$.

Accordingly the following expression is obtained for the whole quantity of heat, ΔQ , added to the liquid during the experiment:

$$\begin{aligned}\Delta Q = (M)_m(C_p)_m &\left\{ \int_{\theta_1}^{\theta_2} \left[(1 - a\theta_m)(1 + b\theta_m) + a(1 - b\theta_m)\theta + \frac{ab}{2}\theta^2 \right] d\theta \right. \\ &+ \int_{\theta_1}^{\theta_1 + \lambda} \left[b(1 - a\theta_m)\theta_1 + \frac{ab}{2}\theta_1^2 \right] d\theta \\ &\left. + \int_{\theta_1 + \lambda}^{\theta_2} \left[b(1 - a\theta_m)(\theta - \lambda) + \frac{ab}{2}(\theta - \lambda)^2 \right] d\theta \right\} \quad (10)\end{aligned}$$

If now for simplicity, θ_1 be taken equal to zero, and the notation $\frac{\theta_2 - \theta_1}{2} = \theta_m = \frac{1}{2}\Delta\theta$ be used, the result of carrying out the integrations in equation (10) is to give the equation:

$$\Delta Q = (M)_m(C_p)_m\Delta\theta \left[1 - b\lambda + \frac{ab\lambda}{4} + \frac{ab(\Delta\theta)^2}{12} + \frac{b\lambda^2}{2\Delta\theta} \left(1 - \frac{a\lambda}{3} \right) \right] \quad (11)$$

The values of a , b , and λ are, within the range of the present experiments, so small that when $\Delta\theta$ is 10° the sum of the last three

terms within the bracket does not exceed 0.0001 of the bracketed expression, and these terms may, therefore, be ignored. The equation can be completely solved for $(C_p)_m$, but for the purpose of computing this quantity from the observed data it is put in the more convenient form

$$(C_p)_m = \frac{1}{(M)_m} \frac{\Delta Q}{\Delta \theta} - \frac{1}{(M)_m} \frac{\Delta M}{\Delta \theta} (C_p)_m \lambda \quad (12)$$

The term $\frac{1}{(M)_m} \frac{\Delta Q}{\Delta \theta}$ represents the heat added to the liquid per unit temperature change, per unit mass in calorimeter, when in equilibrium at the mean temperature of the experiment. It would equal the specific heat at the mean temperature if the experiment proceeded under ideal conditions such that the heat were instantaneously distributed uniformly throughout the liquid, making $\lambda = 0$, for then the second term would vanish.

The term $\frac{1}{(M)_m} \frac{\Delta M}{\Delta \theta} (C_p)_m \lambda$ represents the amount of heat per unit temperature change of contents, per unit mass of contents at the mean temperature of the experiment, required to change the temperature of the expelled portion from the effective mean temperature to the overflow temperature. It is, indeed, the correction to the preceding term for the departure of the actual experiment from the ideal process just described above.

5. CORRELATION OF THE TWO METHODS

Having the specific heat, σ , of the saturated liquid, determined by measurements made under saturation conditions, and the specific heat, C_p , at a constant pressure, p , above the saturation pressure for the highest temperature reached during the determination, the first step in the correlation of these results, for the purpose of an experimental check, is to compute the limiting value which C_p approaches as p approaches the saturation pressure, π , for the given temperature. For this computation a knowledge of the latent heat of pressure variation, l , is required over the range of temperature and pressure including the specific heat determinations.

By definition,

$$C_p = \left(\frac{\partial Q}{\partial \theta} \right)_p$$

$$l = \left(\frac{\partial Q}{\partial p} \right)_\theta$$

since $dQ = \theta d\phi$ where ϕ is entropy,

$$C_p = \theta \left(\frac{\partial \phi}{\partial \theta} \right)_p$$

$$\frac{l}{\theta} = \left(\frac{\partial \phi}{\partial p} \right)_\theta$$

differentiating the first of these identities with respect to p and the second with respect to θ

$$\frac{\partial}{\partial p} (C_p) = \theta \frac{\partial^2 \phi}{\partial \theta \partial p}$$

$$\frac{\partial}{\partial \theta} \left(\frac{l}{\theta} \right) = \frac{\partial^2 \phi}{\partial \theta \partial p}$$

hence it follows that

$$\frac{\partial}{\partial p} (C_p) = \theta \frac{\partial}{\partial \theta} \left(\frac{l}{\theta} \right)_p \quad (13)$$

The change in C_p for a change in pressure from the observed pressure, p , to the saturation pressure, π , at the same temperature, θ , is obtained by integrating equation (13) over this pressure interval. Examination of the experimental data for l showed that within the range of temperature and pressure employed in the experiments the quantity $\frac{\partial}{\partial \theta} \left(\frac{l}{\theta} \right)$ may be treated as independent of p in this integration without introducing any significant error. Therefore, since θ also is constant during this integration, the following equation is obtained for the change in C_p :

$$[C_p]_{\theta, \pi} - [C_p]_{\theta, p} = \theta \frac{\partial}{\partial \theta} \left(\frac{l}{\theta} \right) (\pi - p) \quad (14)$$

by means of which the values of $[C_p]_{\theta, \pi}$ may be computed from the data for C_p and l .

The next step is to compute the specific heat, σ , of the saturated liquid from these limiting values of $[C_p]_{\theta, \pi}$ just found.

Starting with the general equation

$$dQ = C_p d\theta + l dp,$$

which is true for any simultaneous variations of p and θ during which the fluid remains homogeneous, it follows that

$$\frac{dQ}{d\theta} = C_p + l \frac{dp}{d\theta}$$

If the change be restricted to saturation conditions, then by definition $\frac{dQ}{d\theta} \equiv \sigma$ and $\frac{dp}{d\theta} \equiv \frac{d\pi}{d\theta}$,
whence

$$\sigma = \left[C_p \right]_{\theta, \pi} + \left[l \right]_{\theta, \pi} \frac{d\pi}{d\theta}. \quad (15)$$

The values of σ obtained indirectly by the foregoing method of computation from observed values of C_p and l should, of course, agree with those obtained by the method of measurement under actual conditions of saturation, and the comparison of the two series of values affords a valuable test of accuracy.

V. MATERIAL

The material used in these determinations was prepared by Messrs. McKelvy and Taylor of the chemical division of this Bureau by methods described in detail in an independent paper.* A brief description of the process of preparation is here given.

A sample of commercial ammonia was transferred by distillation into a steel container which would hold about a kilogram. From this it was again transferred by distillation into a similar vessel containing metallic sodium, to remove any remaining traces of water. Following this dehydration, the purification was continued by from six to eight consecutive fractional distillations, the first and last tenths of each distillation being rejected. Removal of the rejected first fractions was performed in such a way as to extract the noncondensing gas present.

Two samples purified in the above manner, designated A and C, were used in the determination.

Sample A, prepared March, 1916, and used in the determinations by the first method, was made from commercial anhydrous ammonia manufactured from ammonium sulphate.

Tests showed the following impurities in this sample: Noncondensing gases in the vapor phase at 25° C, 1 part in 10 000 by volume; water, 1 part in 10 000 by weight.

Sample C, prepared July, 1916, and used in the determinations by the second method, was made from commercial anhydrous ammonia manufactured by the synthetic method. The sample showed about the same proportion of noncondensing gases as sample A.

* To be published in this Bulletin.

VI. FIRST METHOD. MEASUREMENTS UNDER SATURATION CONDITIONS

In the determinations by this method the ammonia contained in the calorimeter is part liquid and part vapor, and, therefore, the pressure is that of saturated vapor, and the state and relative amount of each phase when in equilibrium are completely defined by the temperature. The observations yield, as a direct result, the heat added to the contained ammonia per degree temperature rise. For the obvious purpose of minimizing the variation with temperature of the heat content of the vapor present, only enough vapor space is allowed when filling to insure a safe margin at all temperatures to be experienced. From the measured heat capacity of the total contents the specific heat of the liquid may be computed, if the specific volumes of the two phases and the latent heat of vaporization which determine the correction for change in heat content of the vapor are known.

1. EXPERIMENTAL DETAILS

The sample was introduced into the calorimeter and the amount determined by difference between the initial and final weights of the reservoir. Each filling served for a series of experiments. The heat capacity of the filled calorimeter was determined over suitable temperature intervals throughout the range to be covered. The heat capacity of the empty calorimeter was determined by independent experiments. Briefly stated, the manipulations and observations during an experiment occurred as follows:

The calorimeter containing the sample was cooled by means of the special cooling device to the initial temperature of the first experiment of a series. The jacket was brought under control of the thermoregulator at the temperature of the calorimeter. The initial temperature of the calorimeter was determined by observing the resistance of one or both the platinum thermometers. Electric current was then passed through the heating coil of the calorimeter for a measured interval of time. During this time alternate readings of current and potential drop were made periodically to determine the rate of energy supply to the calorimeter. Meanwhile, by hand control of the jacket-heating current, the temperature of the jacket surface was kept nearly equal to the rising temperature of the calorimeter surface, the deviations being observed by means of the thermocouples, and the sum of the deviations during an experiment usually being made zero.

After the interruption of the heating current the jacket was again brought under control of the thermoregulator, and when the calorimeter attained thermal equilibrium the thermometer resistance was again observed to determine the final temperature. This was usually taken as the initial temperature of the succeeding experiment, and the process repeated until completion of the series.

A complete description of the method and the results of the calibration of the calorimeter resistance thermometer and of the determinations of the heat capacity of the calorimeter is given in a preceding paper.⁹

2. RESULTS OF MEASUREMENTS

The experimental data¹⁰ obtained from the observations made by the first method, together with the computations for obtaining the specific heat, σ , of the saturated liquid, are given in Tables 1 and 2. The computations are made following the method described in Section IV.

⁹ This Bulletin, 14, p 133; 1917.

¹⁰ The laboratory scale of temperature actually used in the measurements given in this paper is the scale of a resistance thermometer of Heraeus platinum of highest purity according to the equation:

$$\theta = \frac{R\theta - R_0}{R_{100} - R_0} 100 + \delta \left(\frac{\theta}{100} - 1 \right) \frac{\theta}{100}$$

R_0 and R_{100} are the resistances of the thermometer at the temperatures, under normal atmospheric pressure, of melting ice and saturated steam, respectively, and δ is determined by substituting for R_0 in the above equation the resistance of the thermometer at the temperature of sulphur vapor under normal atmospheric pressure, and 444.6 for θ . The value of δ for the calorimeter thermometer was found by comparison with a standardized thermometer to be 1.48. The departure of the scale so defined from the "thermodynamic" or "ideal gas" scale down to -50° C is not more than the limit of accuracy of existing gas-thermometer data.

For all mathematical relations involving the second law of thermodynamics the temperatures are necessarily referred to the absolute zero. In recording laboratory data numerically it is usually convenient to use the ice point as zero. Sometimes in the mathematical analyses where differences only are involved it is convenient to simplify the algebra by choosing an arbitrary zero for a particular case. In the numerical tables of data and reductions no ambiguity arises on this account, as the experiments were conducted between -50° and $+50^\circ$ from the ice point, so that the recorded temperatures never numerically exceed 50° while the absolute temperatures are never less than 300° .

Where numerical values are given, the joule used in this paper is determined by the relation $\frac{Q}{t} = \frac{E^2}{R}$, where Q is the number of joules transformed into heat in a given electric circuit in t seconds, E the number of volts potential drop, and R the number of ohms resistance; taking 1 volt = $\frac{1}{1.01830} \times \text{emf of mean Weston normal cell at } 20^\circ \text{ C}$, and 1 ohm = resistance at 0° C of 106.300 cm of uniform mercury column 14.4521 g in mass. The difference between the international joule, realized thus, and the absolute joule is, according to present evidence, perhaps 1 part in 3000. (B. S. Circular No. 60, 1st ed., p. 56; 1916.) The ampere is used only as an intermediary unit, being determined by the relation $I = \frac{E}{R}$ where I = number of amperes.

TABLE 1.—First Method—Measurements of Specific Heat of Liquid Ammonia, Under Saturation Conditions

PART 1.—MASS IN CALORIMETER, 274.33 GRAMS																
Calculation of $\frac{\Delta Q}{\Delta \theta}$ $\frac{\Delta Q}{\Delta \theta} = \frac{I_{EV} - B_{th}}{\Delta \theta} - N$																
Date	Ex- peri- ment No.	$\Delta \theta$ Therm. 15.046	$\Delta \theta$ Therm. 4725	$\Delta \theta$ Mean	I Calo- rimeter heating current, value	$P, D.$ across heating cell, mean value	t' Duration of heating current	I_{EV} Total energy supplied electri- cally	h Av. thermo- couple dial.	t_2 Duration of exper.	B_{th} Thermal leakage to cal.	Net energy to cal. and contents	Heat cap., calorimeter and contents	N Heat capacity calorimeter	$\frac{\Delta Q}{\Delta \theta}$ Heat capacity contents	$\frac{\theta_1 + \theta_2 - \theta_m}{2}$ Mean temper- ature
1916																
Mar. 13	1	10.005	10.009	10.007	1.8925	18.958	599.97	21 526	0	42	0	21 526	2151.1	929.0	1222.1	-38.68
	2	9.875	9.872	9.8735	1.8900	18.943	599.87	21 477	0	37	0	21 477	2175.2	941.8	1233.4	-28.74
	3	9.756	9.756	9.756	1.8877	18.930	599.92	21 438	0	39	0	21 438	2197.4	953.8	1243.9	-18.93
	4	9.821	9.824	9.8225	1.9031	19.094	600.00	21 803	0	37	0	21 803	2219.7	964.5	1255.2	-9.13
Mar. 14	5	9.708	9.707	9.7075	1.9013	19.082	599.98	21 768	0	40	0	21 768	2242.4	974.8	1267.2	+0.63
	1	9.787	9.785	9.786	1.9002	19.064	600.00	21 735	0	39	0	21 735	2221.0	965.0	1256.0	-8.56
	2	10.161	10.160	10.1605	1.9457	19.529	600.04	22 800	0	47	0	22 800	2244.0	975.5	1268.5	+1.42
	3	10.048	10.046	10.047	1.9448	19.527	599.97	22 784	0	36	0	22 784	2267.7	985.7	1282.0	+11.52
Mar. 15	4	10.427	10.429	10.428	1.9921	20.008	599.95	23 913	0	41	0	23 913	2283.1	995.8	1297.3	+21.76
	5		10.306	10.306	1.9910	20.006	600.03	23 900	0	40	0	23 900	2319.0	1005.8	1313.2	+32.12
	1	5.252		5.252	1.9280	19.324	300.06	11 185	0	39	0	11 185	2129.7	918.5	1211.2	-46.32
	2	10.993	10.991	10.992	1.9277	19.322	599.96	22 347	0	38	0	22 347	2150.4	929.3	1221.1	-38.49
Mar. 16	3	10.268	10.269	10.2685	1.9264	19.321	600.08	22 335	0	37	0	22 335	2175.1	942.5	1232.6	-28.16
	4	10.152	10.151	10.151	1.9252	19.318	599.98	22 314	0	44	0	22 314	2196.1	954.6	1243.5	-17.95
	5	10.041	10.040	10.040	1.9241	19.316	600.08	22 303	0	42	0	22 303	2221.3	965.8	1255.5	-7.86
	1	10.100	10.099	10.099	1.9863	19.443	600.00	22 588	0	47	0	22 588	2236.5	972.7	1263.8	-1.33
Mar. 16	2	9.989	9.989	9.989	1.9851	19.439	600.09	22 573	0	39	0	22 573	2259.8	983.0	1276.8	+8.72
	3	9.877	9.875	9.876	1.9839	19.435	600.05	22 553	0	37	0	22 553	2283.6	992.8	1290.8	+18.65
	4	9.760	9.760	9.760	1.9824	19.427	600.05	22 526	0	39	0	22 526	2308.0	1002.3	1305.7	+28.47
	5	9.647	9.646	9.6465	1.9814	19.424	600.03	22 511	0	41	0	22 511	2333.6	1011.3	1322.3	+38.17

TABLE 1.—First Method—Measurements of Specific Heat of Liquid Ammonia, Under Saturation Conditions—Continued

PART 2.—MASS AMMONIA IN CALORIMETER, 274.72 GRAMS

Date	Ex- peri- ment No.	$\Delta\theta$ Therm. 15.046	$\Delta\theta$ Therm. 4725	$\Delta\theta$ Mean	I Calo- rimeter heating current, mean value	E P. D. across heating coil, mean value	t' Duration of heating current seconds	$I E t'$ Total energy supplied electrically	A Av. thermo- couple defl.	t_2 Duration of exper.	$B M_2$ Thermal leakage to cal.	Net energy to cal. and contents	Heat cap., calorimeter and contents	N Heat capacity calorimeter	$\frac{\Delta Q}{\Delta\theta}$ Heat capacity contents	$\frac{\theta_1 + \theta_2}{2}$ Mean temper- ature
1916 Mar. 18	1	9.471	9.472	9.4715	1.8420	18.453	599.96	20 393	0	40	0	20 393	2153.1	929.0	1224.1	-38.65
	2	9.366	9.365	9.3655	1.8407	18.448	599.89	20 370	0	39	0	20 370	2175.0	941.2	1233.8	-29.23
	3	9.265	9.269	9.267	1.8391	18.440	600.10	20 351	0	41	0	20 351	2196.1	952.4	1243.7	-19.92
	4	9.168	9.168	9.168	1.8378	18.436	600.21	20 336	0	36	0	20 336	2218.1	962.6	1255.5	-10.70
Mar. 20	1	9.750	9.745	9.7475	1.8956	19.019	600.00	21 631	+2.1	37	+11	21 642	2220.3	964.2	1256.1	-9.37
	2		4.816	4.816	1.8862	9.495	600.02	10 745	+3.6	47	+25	10 770	2236.3	972.0	1264.3	-2.08
	3		4.776	4.776	1.8831	9.487	599.98	10 719	+3.1	38	+17	10 736	2247.9	976.8	1271.1	+2.71
	4		4.729	4.729	1.8810	9.478	600.00	10 697	-2.8	46	-19	10 678	2258.0	981.7	1276.3	+7.46
Mar. 21	1	9.549	9.549	9.549	2.6065	13.099	599.97	20 485	+2.9	44	+18	20 503	2147.1	926.2	1220.9	-40.89
	2	9.449	9.447	9.448	2.6047	13.099	600.05	20 473	+3.6	43	+22	20 495	2169.2	938.5	1230.7	-31.89
	3	9.342	9.343	9.3425	2.6035	13.098	600.05	20 462	+1.7	38	+9	20 471	2191.2	949.8	1241.4	-21.96
	4	9.237	9.237	9.237	2.6013	13.094	600.03	20 438	0	39	0	20 438	2212.6	960.5	1252.1	-12.70
Mar. 22	5	9.140	9.140	9.140	2.5992	13.089	600.09	20 416	+0.4	37	+2	20 436	2233.9	970.5	1263.4	-3.51
	1	9.106	9.104	9.105	2.5948	13.081	600.04	20 396	0	38	0	20 396	2253.5	970.0	1263.5	-4.00
	2	8.983	8.983	8.983	2.5875	13.035	600.07	20 359	0	42	0	20 359	2253.0	979.3	1273.7	+5.05
	3	8.888	8.891	8.8895	2.5853	13.031	600.07	20 216	0	42	0	20 216	2274.1	988.3	1285.8	+13.98
Mar. 23	4	8.793	8.790	8.7915	2.5828	13.026	600.03	20 187	0	44	0	20 187	2296.2	996.8	1298.4	+22.82
	5		8.707	8.707	2.5814	13.026	600.00	20 175	0	42	0	20 175	2317.1	1005.4	1311.7	+31.37
	1	8.698	8.696	8.697	2.5788	13.014	600.12	20 141	0	41	0	20 141	2315.8	1004.6	1311.2	+30.85
	2	8.599	8.600	8.5995	2.5737	13.006	600.10	20 103	0	36	0	20 103	2337.7	1012.4	1325.3	+39.49
	3	4.250	4.250	4.250	2.5709	12.985	299.99	10 015	0	40	0	10 015	2356.5	1018.2	1338.8	+45.92

TABLE 2.—First Method—Corrections for Vapor

$$\sigma = \frac{1}{M} \frac{\Delta Q}{\Delta \theta} - \frac{L_2 x_2 - L_1 x_1}{\Delta \theta} + \left[\frac{Lx}{\theta} \right]_m$$

PART 1

Heat cap. contents from Table 1 $\frac{1}{M} \frac{\Delta Q}{\Delta \theta}$	Correction terms for vapor		Specific heat of liquid $\sigma_{\text{obs.}}$	Calc. from empirical equation A $\sigma_{\text{calc.}}$	Obs.— Calc.	Mean temp. from Table 1 θ_m
	$\frac{L_2 x_2 - L_1 x_1}{\Delta \theta}$	$\left[\frac{Lx}{\theta} \right]_m$				
J/g.deg.	J/g.deg.	J/g.deg.	J/g.deg.	J/g.deg.	J/g.deg.	° C
4.455	+0.016	+0.002	4.441	4.441	±0.000	−38.68
4.496	.018	.002	4.480	4.474	+ .006	−28.74
4.534	.022	.003	4.515	4.511	+ .004	−18.93
4.576	.023	.004	4.557	4.553	+ .004	− 9.13
4.620	.023	.004	4.601	4.599	+ .002	+ .63
4.578	.023	.004	4.559	4.554	+ .005	− 8.56
4.624	.023	.004	4.605	4.603	+ .002	+ 1.42
4.673	.021	.005	4.657	4.657	± .000	+11.52
4.729	.009	.005	4.725	4.723	+ .002	+21.76
4.787	− .010	.005	4.802	4.798	+ .004	+32.12
4.415	+ .011	.001	4.405	4.417	− .012	−46.32
4.451	.016	.002	4.437	4.441	− .004	−38.49
4.493	.019	.002	4.476	4.476	± .000	−28.16
4.533	.021	.003	4.515	4.514	+ .001	−17.95
4.577	.024	.004	4.557	4.558	− .001	− 7.86
4.607	.023	.004	4.588	4.590	− .002	− 1.33
4.654	.021	.005	4.638	4.642	− .004	+ 8.72
4.705	.014	.005	4.696	4.703	− .007	+18.65
4.760	− .003	.005	4.768	4.769	− .001	+28.47
4.820	− .023	.005	4.848	4.848	± .000	+38.17

PART 2

4.456	+0.016	+0.002	4.442	4.441	+0.001	−38.65
4.491	.017	.002	4.476	4.473	+ .003	−29.23
4.527	.021	.003	4.509	4.507	+ .002	−19.92
4.570	.023	.004	4.551	4.546	+ .005	−10.70
4.572	.023	.004	4.553	4.552	+ .001	− 9.37
4.602	.023	.004	4.583	4.585	− .002	− 2.08
4.627	.023	.004	4.608	4.610	− .002	+ 2.71
4.646	.021	.005	4.630	4.635	− .005	+ 7.46
4.444	.014	.001	4.431	4.433	− .002	−40.89
4.480	.017	.002	4.465	4.465	± .000	−31.39
4.519	.021	.003	4.501	4.499	− .002	−21.96
4.558	.022	.003	4.539	4.537	+ .002	−12.70
4.599	.023	.004	4.580	4.579	+ .001	− 3.51
4.599	.023	.004	4.580	4.577	+ .003	− 4.00
4.636	.022	.004	4.618	4.622	− .004	+ 5.05
4.680	.019	.005	4.666	4.673	− .007	+13.98
4.730	.007	.005	4.728	4.729	− .001	+22.82
4.775	− .010	.005	4.790	4.793	− .003	+31.57
4.773	− .008	.005	4.786	4.787	− .001	+30.85
4.824	− .029	.005	4.858	4.860	+ .002	+39.49
4.872	− .050	.004	4.926	4.921	+ .005	+45.92

The approximations involved in the application of this method to the data here reduced result in no error in σ_m greater than 1 part in 5000.

The calculation of Lx and $\frac{Lx}{\theta}$ is given in Table 3, the following formula being used, together with the values of L and θ .

$$x = \frac{\frac{V}{M} - u}{u' - u}$$

TABLE 3.—First Method—Calculation of Corrections for Vapor

PART 1.—FIRST FILLING, 274.33 GRAMS

Temperature θ	Total volume in calorim- eter V	Mean specific volume of contents $\frac{V}{M}$	Specific volume liquid u	Specific volume vapor u'	$\frac{V}{M} - u$	$u' - u$	Fraction which is vapor $x = \frac{\frac{V}{M} - u}{u' - u}$	Latent heat of vaporiza- tion tentative value L	Lx	$\frac{Lx}{\theta}$
° C	cm ³	cm ³ /g	cm ³ /g	cm ³ /g	cm ³ /g	cm ³ /g		joules/g	joules/g	joules/g
-50	500.8	1.8256	1.4241	2550	0.4015	2549	0.000158	1416	0.223	0.0010
-40	501.0	1.8263	1.4489	1514	.3774	1513	.000250	1388	.346	.0015
-30	501.2	1.8270	1.4754	941	.3516	940	.000374	1360	.509	.0021
-20	501.4	1.8277	1.5033	612	.3244	611	.000531	1329	.706	.0028
-10	501.6	1.8285	1.5332	414.0	.2953	412.5	.000716	1297	.929	.0035
0	501.8	1.8291	1.5658	288.7	.2633	287.1	.000917	1263	1.159	.0042
10	502.0	1.8299	1.6006	205.0	.2293	203.4	.001127	1228	1.384	.0049
20	502.2	1.8306	1.6381	148.0	.1925	146.4	.001315	1190	1.564	.0053
30	502.4	1.8313	1.6793	110.2	.1520	108.5	.001401	1147	1.608	.0053
40	502.6	1.8320	1.7251	82.8	.1069	81.1	.001319	1101	1.452	.0046
50	502.8	1.8327	1.7760	63.0	.0567	61.2	.000926	1053	.975	.0030

PART 2.—SECOND FILLING, 274.72 GRAMS

-50	1.8230		0.3989		0.000157		0.222	0.0010
-40	1.8237		.3748		.000248		.344	.0015
-30	1.8244		.3490		.000372		.505	.0021
-20	1.8251		.3218		.000527		.700	.0028
-10	1.8259		.2927		.000710		.921	.0035
0	1.8266		.2607		.000908		1.147	.0042
10	1.8273		.2267		.001115		1.368	.0048
20	1.8280		.1899		.001297		1.543	.0053
30	1.8287		.1494		.001377		1.580	.0052
40	1.8294		.1043		.001287		1.417	.0045
50	1.8301		.0541		.000884		.930	.0029

The values of x for the two fillings are shown graphically as functions of θ in Fig. 2.

The values of $\frac{L_2x_2 - L_1x_1}{\Delta\theta}$ and $\left(\frac{Lx}{\theta}\right)_m$ used in the calculations of Table 2 were interpolated graphically from Table 3.

The volume, V , of the calorimeter was determined before the calorimeter was assembled, by weighing empty and again filled with water at a known temperature. The variation in volume with temperature need not be known accurately. It was obtained from

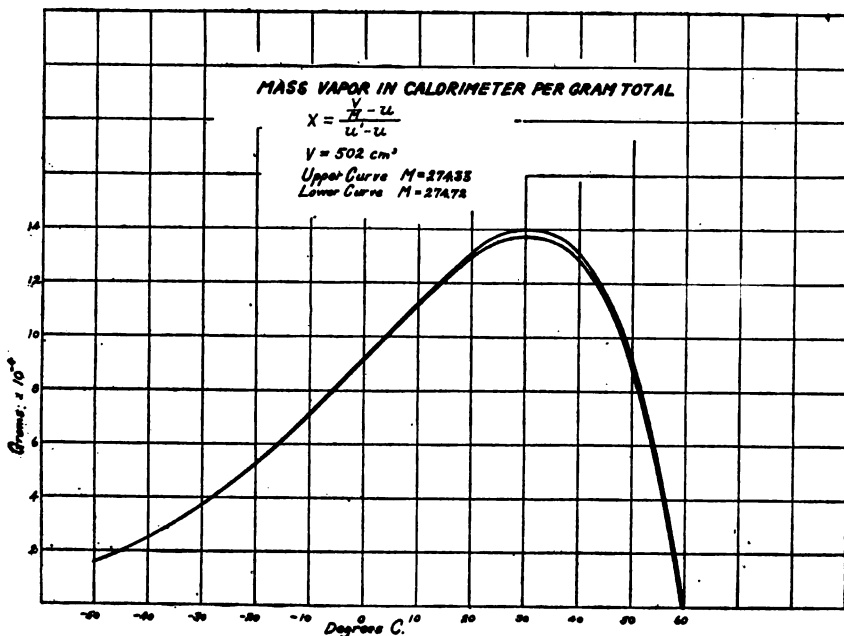


FIG. 2

independent determinations of the change in amount of liquid ammonia contained at various temperatures and pressures.

The values for the specific volumes of ammonia liquid and vapor are preliminary values from the measurements made at this Bureau by Harper, Cragoe, and O'Connor, the final results of which will be published in a separate paper.

The values of the latent heat of vaporization are from data published under a separate title.¹¹

¹¹ Osborne and Van Dusen, Latent Heat of Vaporization of Ammonia. This Bulletin, 14; 1917.

The results of the foregoing direct measurements of the specific heat of liquid ammonia under saturation conditions may be represented by the empirical equation

$$\sigma = 3.0931 - 0.00064\theta + \frac{17.329}{\sqrt{133 - \theta}} \quad (\text{A})$$

in which 133° is taken as the temperature at the critical point. The form of this equation is discussed in Section VIII. A comparison between the observed values and those calculated by this equation is given in Table 2. The possible sources of error in the first method are discussed in Section IX.

VII. SECOND METHOD—MEASUREMENT AT CONSTANT PRESSURE

The primary object of this series of measurements was to afford a check upon the accuracy of the previous series of measurements under saturation conditions. In the first method distillation in the connecting tubes might occur, thus introducing an error¹² which could not be readily evaluated. The method of measurement at constant pressure, however, in which the tubes are kept filled with liquid, eliminates this possibility. The two methods are fundamentally independent. In the first method a fixed mass of material consisting of both liquid and vapor phases is confined in the calorimeter. The relative amounts of these two phases depend upon the temperature, and corrections involving a knowledge of the specific volumes and the latent heat of vaporization are, therefore, necessary. In the second method the liquid phase alone entirely fills the calorimeter space, and, since the thermal expansion of the material differs from that of the calorimeter itself, the amount contained varies with the temperature. It was scarcely anticipated when the second method was undertaken that it would yield results comparable in precision with the first method, on account of greater experimental difficulty, although it was expected to disclose any serious error in the first method due to distillation in the connecting tubes. As eventually carried out, however, this method gave a precision equal to that of the first, and the possible sources of systematic error, while different from those in the first method, appeared no greater.

¹² It is estimated that the maximum error due to this cause could not have exceeded 1 part in 1000, and that the average error was much less than this, but the comparison between the results of the two methods furnishes the most convincing evidence as to its limit.

1. EXPERIMENTAL DETAILS

In filling the calorimeter, the reservoir used to hold the sample was provided with a tube reaching to the bottom, so that the transfer occurred by liquid flow induced by differences of vapor pressures inside the calorimeter and in the reservoir. By keeping the reservoir somewhat warmer than the calorimeter, the latter could always be kept full of liquid when the valves in the connecting tube were open, provided, of course, sufficient material were present to more than fill the calorimeter at all temperatures. In making a determination the reservoir was kept at a constant temperature by means of the thermoregulator. This temperature was observed, and it furnished, when supplemented with data for vapor pressure of ammonia at various temperatures, a means of determining the pressure, which for this purpose need not be known with great exactness. Thus, the pressure in the calorimeter remained constant, whereas the mass contained varied with the temperature. The manner of measurement of the heat added to the calorimeter and the initial and final equilibrium temperatures was identical with that already described under the first method.

The mass of liquid contained in the calorimeter at various temperatures and pressures was determined by a separate series of experiments. By keeping a fixed total mass in calorimeter and reservoir the variation in mass contained in the calorimeter was found by making weighings of the reservoir, closed off and detached from the calorimeter when the latter was in equilibrium. The total amount contained in the calorimeter when full at a single temperature and pressure was determined, as in the first method, by emptying.

It is shown (Sec. IV, 4; p. 403) where the theory of this method is discussed that the computation of the specific heat from the observed quantities can be made with sufficient accuracy, by means of a very simple formula, if the relation between the temperature, θ' , of the outflow and the instantaneous temperature, θ , of the contents has the characteristics of an ordinary case of lag. The survey of these temperatures, by which this relation was established, was made the object of the series of supplementary experiments, which are now to be described. In each of these experiments the operations were the same as in a specific heat measurement, with the exception that the usual calorimetric observations were omitted and, instead, the two temperatures θ

and θ' were each observed periodically. The place of departure of the liquid from the calorimeter was taken¹³ as the section of the outflow tube midway between the calorimeter and the jacket. The exactness of this location is not of great importance. To obtain the temperature θ' of the liquid at this place, the auxiliary thermocouples, with junctions on the outflow tubes and calorimeter top, were used. These couples were part of the permanent equipment of the calorimeter and have been described in connection therewith. The reference junction was located on the jacket. The temperature of the jacket was obtained periodically by observing the resistance of the platinum thermometer in the envelope and simultaneously observing, by means of the surface thermocouples, the difference in temperature between this thermometer and the jacket. A record was thus obtained with respect to time of the temperatures of three points in close proximity to the outflowing liquid, namely, the calorimeter top and the two points on the outflow tube one-fourth and three-fourths the distance along the tube from the calorimeter to the jacket and designated respectively *A*, *B*, and *C*.

To obtain a value for the temperature θ , the obvious method of observing the overflow of the liquid itself was employed. For this purpose a vertical glass tube closed at the top, graduated to tenths of cubic centimeters and surrounded by a stirred oil bath thermally controlled, was attached by a small tube, terminated by a valve, to the outlet provided for the manometer connection. This graduated tube constituted the isothermal stem of a thermometer, the bulb of which was the calorimeter itself, the two being connected by a tube of small volume. The fact that the stem was less than a centimeter in diameter made the thermometer sensitive to 0.01 C with ammonia as the liquid, since the volume of the bulb was about 500 cm³. The volume of the stem was sufficient to accommodate the expansion of the ammonia in any experiment covering a range of 10 degrees. This dilatometric thermometer was calibrated in place directly in terms of the equilibrium temperature of the calorimeter as a part of each experiment and for the conditions then existing, and except for corrections far within the precision of the readings¹³ it indicated the instantaneous temperature of the liquid contained in the

¹³ Assuming that the specific heat and specific volume are linear functions of the temperature, and also assuming that the volume of the calorimeter is the same as for equilibrium conditions, the indicated temperature would be identical with the mean effective temperature, θ , as defined on page 404. These assumptions are all so nearly true for the conditions of the experiments that the difference between the two temperatures is insignificant.

calorimeter, whether in equilibrium or not, provided this temperature be defined as above; that is, as the effective mean temperature of the contents or the equivalent equilibrium temperature. The calibration was found to be very nearly linear.

These experiments were carried out at various temperatures and pressures in the range of the specific heat determinations and also at various rates of heating. The jacket was kept as nearly as possible at the same temperature as the calorimeter, just as in the specific heat determinations.

The results of a single experiment are shown graphically in Fig. 3. The temperatures are all plotted with respect to time. The mean temperature, θ , of the calorimeter contents is shown by curve 1. The ideal temperature, supposing the heat added at a uniform rate to be instantaneously distributed, is shown by the dotted line, which is very nearly straight. The jacket temperature is shown by curve 2. This curve also represents the average temperature of the calorimeter surface, since the temperatures of these two surfaces were kept very closely together. The temperatures of the points *A*, *B*, and *C* are shown by curves 5, 3, and 4. (The points *A* and *B* were found to have always essentially the same temperature during the period of heating, and hence curves 5 and 3 coincide for this period.) These curves were plotted by reference to curve 2, using the reduced thermocouple readings.

By study of these curves the temperature changes may be readily analyzed. Initially all parts of the system are at the same temperature, θ_1 . In a short time after starting the heating current, the rate of rise of the temperature θ , as indicated by the outflow, has become constant and remains so until the interruption of the heating current, after which the outflow almost immediately ceases and a slight recession of the liquid gradually occurs while equilibrium is being reestablished. The calorimeter top and the observed points of the outflow tube are more remote from the source of heat, and the temperatures at these points therefore lag behind θ . After a few minutes, during which the convection currents are being established, this lag becomes nearly constant and remains so during the addition of heat. The manner in which the various parts resume equilibrium is shown in the portion beyond the vertical dotted line which marks the time of interruption of the current. The example given is characteristic of what was observed in each of the entire series of experiments.

For the purpose of evaluating the deficiency of heat in the outflow below what it would have contained were there no lag,

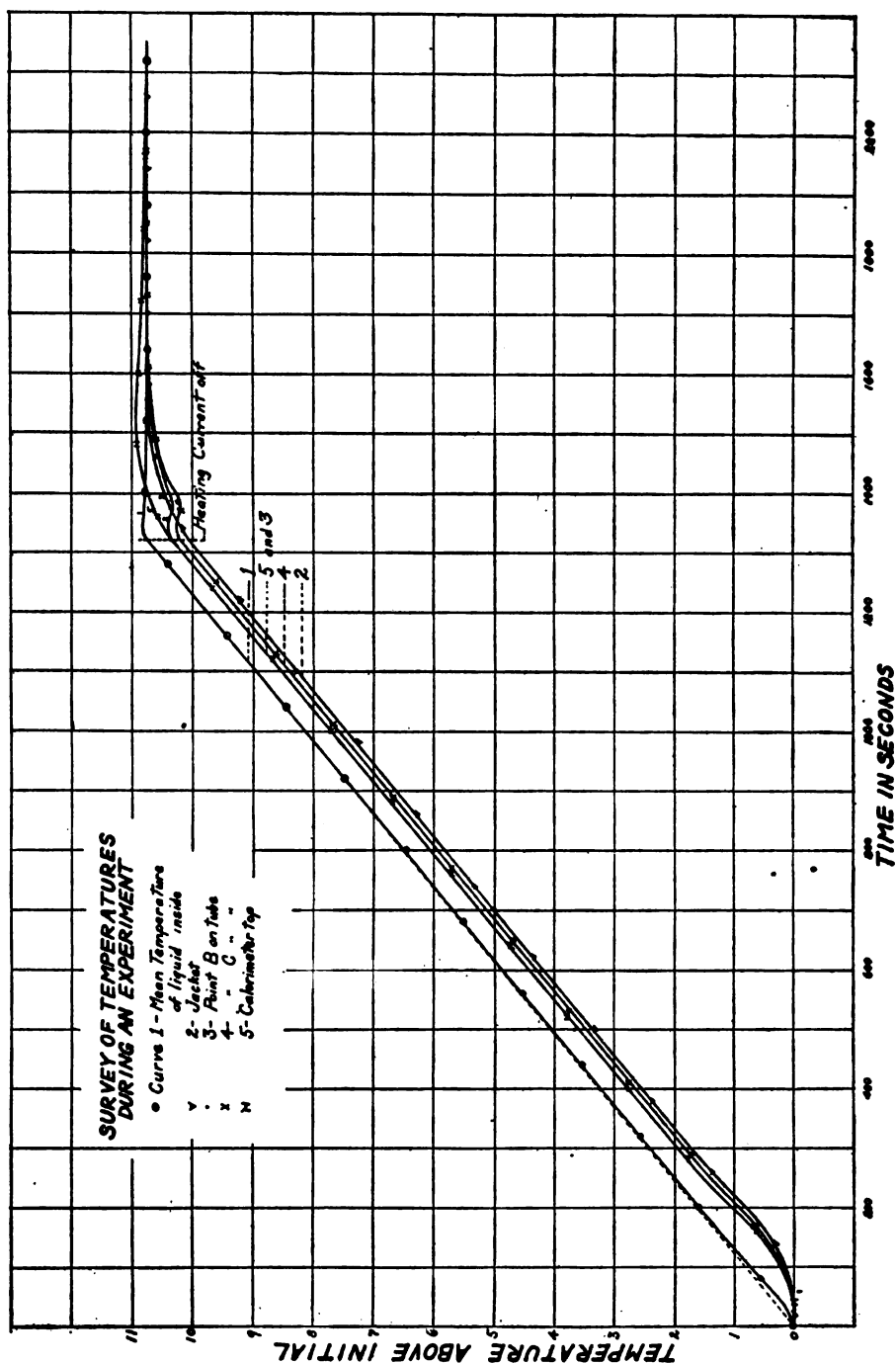


FIG. 3

a sufficiently accurate, simple statement of the results is possible if two periods are considered, the first relatively short, in which the outflow temperature θ' is assumed to remain equal to the initial temperature θ_1 , and a second, lasting until θ_2 is reached at the end of the outflow, in which $\theta - \theta'$ remains nearly constant. Accordingly, using symbols

$$\begin{aligned} \text{for } \theta_1 < \theta < \theta_1 + \lambda, & \quad \theta' = \theta_1 \\ \text{for } \theta_1 + \lambda < \theta < \theta_2, & \quad \theta' = \theta - \lambda \end{aligned}$$

The value of λ was found to depend only slightly on the rate of heating. For the rate of 0.5° per minute, within the range of temperature -40° to $+40^\circ$ C., i. e., for the conditions under which the determinations of C_p were made, λ in degrees as determined in the supplementary experiments may be expressed by the equation:

$$\lambda = 0.53 - 0.0028 \theta \quad (16)$$

It is readily seen that in any of the determinations of C_p , each of which was for a temperature interval of about 10 degrees, the change in λ is less than 0.03 , and therefore in any experiment, λ , which enters as a factor in a small correction term, can be regarded as constant.

2. RESULTS OF MEASUREMENTS

The experimental data obtained from the observations for the determinations of the heat capacity of the liquid at constant pressure made by the second method are given in Table 4, together with the computations for obtaining the heat added to the liquid per unit mass at mean temperature per unit temperature change; i. e., $\frac{1}{(M)_{m\Delta\theta}} \frac{\Delta Q}{\Delta\theta}$. The computations were made following the method described in Section IV.

TABLE 4.—Second Method—Measurements of Specific Heat of Liquid Ammonia at Constant Pressure
 (Calculation of net heat, per unit temperature change, added to liquid per unit mass at mean temperature)

Date	Experi- ment No.	$\Delta\theta$ Therm 15 046	$\Delta\theta$ Therm 4725	Mean $\Delta\theta$	Current in calor- imeter heating coil I	P. D. across heating coil E	Duration of heating current t	Total electric energy I/E	Heat added to calor. contents	Heat capacity calori- meter N	Heat added to contents $\frac{\Delta Q}{\Delta\theta}$	Tempera- ture of reservoir θ_r	Pressure in calor- imeter p	Mean tempera- ture θ_m	Mass in calori- meter at mean tem- perature M_m	Net heat added $\frac{1}{M_m} \frac{\Delta Q}{\Delta\theta}$
1916		degrees	degrees	degrees	amperes	volts	seconds	joules	joules/deg	joules/deg	joules/deg	°C	kg/cm ²	°C	grams	joules/deg
July 21	3	9.966	9.973	9.970	1.4231	14.279	1200.0	24 384	2445.7	990.7	1455.0	+30.7	12.24	+18.05	308.43	4.717
	1	10.346	10.353	10.349	1.4568	14.571	1200.0	25 472	2461.3	926.8	1534.5	15.72	7.69	-38.95	345.64	4.440
	3	10.426	10.428	10.427	1.4611	14.636	1200.0	25 644	2459.4	943.0	1516.4	15.68	7.69	-26.32	337.85	4.488
29	4	10.419	10.421	10.420	1.4594	14.617	1200.0	25 598	2456.6	955.5	1501.1	15.70	7.69	-15.90	331.36	4.530
	2	10.413	10.417	10.415	1.4573	14.605	1200.1	25 543	2452.5	966.4	1486.1	36.2	14.34	-5.94	325.14	4.571
	3	11.468	11.469	11.468	1.4565	14.605	1320.2	28 063	2448.7	977.9	1470.8	36.2	14.34	+5.00	317.78	4.628
31	1	10.445	10.447	10.446	1.4566	14.614	1200.0	25 544	2445.3	987.6	1457.7	39.63	15.78	+14.86	310.89	4.689
	2	10.423	10.425	10.424	1.4543	14.597	1199.9	25 471	2443.5	997.5	1446.0	39.62	15.78	+25.29	303.21	4.769
	3	10.390	10.390	10.390	1.4517	14.579	1200.0	25 397	2444.4	1007.0	1437.4	42.85	17.25	+35.72	295.28	4.868

The pressure, p , corresponding to saturated vapor pressures at temperatures, θ , in the reservoir were obtained from data for pressure of saturated ammonia vapor given by Keyes and Brownlee.¹⁴

The data obtained from the observations for determining at various temperatures the mass contained in the calorimeter and the variations with pressure of the mass contained appear in Table 5.

TABLE 5.—Method 2—Mass of Liquid in Calorimeter when Full at Saturation Pressure

Temper- ature θ	Mass Liquid M	Change of mass with temper- ature $\frac{dM}{d\theta}$	Change of mass with pressure $\left(\frac{\partial M}{\partial p}\right)_\theta$	Temper- ature θ	Mass Liquid M	Change of mass with temper- ature $\frac{dM}{d\theta}$	Change of mass with pressure $\left(\frac{\partial M}{\partial p}\right)_\theta$
$^{\circ}\text{C}$	grams	grams deg.	grams kg/cm ²	$^{\circ}\text{C}$	grams	grams deg.	grams kg/cm ²
-50	352.0	0.60	0.033	10	313.8	0.69	0.046
-40	345.9	.61	.034	20	306.7	.72	.052
-30	339.8	.62	.035	30	299.3	.76	.059
-20	333.6	.63	.036	40	291.5	.80	.069
-10	327.3	.65	.038	50	283.2	.86	.080
0	320.7	.67	.041				

The computation of the values of C_p at pressures above saturation is found in Table 6, the correction for heat to expelled liquid being computed, using for λ equation (16).

The further computation of the limiting value of C_p at pressure π , corresponding to saturation at temperature θ_m , follows in Table 6, for this computation the coefficient $\theta \frac{\partial}{\partial \theta} \left(\frac{l}{\theta} \right)_p$, obtained from independent measurements of the latent heat of pressure variation and published elsewhere,¹⁵ being used.

¹⁴ Thermodynamic Properties of Ammonia, p. 13, John Wiley & Son; 1916.

¹⁵ To appear in this Bulletin.

TABLE 6.—Method 2—Calculation of Specific Heat at Constant Pressure Equal to Saturation Pressure

Heat added to liquid $\frac{1}{M_m} \frac{\Delta Q}{\Delta \theta}$	Mean temp. θ_m	Corr. for heat to expelled liquid $\frac{1}{M_m} C_p \lambda \times \frac{\Delta M}{\Delta \theta}$	Specific heat at press. p temp. θ_m C_p	Pressure p	Saturation pressure at θ_m p_r	$p - p_r$	Change in C_p with pressure $\frac{\partial}{\partial p} \left(\frac{l}{\theta} \right)$	$\left(\frac{\partial}{\partial p} \right)_{\theta} \left(\frac{l}{\theta} \right)$	Specific heat at const. press. equal to saturation press. $C_p]_{\theta, p_r}$	C_p Calc.	Obs.—Calc.
							$\frac{J}{g.deg/cm^2}$				
J/g.deg	° C	J/g.deg	J/g.deg	kg/cm ²	kg/cm ²	kg/cm ²	g.deg/cm ²	J/g.deg	J/g.deg	J/g.deg	J/g.deg
4.440	-38.95	+0.005	4.445	7.69	0.77	6.92	-0.00031	-0.002	4.447	4.450	-0.003
4.488	-26.32	+ .005	4.493	7.69	1.26	6.43	.00041	.003	4.496	4.494	+ .002
4.530	-15.90	+ .005	4.535	7.69	2.32	5.37	.00052	.003	4.538	4.536	+ .002
4.571	- 5.94	+ .005	4.576	14.34	3.53	10.81	.00064	.007	4.583	4.582	+ .001
4.628	+ 5.00	+ .005	4.633	14.34	5.29	9.05	.00079	.007	4.640	4.641	- .001
4.689	+14.86	+ .005	4.694	15.78	7.44	8.34	.00099	.008	4.702	4.703	- .001
4.717	+18.05	+ .005	4.722	12.24	8.28	3.96	.00105	.004	4.726	4.726	± .000
4.769	+25.29	+ .005	4.774	15.78	10.41	5.37	.00123	.006	4.780	4.782	- .002
4.868	+35.72	+ .006	4.874	17.25	14.14	3.11	.00157	.005	4.879	4.877	+ .002

This limiting value of C_p may be represented by the empirical equation:

$$C_p]_{\theta, p_r} = 3.8927 + \frac{95.779}{133 - \theta} \quad (17)$$

Values of C_p computed by means of this equation and comparison of these computed with the observed values are given in the same table.

In Table 7 the computation of the specific heat of the saturated liquid from the limiting values of C_p corresponding to saturation is made, using the limiting values of l obtained from the independent measurements mentioned above.

The values of the specific heat, σ , of the saturated liquid computed from the measurements made at constant pressure may be represented by the empirical equation:

$$\sigma = 3.1800 - 0.00050\theta + \frac{16.356}{\sqrt{133 - \theta}} \quad (B)$$

A comparison between the values of σ computed by this equation and those actually determined is given in the same table.

TABLE 7.—Method 2—Calculation of Specific Heat of Saturated Liquid Ammonia

Temp. θ	Specific heat at const. press. $C_p]_{\theta, \tau}$	Heat of pressure variation $[H]_{\theta, \tau} \frac{d\tau}{d\theta}$	Specific heat at satura- tion σ	Specific heat calc. by equation (B) $\sigma_{calc.}$	Obs- calc
°C	J/g.deg	J/g.deg	J/g.deg	J/g.deg	J/g.deg
-38.95	4.447	-0.002	4.445	4.447	-0.002
-26.32	4.496	.005	4.491	4.489	+ .002
-15.90	4.538	.008	4.530	4.529	+ .001
- 5.94	4.583	.012	4.571	4.571	± .000
+ 5.00	4.640	.018	4.622	4.623	- .001
14.86	4.702	.026	4.676	4.678	- .002
18.05	4.726	.029	4.697	4.697	± .000
25.29	4.780	.038	4.742	4.743	- .001
35.72	4.879	.055	4.824	4.821	+ .003

VIII. FORM OF EMPIRICAL EQUATION FOR SPECIFIC HEAT OF THE SATURATED LIQUID

A form of empirical equation was sought which would closely fit the experimental data and which would also be consistent with other known physical facts. Keyes and Brownlee¹⁸ have given an empirical equation for the specific heat of saturated liquid ammonia, the form of which they chose because "it seems probable that the heat capacity becomes infinite at the critical temperature." This conclusion appears as certain as any physical fact well can be which is not susceptible of direct experimental proof. It is, however, not of more importance than the manner of variation by which the infinite value is reached, and if used alone may easily lead to a form of empirical equation which gives the absurd result of an infinite value for the heat content of the liquid at the critical point. Therefore the limitation should also be imposed that the heat added for any finite temperature increment must be finite.

An empirical equation for σ over an extended range should, therefore, conform to two criteria in addition to adaptability to the experimental data, namely,

$$\sigma_c = \infty$$

$$\int_0^{\theta_c} \sigma d\theta \neq \infty$$

where θ_c is the temperature of the critical point.

¹⁸ Thermodynamic Properties of Ammonia. John Wiley & Son; 1916.

An equation of the form $\sigma = A + B\theta + \frac{C}{(\theta_c - \theta)^{1/2}}$ has been found to meet these requirements, and has been used to represent the results of the present investigation. The equation has the further advantage of giving no real values above the critical temperature.

The form of equation used by Keyes and Brownlee¹⁷ was found to be unsuited to represent these results; indeed, when applied to the data here given the constants found were such as to give a maximum for σ at about 100° and $-\infty$ at the critical temperature 133° .

IX. CONCLUSIONS

The results of the determinations by the two independent methods have been expressed by the following two empirical equations:

$$\text{First method: } \sigma = 3.0931 - 0.00064\theta + \frac{17.329}{\sqrt{133 - \theta}} \quad (\text{A})$$

$$\text{Second method: } \sigma = 3.1800 - 0.00050\theta + \frac{16.356}{\sqrt{133 - \theta}} \quad (\text{B})$$

where the positive value of the square root is to be used.

The agreement of the results by the two methods is shown by the following values computed from the equations:

Temperature.....	-40°	-20°	0°	20°	40°
Equation (A).....	4.438	4.508	4.599	4.711	4.864
Equation (B).....	4.444	4.513	4.599	4.710	4.856
Mean.....	4.441	4.510	4.599	4.710	4.860

The greatest difference between the mean results by both methods and the results of either method as represented by empirical equations is seen to be less than 1 part in 1000.

In Fig. 4 the results of all the determinations by both methods are shown graphically.

In Table 8 the average and maximum deviations of individual experiments from mean values are assembled for convenience in comparison.

¹⁷ Thermodynamic Properties of Ammonia. John Wiley & Son; 1916.

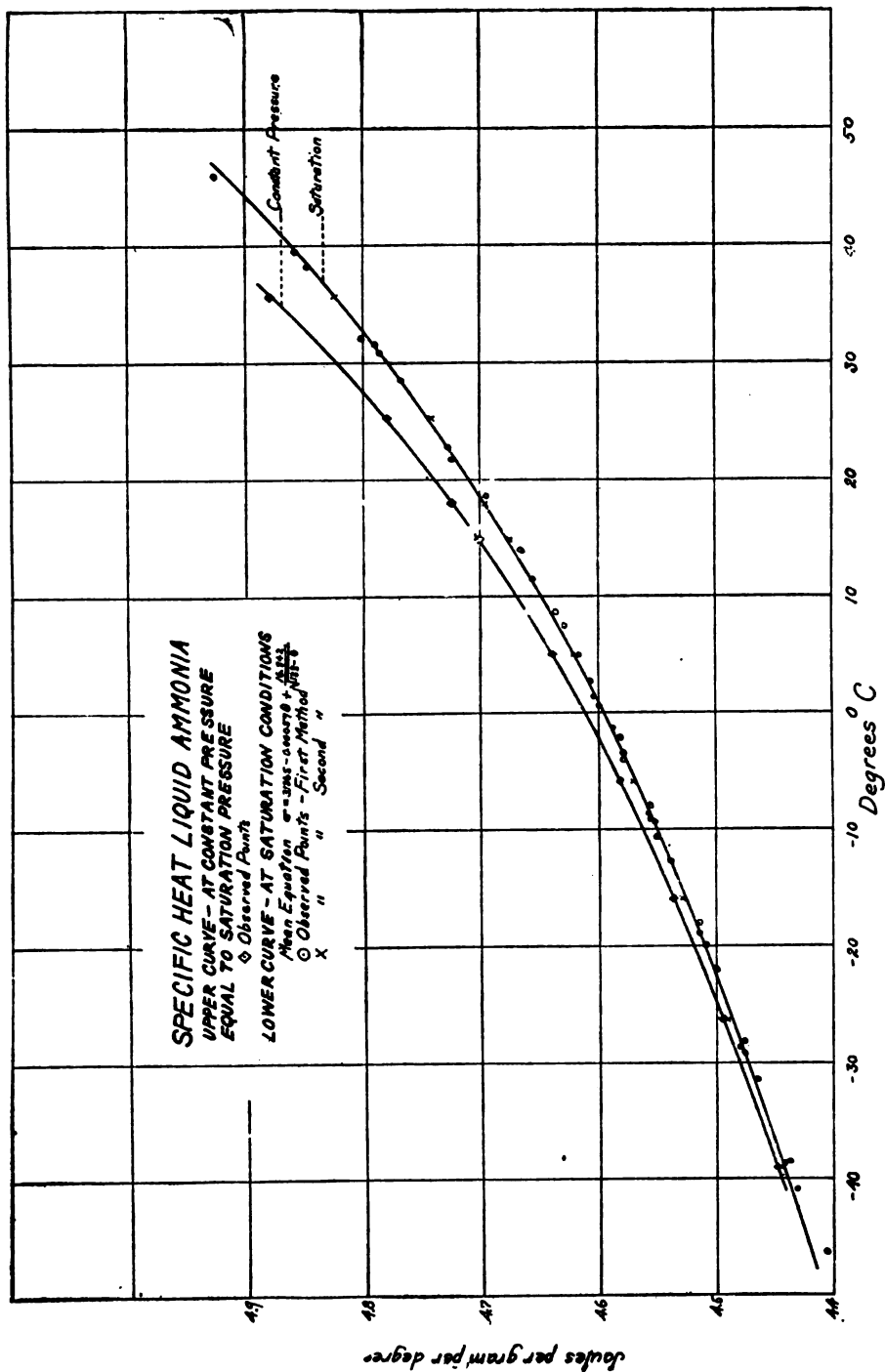


FIG. 4

TABLE 8.—Analysis of Precision—Both Methods

Comparison	Mean deviation— parts per 10 000		Per cent of number of obser- vations within 1 part per 1000	Per cent of number within 2 parts per 1000	Maximum deviation parts per 10 000
	Above 0°.0 C	Below 0°.0 C			
σ (method 1) with empirical equation (A)...	± 6	± 6	90	98	27
σ (method 2) with empirical equation (B)...	± 3	± 3	100	100	6
σ (method 1) with empirical equation (C)...	± 7	± 7	81	98	36
σ (method 2) with empirical equation (C)...	± 3	± 7	100	100	10
σ (both methods) with empirical equation (C).....	± 6	± 7	84	96	36

In Table 9, by assigning to the elements which enter into the determination of the result, σ , estimated values of the *average* and *maximum* accidental errors, the corresponding accidental errors which would be produced in σ were estimated.

For method 1, comparison of the calculated with the observed deviations shows that the maximum deviations observed were to be expected, while the average deviations observed were slightly larger than were expected, unless they are to be explained by reason of distillation in the connecting tubes. If an average deviation of about 6 parts in 10 000 due to this cause were assumed, it would account for the observed average deviation.

A similar comparison for method 2 showed excellent agreement for the average deviations, while the observed maximum deviation was less than that estimated, as it might likely happen in so limited a series of observations.

A similar calculation applied to the *total* errors enables limits to be established within which the actual errors should be. As a result of this calculation it is found that total errors to be expected should not exceed 6 parts in 10 000 and the extreme total error possible should not exceed 55 parts in 10 000. These figures agree very well with those obtained from a comparison of the two methods with the mean.

TABLE 9.—Estimation of Errors

[All errors are parts in 10 000 produced in the final result σ]

Source of error. Quantity measured or calculated	First method				Second method				Both methods. Total errors	
	Fortuitous errors		Total errors		Fortuitous errors		Total errors			
	Aver- age	Poss- ible	Aver- age	Poss- ible	Aver- age	Poss- ible	Aver- age	Poss- ible	Aver- age	Poss- ible
Measurement of energy.....	±4	±8	±4	±8	±4	±8	±4	±8	±4	±8
Measurement of temperature change.....	2	10	2	10	2	10	2	10	2	10
Thermal leakage....	1	5	1	5	1	5	1	5	1	5
Correction for change of phase in- side closed calori- meter.....	0	2	6	10					6	10
Heat capacity calori- meter.....	0	1	3	5	0	1	3	5	3	5
Determination of mass in calori- meter.....	0	0	0	0	0	2	2	4	2	4
Correction for heat of pressure varia- tion.....					0	2	2	10	3	10
Correction for lag of outgoing liquid (second method)...					0	2	2	3	2	3
Total, calcu- lated from above esti- mations.....	±4	±26	±6	±38	±4	±30	±4	±45	±6	±55
Total, by com- parison of observed points with least square reductions...	±6	±27	±7	±36	±3	±6	±5	±10	±7	±36

The order of accuracy of the two methods seems to be about the same, with a slight advantage in favor of the second. Giving each method equal weight, the final mean value of σ in joules per gram per degree is expressed in the range -45° to $+45^{\circ}$ C by the equation:

$$\sigma = 3.1365 - 0.00057 \theta + \frac{16.842}{\sqrt{133 - \theta}} \quad (C)$$

If the relation between the joule and the 20° calorie¹⁸ be taken as 1 calorie₂₀ = 4.183 joules, the specific heat, σ , in Cal₂₀/g. deg., of liquid ammonia under saturation conditions is given by the equation:

$$\sigma = 0.7498 - 0.000136 \theta + \frac{4.0263}{\sqrt{133 - \theta}} \quad (D)$$

X. SUMMARY

Using a calorimeter of the aneroid type specially designed for the peculiar conditions, the specific heat of saturated liquid ammonia has been determined throughout the temperature interval -45° to $+45^{\circ}$ C.

A brief description of the instrument is given in this paper. A detailed description of the design and construction are given in a separate paper.

Two distinct and independent methods were used, each of which avoids sources of error present in the other. In the first method the heat added to a fixed amount confined in the calorimeter under saturation conditions and the resulting change in temperature are measured. By using data for the specific volumes of the two phases and the latent heat of vaporization, the corrections for the vapor are applied, giving the specific heat of the liquid kept saturated.

In the second method the calorimeter is kept full of liquid at a constant pressure. The heat added to the variable amount in the calorimeter and the resulting change in temperature are measured. A correction for the heat withdrawn in the expelled liquid is determined by special experiments. By use of the data for the latent heat of pressure variation of the liquid, obtained from separate measurements, made with the same apparatus and material, the corrections for pressure variation are applied, the result being to give a second determination of the specific heat of the saturated liquid.

¹⁸ The 20° calorie is taken as the quantity of heat per gram (mass) per degree centigrade required to raise the temperature of water at 20° C. at normal atmospheric pressure.

The greatest difference between the mean results of both methods and the results of either method as represented by empirical equations is less than 1 part in 1000.

As a final result, the specific heat σ , in joules per gram per degree centigrade, of liquid ammonia, kept saturated, at the temperature θ , is given in the range -45° to $+45^{\circ}$ C by the equation

$$\sigma = 3.1365 - 0.00057\theta + \frac{16.842}{\sqrt{133 - \theta}}$$

WASHINGTON, February 10, 1917.

APPENDIX

TABLE 10.—Specific Heat of Liquid Ammonia Under Saturation Conditions

[Expressed in Calories₉₉ per Gram per Degree C]

Temp., °C	0	1	2	3	4	5	6	7	8	9
—40	1.062	1.061	1.060	1.059	1.058	1.058	1.057	1.056	1.055	1.055
—30	1.070	1.069	1.068	1.067	1.066	1.065	1.064	1.064	1.063	1.062
—20	1.078	1.077	1.076	1.075	1.074	1.074	1.073	1.072	1.071	1.070
—10	1.088	1.087	1.086	1.085	1.084	1.083	1.082	1.081	1.080	1.079
— 0	1.099	1.098	1.097	1.096	1.094	1.093	1.092	1.091	1.090	1.089
+ 0	1.099	1.100	1.101	1.103	1.104	1.105	1.106	1.108	1.109	1.110
+10	1.112	1.113	1.114	1.116	1.117	1.118	1.120	1.122	1.123	1.125
+20	1.126	1.128	1.129	1.131	1.132	1.134	1.136	1.137	1.139	1.141
+30	1.142	1.144	1.146	1.148	1.150	1.152	1.154	1.156	1.158	1.160
+40	1.162	1.164	1.166	1.169	1.171	1.173	1.176	1.178	1.181	1.183

TABLE 11.—Heat Content of Saturated Liquid Ammonia ¹⁰

[Reckoned from the temperature of melting ice]

CALORIES PER GRAM

Temp., °C	0	1	2	3	4	5	6	7	8	9
—40	—43.3	—44.3	—45.4	—46.4	—47.5	—48.6	—49.6	—50.7	—51.7	—52.8
—30	32.6	33.6	34.7	35.8	36.8	37.9	39.0	40.0	41.1	42.2
—20	21.8	22.9	24.0	25.1	26.2	27.2	28.3	29.3	30.4	31.5
—10	11.0	12.1	13.1	14.2	15.3	16.4	17.5	18.6	19.7	20.8
— 0	0.0	1.0	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
+ 0	+ 0.0	+ 1.1	+ 2.2	+ 3.3	+ 4.4	+ 5.5	+ 6.7	+ 7.8	+ 8.9	+10.0
+10	11.1	12.2	13.4	14.5	15.6	16.7	17.9	19.0	20.1	21.3
+20	22.4	23.5	24.7	25.8	27.0	28.1	29.3	30.4	31.6	32.7
+30	33.9	35.0	36.2	37.4	38.5	39.7	40.8	42.0	43.2	44.4
+40	45.5	46.7	47.9	49.1	50.3	51.5	52.7	53.8	55.0	56.2

BTU PER POUND

Temp., °F	0	1	2	3	4	5	6	7	8	9
— 40	—77.9	—78.9	—80.0	—81.1	—82.1	—83.2	—84.3	—85.3	—86.4	—87.4
— 30	67.2	68.3	69.4	70.4	71.5	72.6	73.6	74.7	75.8	76.8
— 20	56.5	57.6	58.7	59.8	60.8	61.9	63.0	64.0	65.1	66.2
— 10	45.8	46.9	48.0	49.0	50.1	51.2	52.3	53.3	54.4	55.5
— 0	35.0	36.1	37.2	38.2	39.3	40.4	41.5	42.6	43.6	44.7
+ 0	35.0	33.9	32.8	31.7	30.7	29.6	28.5	27.4	26.3	25.2
+10	24.1	23.0	21.9	20.9	19.8	18.7	17.6	16.5	15.4	14.3
+20	13.2	12.1	11.0	9.9	8.8	7.7	6.6	5.5	4.4	3.3
+30	— 2.2	— 1.1	0.0	+ 1.1	+ 2.2	+ 3.3	+ 4.4	+ 5.5	+ 6.7	+ 7.8
+40	+ 8.9	+10.0	+11.1	12.2	13.3	14.4	15.6	16.7	17.8	18.9
+50	20.0	21.1	22.3	23.4	24.5	25.6	26.8	27.9	29.0	30.1
+60	31.3	32.4	33.5	34.7	35.8	36.9	38.1	39.2	40.3	41.5
+70	42.6	43.8	44.9	46.0	47.2	48.3	49.5	50.6	51.8	52.9
+80	54.1	55.2	56.4	57.5	58.7	59.8	61.0	62.1	63.3	64.4
+90	65.6	66.8	67.9	69.1	70.3	71.4	72.6	73.8	74.9	76.1
+100	77.3	78.5	79.6	80.8	82.0	83.2	84.3	85.5	86.7	87.9
+110	89.1	90.3	91.5	92.6	93.9	95.1	96.3	97.5	98.7	99.9

¹⁰ Heat content as used here is defined by the relation:

$$H = \epsilon + p\theta$$

Where H = heat content, taken as zero at the temperature of melting ice, ϵ = internal or "intrinsic" energy, and H , ϵ , and $p\theta$ are all expressed in the same units.

THE LATENT HEAT OF PRESSURE VARIATION OF LIQUID AMMONIA

By Nathan S. Osborne and Milton S. Van Dusen

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I. INTRODUCTION

When a fluid undergoes a change of pressure, there occurs a transformation of energy into heat or vice versa, which results in a change of temperature of the substance unless a like amount of heat is abstracted or added. This change expressed as the heat so transformed per unit change of pressure will be called "latent heat of pressure variation." For most liquids under usual conditions of temperature and pressure this quantity, which depends on the thermal expansivity, is small compared with the other quantities of heat which are usually observed, but for liquid ammonia in the range -40 to $+40^{\circ}$ C and corresponding saturated vapor pressures it is sufficiently large to be taken into account in calorimetric determinations of specific heat; and, in consequence, the measurements here described were made as a supplement to a series of such determinations in order to correlate measurements of specific heat of liquid ammonia¹ made at constant pressure with others made under saturation conditions.

¹ This Bulletin, 14, p. 397; 1917.

The latent heat of pressure variation has been determined in two ways, namely, by direct calorimetric observations and by computation from the expansivity, using for the latter two independent sources of experimental data. Thus, three independent determinations were obtained. On account of the fact that the determinations of specific heat at constant pressure were, when made, regarded as secondary in importance to the others, these measurements of the latent heat of pressure variation as well as the specific heat at constant pressure were not executed with the nicety of which the method finally proved worthy, and, in consequence, the accuracy, while sufficient for the intended purpose, could probably have been considerably bettered with moderate additional refinements in the control and measurement of pressures.

II. NOTATION

M = mass, in grams, of ammonia in the calorimeter.

θ = temperature, in centigrade degrees, of the thermodynamic scale.

Q = heat added in joules.

N = heat capacity, in joules per degree, of calorimeter.

$\Delta\theta$ = total change of temperature in any experiment from initial equilibrium temperature to final equilibrium temperature.

θ' = temperature at which overflow of liquid leaves the calorimeter.

p = pressure, in kilograms per square centimeter.

C_p = specific heat at constant pressure, p , and temperature, θ .

l = latent heat of pressure variation, in joules per gram per unit pressure change.

v = specific volume in cubic centimeters per gram of liquid at pressure p .

III. DETERMINATIONS BY COMPUTATION FROM VOLUMETRIC DATA

The latent heat of pressure variation is quantitatively defined by the equation

$$l = \frac{1}{M} \left(\frac{\partial Q}{\partial p} \right)_\theta$$

where ∂Q is the heat added to mass M to maintain constant temperature simultaneously with a change of pressure ∂p . The latent heat of pressure variation may be computed by means of the general relation:

$$l = -\theta \left(\frac{\partial v}{\partial \theta} \right)_p \quad (1)$$

if the expansivity, $\left(\frac{\partial v}{\partial \theta}\right)_p$, is known. This was obtained in two ways from independent experimental sources—first, from the specific volumes of the saturated liquid and the compressibilities at various temperatures and, second, from the direct observations of the thermal expansion at constant pressure made with a dilatometer specially designed for use in connection with the calorimeter in the measurements of specific heat and described therewith.²

The compressibilities were obtained from the determinations of mass contained in the calorimeter at various temperatures and pressures, corrected for the expansion of the calorimeter itself. The values for specific volume of the liquid are preliminary values from measurements made at this Bureau by Messrs. Harper, Cragoe, and O'Connor, the final results of which will be published in a separate paper. By this method of computation it was found that within the range of temperature and pressure covered by the present experimental work the expansivity, $\left(\frac{\partial v}{\partial \theta}\right)_p$, and, consequently, the latent heat of pressure variation, l , were independent of the pressure to the degree of accuracy sought.³ The results of the direct dilatometric observations, while covering too limited a range of pressure to permit this same conclusion independently, were found to be in agreement with it over the range actually covered.

From each independent series of values of $\left(\frac{\partial v}{\partial \theta}\right)_p$, determined as described above, the corresponding values of l were computed, using equation (1). Only the final results appear in the appended Table 4.

IV. DETERMINATIONS BY DIRECT CALORIMETRIC OBSERVATIONS

The apparatus used for these determinations was identical with that used in the measurements of specific heat at constant pressure⁴ and has been previously described.⁵

As used in the measurements here described, the essential features consist of a metal shell containing the material, suspended for thermal insulation in an air space within a thermally controlled

² This Bulletin, 14, p. 397; 1917.

³ The expansivity was computed by means of the formula:

$$\left(\frac{\partial v}{\partial \theta}\right)_p = \frac{dv}{d\theta} + \left(\frac{\partial v}{\partial p}\right)_{\theta, \tau} \frac{d\tau}{d\theta} - \int_{\tau}^p \frac{\partial}{\partial \theta} \left(\frac{\partial \tau}{\partial p}\right)_{\theta} dp$$

where v is specific volume of saturated liquid and τ is saturation pressure at temperature θ .

⁴ This Bulletin, 14, p. 397; 1917.

⁵ This Bulletin, 14, p. 133; 1917.

jacket. A platinum resistance thermometer located in the center serves to indicate the temperature of the system when in equilibrium. A heating coil likewise in the center may be used to add heat electrically in order to determine the heat capacity of the system. For evaluating the thermal leakage thermocouples with multiple junctions distributed on the surfaces indicate temperature differences between calorimeter and jacket surfaces. By keeping the average jacket and calorimeter surface temperatures equal the thermal leakage is annulled. The calorimeter full of the liquid connects with a reservoir outside, which is immersed in a thermally controlled bath and contains both liquid and vapor. The temperature of the free surface in the reservoir determines the vapor pressure therein, and, therefore, the pressure in the calorimeter may be controlled by manipulation of the aforementioned bath. The measurements of the heat of pressure variation, l , could readily have been made by simultaneously adding heat electrically and decreasing the pressure at such a rate that no change in temperature occurred, the heat added and the change in pressure being observed and the determination would then be independent of the heat capacity of the system. Since the actual quantity of heat transformed was small in the present experiments, it was found more convenient practically to observe the small change in temperature produced by a change of pressure with the addition of no heat electrically. The change in pressure was produced by changing the temperature of the outside reservoir in which the free surface of the liquid was maintained. By observing this temperature when the calorimeter was in equilibrium at the beginning and again at the end of an experiment, and using data for the relation between temperature and pressure of saturated ammonia vapor, the change in pressure Δp was obtained. $\Delta \theta$ was obtained from observations of the initial and final temperatures of the calorimeter when in equilibrium, just as in the specific-heat determinations. The quantity l was then computed, using the heat capacity of the calorimeter and contents determined in separate experiments. The method of making this computation is as follows:

The quantity of heat, dq , added to the calorimeter and contents to produce a given change in temperature, $d\theta$, accompanied by a change in pressure, dp , can be expressed in three parts as follows:

1. Amount required to produce the change, $d\theta$, dp , in the amount M , in the calorimeter at any mean temperature θ of the contents

$$dQ_1 = M (C_p d\theta + l dp)$$

2. Amount required to heat from θ to θ' the amount dM expelled during the change $d\theta$, dp

$$dQ_{II} = C_p dM (\theta' - \theta)$$

3. Amount required to heat the calorimeter through the temperature increment $d\theta$

$$dQ_{III} = Nd\theta$$

The entire amount is, therefore, by addition of the above separate parts

$$dQ = (MC_p + N) d\theta + Mldp + C_p dM (\theta' - \theta) \quad (2)$$

The total amount of heat, ΔQ , added to the system during an experiment is obtained by integrating equation (2) over the corresponding increments of temperature, pressure, and mass experienced. For such small increments of temperature and pressure as were observed the quantities M , C_p , N and l were all found to be so nearly linear functions of θ and p that for the purpose of this integration they may without significant error be regarded as constants and taken equal to their respective values at the mean temperature of the experiment. Furthermore the term $C_p dM(\theta' - \theta)$ is negligible in comparison with the other terms, since dM and $\theta' - \theta$ are each small. By integrating after making these approximations equation (2) then becomes

$$\Delta Q = (M C_p + N) \Delta\theta + M l \Delta p \quad (3)$$

Solving for l

$$l = \frac{\Delta Q}{M \Delta p} - \frac{M C_p + N}{M} \frac{\Delta\theta}{\Delta p} \quad (4)$$

In the experiments ΔQ was made zero. The equation, therefore, may be written

$$l = - \left(\frac{N}{M} + C_p \right) \frac{\Delta\theta}{\Delta p} \quad (5)$$

which is suited for the computation of l from the observed quantities.

The results of these direct determinations upon liquid ammonia are given in the following table. The adjusted values of the mean of the three independent methods are also given in this table, equal weight being assigned to each series of values.

V. RESULTS

Latent Heat of Pressure Variation of Liquid Ammonia

Temperature of calorimeter θ	Initial pressure p_1	Final pressure p_2	Pressure change Δp	Mean pressure $\frac{p_1+p_2}{2}$	Change in temperature of calorimeter $\Delta \theta$	Energy transformed $\left(\frac{N}{M} + C_p\right) \Delta \theta$ ^a	l determined calorimetrically ^b	l computed from specific volumetric data	l computed from dilatometer data	l adjusted value ^c
°C	Kg/cm ²	Kg/cm ²	Kg/cm ²	Kg/cm ²	°C	J/g	J/Kg/cm ³	J/Kg/cm ³	J/Kg/cm ³	J/Kg/cm ³
-44.1	11.41	7.67	-3.74	9.54	-0.029	+0.206	-0.055	-0.056	-0.054	-0.055
-39.0	7.67	.86	-6.81	4.26	-.053	+ .377	-.055	-.059	-.057	-.057
-24.2	16.12	2.01	-14.11	9.06	-.126	+ .920	-.065	-.069	-.067	-.068
- .2	12.03	18.85	+ 6.82	15.44	+ .079	- .599	-.088	-.089	-.089	-.088
- .2	18.85	4.58	-14.27	11.71	-.161	+1.219	-.085	-.089	-.089	-.088
- .2	4.58	18.80	+14.22	11.69	+ .166	-1.262	-.088	-.089	-.089	-0.88
+16.5	20.94	8.17	-12.77	14.55	-.175	+1.372	-.107	-.109	-.109	-.107
+26.5	20.83	11.04	- 9.79	15.93	-.149	+1.191	-.122	-.124	-.124	-.123
+35.4	20.83	14.30	- 6.53	17.56	-.112	+ .923	-.141	-.141	-.141	-.140
+40.3	20.83	16.18	- 4.65	18.50	-.085	+ .705	-.152	-.152	-.151	-.150

^a $N=1000$ J/deg. approximately; $M=300$ g. approximately; $C_p=4.5$ J/g. deg. approximately.

^b Certain observations with smaller pressure increments than those here given appeared to indicate an increase in the numerical value of l with pressure, in apparent contradiction to the result of the computations from the data on specific volume and compressibility. Since the discrepancy was only of the order of the precision of the measurements and was not significant in the application of the results to the specific heat measurements, its investigation was postponed. In future measurements by this method, with better provision for control and observation of the pressure, better concordance could be expected.

^c The adjusted value of l is that given by the empirical equation $l=0.0371-\frac{14.97}{120-\theta}$

WASHINGTON, June 14, 1917.

LATENT HEAT OF VAPORIZATION OF AMMONIA

By Nathan S. Osborne and Milton S. Van Dusen

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I. INTRODUCTION

In tables of heat content of ammonia, such as engineers require, the latent heat of vaporization constitutes the major part. Nevertheless, the direct measurements of this property are among the rarest of the available experimental data. This is attributable probably to the fact that the latent heat of vaporization may, by thermodynamic formulas, be computed from other properties, more easily measurable; however, the data which have heretofore been available for this calculation have not been of a precision such as to yield satisfactory values for the latent heat. The measurements here presented have been carried out in response to the requests of the associations of refrigerating engineers in this

country for more accurate data upon which to base calculations for machinery using ammonia in the production of artificial refrigeration.

II. PREVIOUS DETERMINATIONS

Results of previous determinations of the latent heat of vaporization of ammonia are represented graphically in Fig. 1. The three curves also shown in this figure represent values computed from other data by Keyes,¹ Goodenough and Mosher,² and Holst³.

Regnault⁴ published a record of 12 experiments saved from

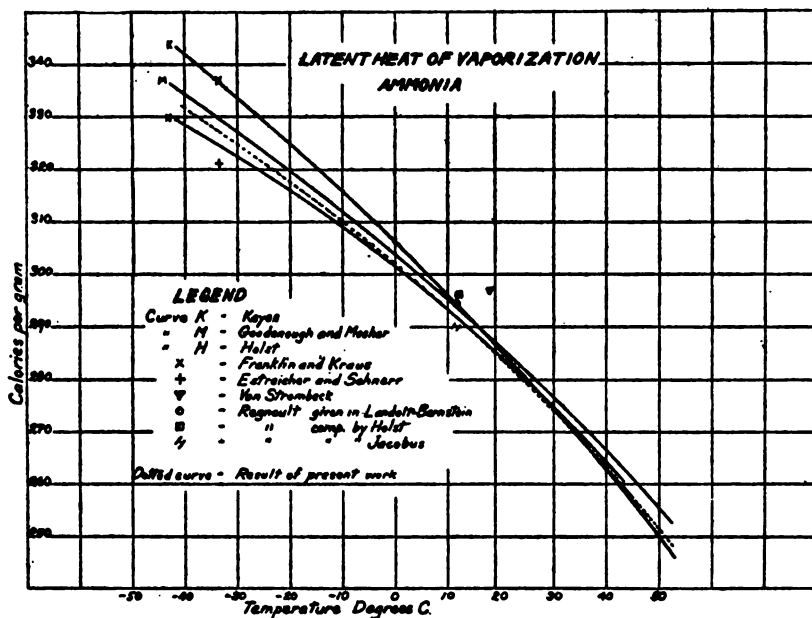


FIG. 1

the ruins of his laboratory, destroyed during the siege of Paris in 1870. The apparatus consisted of two calorimeters, the first, or evaporation calorimeter, in which the ammonia was allowed to evaporate from a steel container and flow through a chamber containing baffle plates, and the second, or expansion calorimeter, in which the ammonia vapor from the first calorimeter was allowed to expand to atmospheric pressure. The capacity of the liquid ammonia container in the first calorimeter was 246 cm³; but it was filled with various amounts, ranging from 17 to 134 grams in

¹ Thermodynamic Properties of Ammonia. John Wiley Sons; 1916.

² Univ. of Ill., Bull. 66; 1913.

³ Bull., Association Internationale du Froid, 51; 1915.

⁴ Ann. Chim. Phys., 24, p. 375; 1871.

different experiments. In each experiment the ammonia was completely evaporated and all vapor expanded to atmospheric pressure. The observed fall in temperature of the water in the first calorimeter varied from $1^{\circ}.7$ to 13° , and in the second it was usually less than 1° . From the data obtained in the first calorimeter Regnault calculated a quantity λ , which is the heat required to change 1 g. of saturated liquid ammonia at the initial temperature and pressure to vapor at the mean temperature of the experiment and at a pressure equal to the pressure in the expansion chamber of the first calorimeter. He stated, however, that it is better to combine the results of the two calorimeters and so proceeded to calculate another quantity, λ' , which is the heat required to change 1 g. of saturated liquid ammonia at the mean temperature and pressure to vapor at the mean temperature and at atmospheric pressure.

Regnault's results have been variously interpreted by different writers. Holst computed the latent heat of vaporization from the observation in the first calorimeter and ignored the partial expansion below saturation pressure which occurred there. As a mean result he obtained 296 calories per gram at 12° C. Jacobus⁶ computed the latent heat of vaporization from the observations in both calorimeters and obtained as a mean value 290 calories per gram at 12° C. Landolt and Börnstein⁶ give values from Regnault's data, the mean value at 12° being 294.5 calories per gram.

Von Strombeck⁷ used the same type of apparatus as Regnault and obtained from 12 experiments a mean value of 296.5 calories per gram at 18° .

Estreicher and Schnerr⁸ (original paper unobtainable), according to Landolt and Börnstein, determined the heat of vaporation at the normal boiling point and obtained a value of 321 calories per gram at $-33^{\circ}.4$ C.

Franklin and Kraus⁹ determined the heat of vaporization at the normal boiling point. The apparatus consisted of a Dewar flask containing a liquid bath and a glass evaporating cell, each supplied with a platinum heating coil. The energy required to evaporate a certain volume of liquid ammonia was measured and the mass computed from the volume evaporated, using the value

⁶ Trans. A. S. M. E., 12, p. 307; 1890.

⁶ Phys. Chem. Tabellen, Ed.; 1912.

⁷ Jour. Franklin Inst., 181, p. 470; 1891.

⁸ Bull. de l'Ac. de Cracovie, p. 345; 1910 (Phys. Chem. Tables; 1912).

⁹ Jour. Phys. Chem., 11, p. 553; 1907.

0.674 for the density. The mean result from three experiments was 341 calories per gram. Recomputed, using more recent data for the density (0.683), the mean value becomes 337 calories per gram. This result is the same as that deduced by Franklin and Kraus from the absolute boiling point and the molecular elevation by Van't Hoff's formula.

III. GENERAL DESCRIPTION OF APPARATUS AND METHOD

The calorimeter used in making the experimental determinations having been previously described in detail, only a brief description is here given.¹⁰ The instrument is of the aneroid type, and was specially designed to meet the requirements of this investigation. A metal shell of sufficient strength is made in the form of a cylinder having a reentrant central tube. The interior annular space contains the material which is the subject of measurement; that is, ammonia, in the present case. An electric heating coil and resistance thermometer are located in the central tube. Heat developed in the coil is transmitted to the surrounding ammonia, the distribution being favored by radial metal vanes. A system of baffle plates in the top of the calorimeter is provided for drying the vapor during its removal in evaporation experiments. The interior surfaces of this calorimeter are tinned and the outside nickerled. The calorimeter is suspended within a thermally-controlled jacket, with an air space between for insulation. For evaluating the thermal leakage, thermocouples with multiple junctions distributed on the surfaces indicate temperature differences between calorimeter and jacket. This leakage can usually be annulled by the method of operation, which consists in keeping the average jacket and calorimeter surface temperatures equal.

In carrying out the experimental work, the heat supplied to the calorimeter for producing evaporation was developed at a nearly constant rate in the heating coil, using current furnished by storage battery. The temperature of calorimeter and contents, when in equilibrium, was measured by the resistance thermometer in the calorimeter. The energy supplied was determined by potentiometer measurements of current and potential drop and by the duration of the heating current as measured by the standard clock.

The sample of ammonia was introduced through one of the two tubes leading into the top of the calorimeter. The material was

¹⁰ This Bulletin, 14, p. 133; 1917.

held in a steel reservoir which, after being weighed, was connected to this tube. The reservoir was immersed in a thermoregulated bath and served either as evaporator or condenser. Transfer occurred by distillation. The amount transferred to or from the calorimeter was determined by reweighing the reservoir and taking the difference. The method employed in the determinations of heat of vaporization was to evaporate, slightly superheat, and withdraw from the calorimeter a measured amount of the ammonia. The approximate amount of heat required to effect this change was added and measured electrically, the small balance being due to the thermal leakage and the resulting change in temperature of the remaining ammonia, both of which were kept relatively small and were measured. The jacket was kept at a constant temperature during the experiment. The withdrawn vapor passed into the reservoir, where it was condensed. The rate of outflow of the vapor was governed by the difference between the vapor pressures in the calorimeter and condenser and by the throttling at the needle valve in the connecting tube outside the calorimeter. This valve was provided with a graduated circle and index for facilitating fine adjustment. The vapor pressure in the condenser was controlled approximately by regulating its temperature. The balance between heat added and heat extracted from the system could be maintained in two independent ways—either by control of the outflow, using the regulating valve, or by control of the heat added electrically.

The temperature of the vapor withdrawn was determined with reference to the initial temperature of the experiment by means of the thermocouples attached to the outflow tube, the common reference junction being on the jacket. Three junctions equidistant along the length of the tube are available for this observation.

The pressure within the calorimeter was indicated by means of a closed manometer, the connection to the interior being similar to that used for transfer of ammonia. This manometer might have been calibrated in terms of actual pressures; but it was found, instead, more convenient to calibrate and use it as a vapor-pressure thermometer to indicate the temperature at the free surface of the liquid in the calorimeter, especially as the evaporation temperature—that is, the free surface temperature during evaporation—enters into the calculation.

IV. THEORY OF METHOD

In the preceding section an outline has been given of the arrangement of apparatus used and of the experimental procedure by which the calorimetric data are obtained for the determination of the latent heat of vaporization. Before passing on to the more complete details of the experiments and the results, the method of computation of the latent heat of vaporization from the observed data will be considered. A brief statement of the problem follows.

Initially, the system consisting of the calorimeter and the ammonia sample, is in thermal equilibrium, the contents comprising both liquid and vapor phases. A portion is withdrawn as superheated vapor with the addition of heat electrically and the remaining portion allowed to resume equilibrium; the aim being to adjust the rate of adding heat and the rate of evaporation so as to avoid large temperature changes throughout the process and to make the final temperature approximately equal to the initial.

The initial and final temperatures and amount removed are observed, and periodic observations made throughout the experiment of the following quantities:

1. Current in electric heating coil.
2. Potential drop in heating coil.
3. Temperature difference between calorimeter and jacket surfaces.
4. Vapor pressure in calorimeter.
5. Temperature of vapor as it leaves the calorimeter at the pressure existing therein.

Given these data and the necessary additional data from independent sources, the problem is to compute the latent heat of vaporization.

1. NOTATION

V = volumetric capacity of calorimeter in cubic centimeters.

N = heat capacity of calorimeter in joules per degree.

Q = general symbol for heat added in joules.

ΔQ = heat added to contents of calorimeter during an experiment.

M = mass in grams contained in calorimeter at any instant.

M_1, M_2 = initial and final mass in calorimeter.

ΔM = mass, in grams, removed from calorimeter as superheated vapor. $\Delta M = M_1 - M_2$.

t = time.

t' = duration, in seconds, of a period of heating while the power, EI , is kept approximately constant.

t_2 = duration, in minutes, of entire experiment between observations of initial and final temperatures.

I = calorimeter heating current in amperes.

E = potential drop in volts across calorimeter heating coil.

h = indication of integrating surface thermocouple in millimeters of galvanometer scale (mean value during experiment).

B = coefficient of thermal leakage in joules per minute per millimeter galvanometer deflection.

θ = temperature in centigrade degrees of the thermodynamic scale.

θ_1 = initial temperature of calorimeter and contents.

θ_2 = final temperature of calorimeter and contents.

p = pressure.

π = saturation pressure.

π_0 = instantaneous pressure in calorimeter.

θ_0 = instantaneous temperature of free surface in calorimeter (determined by π_0).

θ_s = instantaneous temperature of superheated vapor as it leaves the calorimeter.

v = specific volume in cubic centimeters per gram of liquid.

v' = specific volume in cubic centimeters per gram of vapor.

u = specific volume in cubic centimeters per gram of *saturated* liquid (i. e., when in equilibrium with saturated vapor).

u' = specific volume in cubic centimeters per gram of *saturated* vapor.

σ = specific heat in joules per gram per degree of the *saturated* liquid.

σ' = specific heat in joules per gram per degree of the *saturated* vapor.

L = latent heat of vaporization at temperature θ in joules per gram, defined as the heat required to change 1 gram of saturated liquid to saturated vapor at a constant temperature and pressure.

x = mass of vapor in grams per gram total contents, i. e., dryness factor.

2. DETERMINATION OF HEAT ADDED

The heat supplied to the system composed of the calorimeter and the sample of ammonia which is the subject of measurement, consists of two parts, namely, the heat developed in the electric heating coil and the heat transferred by thermal leakage, of which the latter is usually zero or small in amount. The heat thus supplied is distributed in two parts, namely, that which is absorbed by the calorimeter and that absorbed by the contents. Denoting by ΔQ the quantity of heat added to the ammonia in the calorimeter, the above statement is expressed by the equation

$$\Delta Q = \int E I dt + B t_2 h - N(\theta_2 - \theta_1) \quad (1)$$

The coefficient of thermal leakage, B , and the heat capacity, N , of the calorimeter are obtained from the results of supplementary experiments described elsewhere,¹¹ being expressed as empirical functions of the temperature. Both B and N change so slowly with temperature that assuming for each a constant value corresponding to temperature $\frac{\theta_1 + \theta_2}{2}$ for any experiment causes no appreciable error.

3. DETERMINATION OF THE LATENT HEAT OF VAPORIZATION

The process which goes on in the calorimeter can not be followed in detail, for during the period while heat is being added and vapor is being withdrawn the temperature is certainly not uniform, and furthermore, the temperature distribution in the ammonia is unknown. The observations during this period furnish merely knowledge at any instant of the temperature θ_0 of the liquid surface at which the evaporation is occurring, and the temperature θ_v to which the vapor is superheated when withdrawn. It is thus evidently impossible to analyze in detail the phenomena which actually occur and recourse must therefore be had to some other method of interpreting the observations quantitatively.

From the first law of thermodynamics it follows that when a mass of substance passes from a definite initial state to a definite final state the increase of internal energy—i. e., the excess of the heat supplied to the mass over the external work done by it—depends only on the two terminal states and is independent of the nature or the order of the intermediate processes. If in several different processes, each of which begins and ends in the same

¹¹ This Bulletin, 14, p. 533: 1917.

states, the mass of the substance does the same amount of external work, then the quantities of heat supplied to the mass are the same in each, regardless of the precise nature of the several processes.

To take advantage of this principle in the present instance, it is evidently necessary to assume an ideal process which would, if carried out, lead from the observed initial to the observed final state and involve the same amount of external work as the actual process, but which is so simple that the amount of heat required can be expressed in terms of known data and of the unknown latent heat of vaporization which is to be determined. If this expression be equated to the observed quantity of heat supplied during the actual experiment, the result is an equation from which the unknown latent heat L may be found.

It will be simpler to consider first an ideal experiment in which it is supposed that θ_0 and θ_s remain constant throughout the period, and after treating this case proceed to consider the effects of the small variations of θ_0 and θ_s which occur in practice.

Of the several available ideal processes which lend themselves to the method of analysis outlined above, the one chosen by virtue of its physical simplicity consists of the following three consecutive steps:

(a) The M_1 grams of ammonia initially present in the calorimeter are changed in temperature from θ_1 to θ_0 , the volume remaining constant except for the negligible change in the volume of the calorimeter. This change of temperature is to be carried out so slowly that the liquid and the vapor remain sensibly in equilibrium under the vapor pressure corresponding to saturation at the instantaneous temperature.

(b) A mass ΔM grams is withdrawn as vapor. The necessary evaporation occurs at θ_0 so that the vapor comes out at the pressure π_0 corresponding to θ_0 ; but before its exit the vapor is superheated to θ_s .

(c) The M_2 grams of ammonia remaining in the calorimeter are changed in temperature from θ_0 to θ_2 by a process similar to (a).

These three steps taken in succession constitute a process which starts and ends with the same states as in the ideal experiment. Furthermore, the external work done is the same, since the same mass of vapor is withdrawn under the same conditions of pressure and temperature, and the work due to changes of volume of the calorimeter is negligible. Hence, the quantity of heat supplied in an experiment made under the assumed restrictions as to the

constancy of θ_0 and θ_s is equal to the quantity that would have to be supplied if the ideal process were carried out as described. An expression for this quantity will be forthwith obtained.

During a change of temperature of a constant mass M consisting of Mx grams of vapor and $M(1-x)$ grams of liquid, the whole being confined within a constant volume, the heat added is evidently expressed by the relation

$$\frac{dQ}{d\theta} = M \left[(1-x)\sigma + x\sigma' + L \frac{dx}{d\theta} \right] = M \left[\sigma + (\sigma' - \sigma)x + L \frac{dx}{d\theta} \right] \quad (2)$$

By means of the familiar general relation

$$\sigma' - \sigma = \frac{dL}{d\theta} - \frac{L}{\theta} = \theta \frac{d}{d\theta} \left(\frac{L}{\theta} \right)$$

the elimination of σ' from (2) is accomplished and the resulting equation is expressed in the form

$$\frac{dQ}{d\theta} = M \left[\sigma + \theta \frac{d}{d\theta} \left(x \frac{L}{\theta} \right) \right] \quad (3)$$

To eliminate the factor $\frac{d}{d\theta} \left(x \frac{L}{\theta} \right)$, use is made of the relation

$$x = \frac{1}{M} \frac{V - Mu}{u' - u} \quad (4)$$

and of Clapeyron's general equation

$$\frac{L}{\theta} = (u' - u) \frac{d\pi}{d\theta} \quad (5)$$

by multiplying (4) and (5) and differentiating

$$\begin{aligned} \frac{d}{d\theta} \left(x \frac{L}{\theta} \right) &= \frac{V}{M} \frac{d}{d\theta} \left(\frac{d\pi}{d\theta} \right) + \frac{d}{d\theta} \left(u \frac{d\pi}{d\theta} \right) \\ &= \frac{V}{M} \frac{d}{d\theta} \left(\frac{d\pi}{d\theta} \right) - \frac{1}{\theta} \left[\frac{d}{d\theta} \left(\theta u \frac{d\pi}{d\theta} \right) - u \frac{d\pi}{d\theta} \right] \end{aligned} \quad (6)$$

By substitution into (3) from (6) and rearranging,

$$\frac{dQ}{d\theta} = M \left[\sigma - \frac{d}{d\theta} \left(\theta u \frac{d\pi}{d\theta} \right) + u \frac{d\pi}{d\theta} \right] + V \theta \frac{d}{d\theta} \left(\frac{d\pi}{d\theta} \right) \quad (7)$$

whence for a finite change of temperature θ_1 to θ_{II}

$$Q_{I\ II} = M \int_I^{II} \left[\sigma - \frac{d}{d\theta} \left(\theta u \frac{d\pi}{d\theta} \right) + u \frac{d\pi}{d\theta} \right] d\theta + V \int_I^{II} \theta \frac{d}{d\theta} \left(\frac{d\pi}{d\theta} \right) d\theta \quad (8)$$

By application of equation (8) to steps (a) and (c) of the ideal process an expression for the quantity of heat added during the two steps is obtained, which after substitution of $M_2 + \Delta M$ for M_1 and combination of integrals may be written

$$\begin{aligned} Q_a + Q_c = M_2 \int_{\theta_1}^{\theta_2} \left[\sigma - \frac{d}{d\theta} \left(\theta u \frac{d\pi}{d\theta} \right) + u \frac{d\pi}{d\theta} \right] d\theta + V \int_{\theta_1}^{\theta_2} \theta \frac{d}{d\theta} \left(\frac{d\pi}{d\theta} \right) d\theta \\ + \Delta M \int_{\theta_1}^{\theta_2} \left[\sigma - \frac{d}{d\theta} \left(\theta u \frac{d\pi}{d\theta} \right) + u \frac{d\pi}{d\theta} \right] d\theta \end{aligned} \quad (9)$$

Step (b) yet remains to be considered. During this step there is evaporated in addition to the ΔM grams withdrawn, the amount by which the mass of vapor within the calorimeter is increased. If the total mass of ammonia within the calorimeter is changed from M_1 to M_2 , then by equation (4) this increase of vapor is

$$\frac{V - M_2 u_o}{u'_o - u_o} - \frac{V - M_1 u_o}{u'_o - u_o} = \Delta M \frac{u_o}{u'_o - u_o} \quad (10)$$

The whole mass of ammonia evaporated can be written, using equation (5)

$$\Delta M + \Delta M \frac{u_o}{u'_o - u_o} = \Delta M \left[1 + \left(\frac{\theta}{L} u \frac{d\pi}{d\theta} \right)_o \right] \quad (11)$$

and the heat required for this evaporation is L_o times this mass. After the evaporation the mass ΔM is superheated to θ_o , which requires the amount of heat

$$\Delta M \int_{\theta_o}^{\theta_2} C'_p d\theta$$

The whole quantity of heat which must be supplied during step (b) is therefore

$$Q_b = \Delta M \left[L_o + \left(\theta u \frac{d\pi}{d\theta} \right)_o + \int_{\theta_o}^{\theta_2} C'_p d\theta \right] \quad (12)$$

The whole quantity of heat Q_{abo} supplied during the three steps of the process is obtained by adding equations (9) and (12). If $L_1 + \int_{\theta_1}^{\theta_0} \frac{dL}{d\theta} d\theta$ be substituted for L_0 , the resulting equation after cancellation and rearrangement may be written

$$Q_{abo} = \Delta M L_1 + \Delta M \left(\theta u \frac{d\pi}{d\theta} \right)_1 + \Delta M \int_{\theta_1}^{\theta_0} \left(\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta} \right) d\theta \\ + \Delta M \int_{\theta_1}^{\theta_0} C'_p d\theta + M_2 \int_{\theta_1}^{\theta_0} \left[\sigma + \frac{V}{M} \theta \frac{d}{d\theta} \left(\frac{d\pi}{d\theta} \right) - \theta \frac{d}{d\theta} \left(u \frac{d\pi}{d\theta} \right) \right] d\theta \quad (13)$$

In its present form equation (13) admits of verification from a slightly different point of view. By examination of the terms in the second member of this equation it may be shown that thus expressed the heat added to the ammonia corresponds to the heat which would be added in a different program of processes, likewise beginning and ending in the same terminal states and involving the same external work as the experimental case, but in which program the several changes affecting ΔM , the amount removed, occur simultaneously, and independently of the change affecting M_2 , the amount remaining behind. Taking the terms successively the amounts of heat expressed by the several terms may be identified with definite thermal processes, corresponding to the respective terms, as follows:

(1.) Heat required to evaporate the mass ΔM at θ_1 ; (2) heat required to evaporate at θ_1 , the amount to replace with vapor the ΔM grams of liquid evaporated and removed; (3) heat required in the process of throttling from θ_1, π_1 to θ_0, π_0 , the ΔM grams of vapor, kept saturated; (4) heat required to superheat from θ_0 to θ_0 , at pressure π_0 , the amount ΔM removed; (5) heat required to change from θ_1 to θ_0 , under saturation conditions the amount M_2 remaining in the calorimeter.¹³

The first four terms, regardless of the precise manner in which the processes occur, are dependent upon the actual changes produced in the process of removing ΔM grams as vapor, and are independent of the last term, which depends upon the initial and final states of the material which remains in the calorimeter. Nothing physically impossible is implied in the simultaneous operation of these four processes, and supplemented by the process

¹³ The details of this analysis, which is perhaps more interesting than essential, may provide a useful problem for students in thermodynamics.

corresponding to the fifth term, by which the remaining material is brought to the final state of equilibrium, the entire program involves the same changes in state and the expenditure of the same external work as an actual experiment carried out under the assumed ideal conditions. An alternative method of deducing equation (13) is provided by consideration of a process of this kind which, although more complex, corresponds better to the actual process occurring in the calorimeter than the one first chosen.

The equation as developed above expresses the heat, Q_{abo} , added when the entire amount, ΔM , is evaporated at a constant temperature θ_o and removed at the constant pressure π_o and constant temperature θ_o . In the actual experiments, however, the temperatures of evaporation and superheat do vary somewhat during the process of evaporation and removal. It is therefore necessary to generalize equation (13) so as to take account of these variations. Obviously, only the terms which depend upon θ_o and θ_o need be considered with respect to the change which is required in the equation.

For convenience let Q''' denote the part of the heat added which depends on the temperature of evaporation and temperature of superheat. Then from equation (13), since only the third and fourth terms depend on these two temperatures, it follows that

$$Q''' = \Delta M \left[\int_{\theta_1}^{\theta_o} \left(\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta} \right) d\theta + \int_{\pi_o}^{\pi_o} C'_p d\theta \right] \quad (16)$$

The factor ΔM does not depend on the temperatures, θ_o and θ_o , being determined arbitrarily by the operator; hence, by differentiating (16) with respect to M , the quantity of heat, dQ''' , added by virtue of the departure of these temperatures from the initial temperature θ_1 , while an amount, dM , is removed, is given by the equation:

$$dQ''' = dM \left[\int_{\theta_1}^{\theta_o} \left(\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta} \right) d\theta + \int_{\pi_o}^{\pi_o} C'_p d\theta \right] \quad (17)$$

The total amount, Q''' , for the entire mass ΔM , is obtained by integrating (17) with respect to M , which gives the equation

$$Q''' = \int_0^{\Delta M} dM \left[\int_{\theta_1}^{\theta_o} \left(\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta} \right) d\theta + \int_{\pi_o}^{\pi_o} C'_p d\theta \right] \quad (18)$$

in which θ_0 and θ_s are *instantaneous* values. Before this integration with respect to M can be performed, the expression in the brackets must be expressed as a function of M unless it can be evaluated as a constant. Now, if the data which are available for evaluation of the functions $\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta}$ and C'_p , within the temperature range of the present experiments, show that these functions change slowly enough with temperature, then for the small intervals of temperature, $\theta_s - \theta_1$ and $\theta_s - \theta_0$, which occur in the experiments, they may for this integration be treated as constants without introducing significant error. This is found to be true for the present work, and therefore performing these integrations with respect to θ , equation (18) becomes

$$Q''' = \Delta M \left[\left(\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta} \right) \left(-\theta_1 + \frac{1}{\Delta M} \int_0^{\Delta M} \theta_s dM \right) + C'_p \frac{1}{\Delta M} \int_0^{\Delta M} (\theta_s - \theta_0) dM \right] \quad (19)$$

If, furthermore, the experiment be performed in such a way that the vapor is withdrawn from the calorimeter at a constant rate then

$$\frac{dM}{dt} = \frac{\Delta M}{\Delta t} \text{ and hence } \frac{1}{\Delta M} \int_0^{\Delta M} \theta_s dM = \frac{1}{\Delta T} \int_0^{\Delta M} \theta_s dt$$

where Δt is the duration of the outflow. The values of $\frac{1}{\Delta t} \int_0^{\Delta t} \theta_s dt$ and $\frac{1}{\Delta t} \int_0^{\Delta t} \theta_s dt$ for any experiment may be obtained with sufficient accuracy from the periodic observations of these temperatures, and the above-mentioned condition of constant rate of outflow was approximately fulfilled in the present series of experiments. If the average values of θ_0 and θ_s with regard to time be denoted by $\bar{\theta}_0$, and $\bar{\theta}_s$ and substituted for $\frac{1}{\Delta M} \int_0^{\Delta M} \theta_s dM$ and $\frac{1}{\Delta M} \int_0^{\Delta M} \theta_s dM$, respectively, in equation (14), this equation assumes the form

$$Q''' = \Delta M \left[\left(\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta} \right) (\bar{\theta}_s - \theta_1) + C'_p (\bar{\theta}_s - \bar{\theta}_0) \right] \quad (20)$$

which expresses the portion of the heat added, Q''' , which is determined by the conditions under which the evaporation and super-

heating proceeds in the actual experiment. If the second member of this equation be substituted for the third and fourth terms of equation (13), ΔQ , the total quantity of heat added in the actual experiment may be substituted for Q_{abc} , the total quantity of heat added in the assumed ideal case, and the resulting equation will be applicable to the observed data. In order to further simplify the equation and adapt it to numerical computation, the integration indicated in the fifth term should be performed. For this integration the function, $\left[\sigma + \theta \frac{d}{d\theta} \left(\frac{V}{M} \frac{d\pi}{d\theta} \right) - \theta \frac{d}{d\theta} \left(u \frac{d\pi}{d\theta} \right) \right]$, may without significant error be considered constant since it changes so slowly with temperature, and since the temperature change, $\theta_2 - \theta_1$ was in every experiment made very small. By performing this integration and making the two aforementioned substitutions equation (13) becomes:

$$\Delta Q = \Delta M \left[L_1 + \left(\theta u \frac{d\pi}{d\theta} \right)_1 + \left(\sigma + \frac{dL}{d\theta} + u \frac{d\pi}{d\theta} \right) (\bar{\theta}_s - \theta_1) + C'_p (\bar{\theta}_s - \bar{\theta}_s) \right]_{(21)} \\ + M_2 \left[\sigma + \theta \frac{d}{d\theta} \left(\frac{V}{M_2} \frac{d\pi}{d\theta} \right) - \theta \frac{d}{d\theta} \left(u \frac{d\pi}{d\theta} \right) \right] (\theta_2 - \theta_1)$$

Instead of completely solving this equation for L_1 , by rearranging, and then substituting $\frac{L}{\theta} \frac{1}{u' - u}$ for $\frac{d\pi}{d\theta}$, the following equation, adapted to the numerical computation of L_1 , is obtained:

$$L_1 = \frac{\Delta Q}{\Delta M} - L_1 \frac{u_1}{u'_1 - u_1} - \left(\sigma + \frac{dL}{d\theta} + \frac{L}{\theta} \frac{u}{u' - u} \right) (\bar{\theta}_s - \theta_1) - C'_p (\bar{\theta}_s - \bar{\theta}_s) \\ - \left\{ \frac{M_2}{\Delta M} \left[\sigma - \theta \frac{d}{d\theta} \left(\frac{L}{\theta} \frac{u}{u' - u} \right) \right] + \frac{V}{\Delta M} \theta \frac{d}{d\theta} \left(\frac{L}{\theta} \frac{1}{u' - u} \right) \right\} (\theta_2 - \theta_1) \quad (22)$$

The expression, $\frac{\Delta Q}{\Delta M} - L_1 \frac{u_1}{u'_1 - u_1}$, in equation (22) represents the quantity of heat which would be added to the ammonia per gram evaporated at the initial temperature θ_1 . It would equal the latent heat of vaporization at that temperature if the experiment proceeded under ideal conditions, such that the heat added was instantaneously distributed over the free surface so as to avoid any change in temperature and pressure whatever, for then all the other terms would vanish. The remaining terms in the right member of equation (22) represent corrections to the above expression for the departure of the actual experiment from the ideal conditions just specified above.

V. MATERIAL

The ammonia used in these determinations was prepared by Messrs. McKelvy and Taylor, of the chemical division of this Bureau, by methods to be described in detail in an independent paper. A brief description of the process of preparation is here given.

The sample which was used in all the determinations of latent heat of evaporation and designated "B" was prepared in May, 1916, from commercial anhydrous ammonia manufactured by the synthetic method.

A sample was taken from the original container by distillation into a steel container which would hold about a kilogram. From here it was transferred by distillation into a similar vessel containing metallic sodium to remove any remaining traces of water. Following this dehydration the purification was continued by from six to eight consecutive fractional distillations, rejecting in each distillation the first and last tenth. Removal of the rejected first fractions was performed in such a way as to extract the noncondensing gas present, either after vigorous shaking or during active boiling of the liquid.

Tests of the purified sample showed about 1 part in 10 000 by volume of noncondensing gases in the vapor phase, and about 1 part in 10 000 by weight of water.

VI. EXPERIMENTAL DETAILS

Preparatory to a series of determinations, a quantity of ammonia was introduced into the calorimeter by distillation. The amount was determined by difference between the initial and final weight of the outside container. Enough was put in to bring the liquid surface near the top of the central tube which contained the heater. The baffle plates for insuring dryness of the vapor were entirely above the liquid. After introducing the ammonia, the large reservoir was replaced by a smaller one, more convenient for determining the amount of the portions withdrawn in the separate experiments. By use of the heating coil or special cooling device the calorimeter and contents were brought to the temperature chosen for a determination and the jacket brought under the control of the thermoregulator at the same temperature. The auxiliary thermoregulated bath in which the condenser was immersed was set at a temperature enough below the temperature of the calorimeter to provide a sufficient difference of vapor

pressure to effect the desired rate of flow. The initial temperature of the calorimeter was determined by observing the resistance of one or both the platinum resistance thermometers. Initial readings were made of the manometer, the integrating thermocouples, and the auxiliary thermocouples on the outflow tubes. The valve in the outflow tube connecting to the condenser was opened, permitting the ammonia to distill from the calorimeter. Electric current was passed through the heating coil for a measured interval of time. During this interval alternate readings of current and potential drop were made periodically to determine the rate of energy supply to the calorimeter. In most experiments the energy was supplied continuously. In a few, however, in order to effect a variation in manner of manipulation, the current was interrupted occasionally and changed in order to control the conditions in the calorimeter.

During the experiment readings were made at equal intervals of time, of the manometer, the integrating thermocouples, and the auxiliary couples on the outflow tubes. These readings furnish a record of the pressure, the temperature of the vapor where it leaves the calorimeter, and the temperature difference between calorimeter and jacket which determines the thermal leakage. After the interruption of the heating current the evaporation and removal of ammonia was continued long enough to bring the temperature back approximately to the original temperature, and the valve then closed. After the practice obtained in a few preliminary experiments it was not difficult to keep the experimental conditions under satisfactory control so as to make the corrections for change from the initial temperature to the temperature of evaporation and to the final equilibrium temperature of calorimeter and contents sufficiently small. After the period of heating and removal of evaporated ammonia the calorimeter was allowed to repose long enough to come to a state of equilibrium and then readings were made of final equilibrium temperature and pressure. The reservoir was detached and reweighed, the mass removed being determined by difference. This program or procedure could be repeated in successive experiments until the ammonia in the calorimeter was so nearly exhausted as to necessitate refilling.

Considerable variation in the manipulation was possible and such variations were purposely introduced as a means of detecting possible systematic errors. Experiments were made in which the aim was to keep the temperature of evaporation as nearly as

possible equal to the initial temperature. In these experiments the thermal leakage was relatively large. Usually the thermal leakage was kept small by keeping the average surface temperature of the calorimeter nearly constant, and permitting the temperature of evaporation to vary. Different rates of evaporation were used as a test for the dryness of the vapor, on the supposition that if spray were carried out with the vapor the amount would vary with the rate. Different parts of the heating coil were used, that part usually being used which would develop the heat nearest to the place where evaporation occurred.

The manometer as used constituted a vapor pressure thermometer, the readings of which determined the departure of the temperature of evaporation θ_0 from the initial temperature θ_1 . The variation in θ_0 was never very large and the correction term in which it enters is small, consequently θ_0 need be known with only moderate precision.

The manometer was not jacketed and consequently its temperature was approximately that of the room. The changes in its temperature during any experiment were too small to produce a significant error in $\bar{\theta}_0 - \theta_1$. The lack of temperature control, however, precluded use of the manometer in experiments where the temperature θ_0 exceeded the temperature of the room because distillation would then have occurred. For these experiments, therefore, the value of $\bar{\theta}_0 - \theta_1$ could not be directly observed. By noting that in the experiments where observed, $\bar{\theta}_0 - \theta_1$ seemed to be independent of the actual temperature, and that after the experimental technique had become fairly uniform the variations in this quantity in the range -50° to $+20^\circ$ were small, it was seen that the mean value for the range $+20^\circ$ to $+50^\circ$ could be assumed without serious uncertainty.

Some significant discrepancies in the earlier results were found to be due to the condensation of ammonia in the outflow tubes previous to beginning an experiment. Such an occurrence would result in transferring to the condenser more material than was evaporated during the period in which the heat added was measured and would therefore lead to a too low result for the latent heat of vaporization. This source of error was detected through the peculiarity that the initial experiment of a series frequently gave a result lower than those immediately following, the discrepancy in some cases being as much as 0.3 per cent. In order to ascertain that precondensation caused the error, special experi-

ments were made which differed from the regular determinations in having the preliminary conditions such as to favor the condensation of ammonia in the tubes, and then evaporating a much smaller sample than usual so that the error, if due to this cause, would be proportionately increased. In this way the discrepancy from the normal value was so accentuated as to be four times as great as before, or about 1.2 per cent. This result left no doubt as to the nature of the error, although additional confirmation was obtained by further investigation of the reasons why the means provided in the design of the calorimeter for avoiding this very difficulty had occasionally failed to accomplish the object.

Upon this occasion the discovery was made that the thermocouple leads entering the air space adjacent to the outflow tube provided a tiny avenue for the transfer of heat between the room outside and the central portion of the tube. It was thus possible for the temperature of this part of the outflow tube to assume a temperature differing from the temperature of the calorimeter in the same sense as the temperature of the room. Consequently, when the calorimeter was above the room in temperature, condensed ammonia would persist at this place, even though the part of the tube beyond was dried by means of the heating coils thereon. Upon applying heat by means of a coil to the conduit containing the leads, it was possible to overcome this defect and insure the complete drying of the outflow tube. However, the heat conveying ability of this path was so small that, although effective as a preventative measure during experiments, this remedy was not expeditious for expelling liquid, and for this purpose an equally effective and more convenient means was adopted, namely, to precede the first determination in a day's series with a blank experiment by which all liquid condensed in the outflow tube during the preliminary cooling would be expelled and the tube dried. The results of all previous initial experiments where this procedure had been omitted should, of course, be excluded from the authentic determinations because, even when operating at low temperatures, although the natural tendency would be for the tube ultimately to become dry, nevertheless there was no assurance that sufficient time had elapsed for this to occur, and therefore all unpreceded experiments in a series are open to doubt.

VII. RESULTS OF MEASUREMENTS

1. CALORIMETRIC DATA

The data¹³ obtained from the calorimetric measurements are given in Table 1. ΔQ , the heat added to the ammonia, is computed according to the formula

$$\Delta Q = \Sigma(IEt') + Bht_2 - N(\theta_2 - \theta_1)$$

The total energy supplied electrically, $\Sigma(IEt')$, is the sum of the amounts for all the periods of heating during the experiment. In most of the experiments the energy was supplied during a single period t' , the current, I , and potential drop, E , being very nearly constant, so that the product of the average values obtained from the periodic readings of the potentiometer multiplied by the time, t' , gives the total energy within the limit of significant error. For the few experiments where the energy was supplied in several installments, each portion was computed separately and the sum taken. The thermal leakage was computed on the basis of the coefficient of thermal leakage, B , determined by supplementary experiments, the average deflection of the galvanometer, h , and the total time, t_2 , elapsing between the initial and final readings of temperature θ_1 and θ_2 . The energy absorbed by the calorimeter was computed on the basis of the heat capacity of the calorimeter, N , determined by supplementary experiments,¹⁴ and the total change in temperature, $\theta_2 - \theta_1$, during the experiment.

¹³ The laboratory scale of temperature actually used in the measurements given in this paper is the scale of a resistance thermometer of Heraeus platinum of highest purity according to the equation

$$\theta = \frac{R_\theta - R_0}{R_{100} - R_0} 100 + \delta \left(\frac{\theta}{100} - 1 \right) \frac{\theta}{100}$$

R_0 and R_{100} are the resistances of the thermometer at the temperatures, under normal atmospheric pressure, of melting ice and saturated steam, respectively, and δ is determined by substituting for R_θ in the above equation the resistance of the thermometer at the temperature of sulphur vapor under normal atmospheric pressure and 444.6 for θ . The value of δ for the calorimeter thermometer was found by comparison with a standardized thermometer to be 1.48. The departure of the scale so defined from the "thermodynamic" or "ideal gas" scale, down to -50°C is not more than the limit of accuracy of existing gas thermometer data.

For all mathematical relations involving the second law of thermodynamics the temperatures are necessarily referred to the absolute zero. In recording laboratory data numerically it is usually convenient to use the ice point as zero. In the numerical tables of data and reductions no ambiguity arises on this account, as the experiments were conducted between -50° and $+50^\circ$ from the ice point, so that the recorded temperatures never numerically exceed 50° while the absolute temperatures are never less than 200° .

Where numerical values are given, the joule used in this paper is determined by the relation $\frac{Q}{t} = \frac{E^2}{R}$, where Q is the number of joules transformed into heat in a given electric circuit in t seconds, E the number of volts potential drop, and R the number of ohms resistance; taking 1 volt = $\frac{1}{1.01830} \times \text{emf of mean Weston normal cell at } 20^\circ \text{C}$, and 1 ohm = resistance at 0°C , of 106,300 cm. of uniform mercury column 14.4521 g. in mass. The difference between the international joule, realized thus, and the true joule is, according to present evidence, perhaps 1 part in 3000. (B. S. Circular No. 60, 1st ed., p. 56; 1916.) The ampere is used only as an intermediary unit, being determined by the relation $I = \frac{E}{R}$, where I = number of amperes.

¹⁴ This Bulletin, 14, p. 133; 1917.

$\bar{\theta}_e$, θ_1 , and $\bar{\theta}_s$ are mean values of these temperatures obtained by averaging the readings taken periodically as explained in a previous section. In order to show how these temperatures varied during an experiment, the values of θ_e and θ_s during several experi-

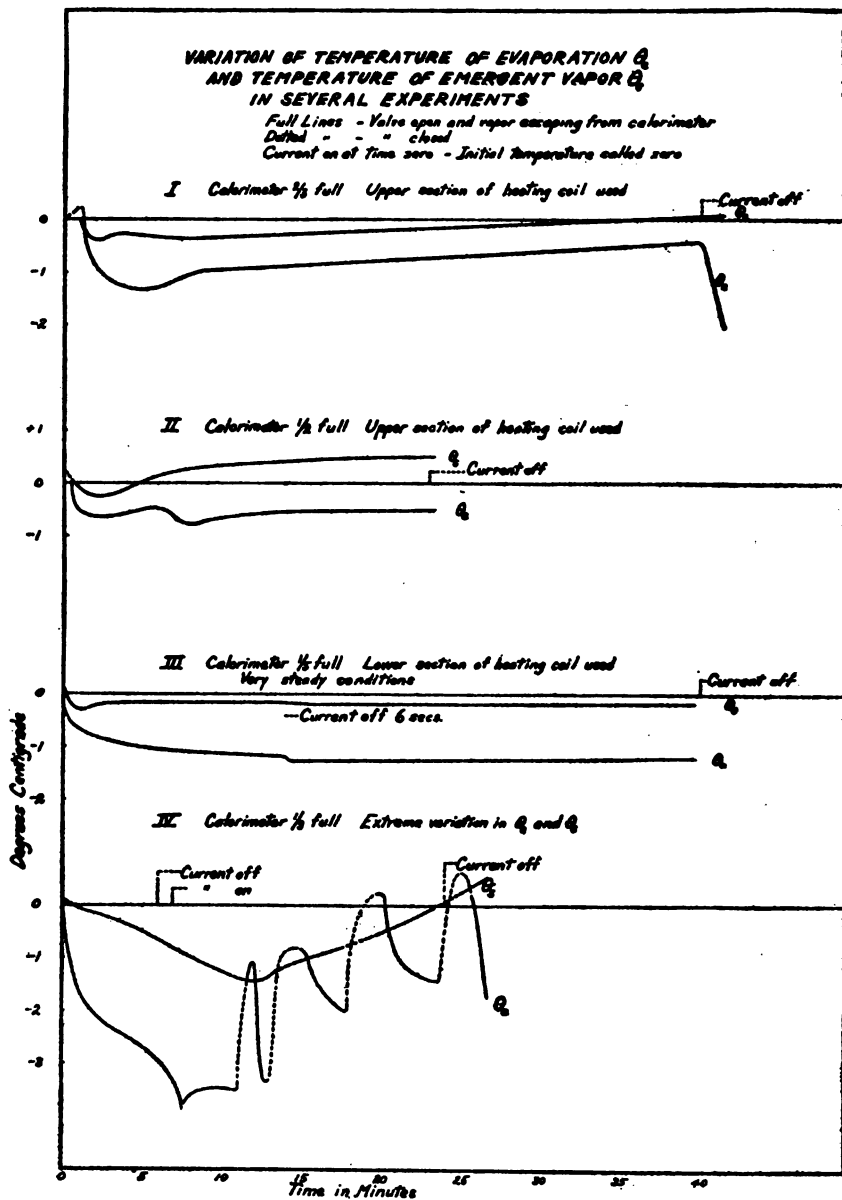


FIG. 2

ments were recorded graphically in Fig. 2. The other items in Table I are self-explanatory when taken in connection with Section VI on experimental procedure.

TABLE 1.—Calorimetric Data

Date	Experi- ment No.	Heat added electrically Z/Ey	Thermal leakage B/Mn	Heat to calorimeter N/(h-A)	Heat to anion ΔQ	Mass ammonia removed ΔM	Final mass in calorimeter M_f	M_f ΔM	$\frac{V}{\Delta M}$	Initial tem- perature θ_i	$\theta_i - \theta_f$	$\theta_f - \theta_i$	$\theta_i - \theta_f$
		Joules	Joules	Joules	Joules	Grams	Grams		cm ³ /g	°C	°C	°C	°C
Apr. 29, 1916.....	2	25 500	-232	+ 4	25 272	19.619	176.72	9.006	26	-5.39	-0.10	+0.04	+0.66
Do.....	3	25 492	-15	+123	25 369	19.610	157.11	8.012	26	-5.40	-1.12	+1.11	+1.01
Do.....	4	25 446	-5	+137	25 578	19.950	137.16	6.875	26	-5.27	-1.10	+1.42	+1.61
May 8, 1916.....	2	17 501	+225	+138	17 639	13.940	108.48	7.762	36	-4.84	-2.23	+1.61	+1.99
May 22, 1916.....	2	24 626	+191	-17	24 800	19.285	54.76	2.839	26	-4.98	-1.17	+0.18	+0.99
Do.....	3	24 638	+84	+145	24 777	19.228	35.53	1.848	26	-4.96	-69	+1.90	+1.75
May 26, 1916.....	2	23 780	+210	+140	24 150	18.182	211.03	11.683	28	-21.83	-1.95	+1.55	+1.35
June 1, 1916.....	3	13 330	+68	+91	14 239	16.625	137.14	14.790	47	-22.09	-26	+0.32	+0.38
June 2, 1916.....	3	13 330	-8	+30	13 432	16.072	137.62	13.634	50	-22.48	-20	+0.06	+0.06
June 2, 1916.....	3	27 783	+34	+4	27 821	20.775	116.84	5.634	24	-22.58	-1.32	+0.04	+1.30
June 3, 1916.....	2	26 992	+45	-31	26 606	19.297	78.38	4.062	26	-35.75	-95	+0.04	+1.05
June 5, 1916.....	2	20 286	+131	-47	20 370	17.085	28.65	1.682	30	+22.23	-81	+0.07	+1.85
June 9, 1916.....	2	22 492	+91	+81	22 664	18.976	210.97	11.122	27	+20.84	-77	+0.01	+0.60
June 10, 1916.....	2	23 545	+34	+26	23 605	19.697	170.90	8.690	26	-20.38	-85	+0.05	+0.98
Do.....	3	23 508	-112	+41	23 432	19.531	151.38	7.755	26	+20.85	-83	+0.01	+0.79
June 13, 1916.....	3	22 794	-78	+20	22 676	18.970	126.51	6.704	27	+19.74	-44	+0.00	+0.53
June 15, 1916.....	3	24 850	-14	+67	25 033	20.110	103.89	5.166	26	+7.60	-54	+0.30	+0.60
Do.....	3	24 462	-5	+20	24 436	19.612	84.28	4.297	26	+7.38	-68	+0.21	+0.52
June 16, 1916.....	2	16 268	-44	+1	16 220	13.014	55.85	3.881	39	+4.09	-25	+0.01	+0.64
June 19, 1916.....	2	22 356	+4	-57	22 303	17.367	15.85	.913	29	-4.09	-35	+0.59	+0.73
Do.....	3	11 161	+10	-20	11 150	8.694	7.16	.824	58	-4.03	-46	+0.21	+0.60
June 21, 1916.....	3	19 008	-12	+6	19 004	9.323	226.66	24.312	54	-42.01	-63	+0.07	+0.75
Do.....	3	12 885	+10	+170	13 067	9.494	217.17	22.874	53	-42.01	-43	+0.14	+0.69
June 22, 1916.....	2	14 925	0	+2	14 954	11.008	185.05	17.674	46	-28.28	-39	+0.01	+0.60
Do.....	3	14 908	0	+5	14 913	11.002	184.05	16.780	46	-28.31	-44	+0.05	+0.68
June 23, 1916.....	3	12 247	-24	+87	12 310	9.396	165.51	17.615	53	-14.93	-44	+0.01	+0.80
Do.....	3	12 245	0	+68	12 313	9.394	156.11	16.618	54	-14.70	-60	+0.07	+0.80
June 24, 1916.....	2	11 851	+9	+61	11 921	10.489	134.94	12.870	48	+36.43		+0.61	+0.36
June 27, 1916.....	2	11 524	+0	+54	11 589	10.807	113.17	10.472	46	+51.80		+0.64	+0.02
June 28, 1916.....	2	11 506	+10	+43	11 549	11.112	90.92	8.182	45	+50.61		+0.42	+0.01
Do.....	3	11 888	0	+38	11 926	11.049	79.87	7.229	46	+50.56		+0.37	+0.03
Do.....	3	11 840	-3	+6	11 831	10.964	68.94	6.305	46	+50.53		+0.06	+0.07
June 30, 1916.....	3	11 657	-20	+18	11 618	10.185	46.49	4.565	40	+36.21		+0.18	+0.19
Do.....	4	11 665	-25	+12	11 652	10.214	36.28	3.551	49	+36.22		+0.12	+0.24

TABLE 2.—Calculation of L_1

Date	Experi- ment No.	$\frac{dH}{d\theta}$	C_p	Z	Y	$\frac{\Delta Q}{\Delta M}$	$L_1 \frac{w_1}{w_1 - w_2}$	$\frac{dH'}{d\theta} (\delta_2 - \delta_1)$	$C_p (\delta_2 - \delta_1)$	$\frac{M_2}{\Delta M} Z (\delta_2 - \delta_1)$	$\frac{V}{\Delta M} Y (\delta_2 - \delta_1)$	L_1	θ	Deri- vation equation
		$\frac{\text{Joules}}{\text{g. deg.}}$	$\frac{\text{Joules}}{\text{g. deg.}}$	$\frac{\text{Joules}}{\text{g. deg.}}$	$\frac{\text{Joules}}{\text{cm}^2 \text{ deg.}}$	$\frac{\text{Joules}}{\text{gram}}$	$\frac{\text{Joules}}{\text{gram}}$	$\frac{\text{Joules}}{\text{gram}}$	$\frac{\text{Joules}}{\text{gram}}$	$\frac{\text{Joules}}{\text{gram}}$	$\frac{\text{Joules}}{\text{gram}}$	$\frac{\text{Joules}}{\text{gram}}$	$^{\circ}\text{C.}$	Parts in 10 000
Apr. 29, 1916.....	2	+1.20	2.33	4.38	0.11	1286.1	3.7	-0.1	1.5	-0.2	+0.0	1281.2	-3.33	+3
Do.....	3	+1.20	2.33	4.38	.11	1285.0	3.7	1.5	2.6	+4.4	+0.0	1281.4	-3.40	+2
Do.....	4	+1.20	2.33	4.38	.11	1285.0	3.7	1.5	2.6	+4.4	+0.0	1281.4	-3.27	+5
May 2, 1916.....	2	+1.20	2.33	4.38	.11	1281.5	3.8	2.7	2.3	+4.9	+0.6	1280.1	-4.64	+8
May 24, 1916.....	2	+1.20	2.33	4.38	.11	1286.0	3.7	1.4	2.3	+0.2	+0.0	1279.1	-4.96	+4
Do.....	3	+1.20	2.33	4.38	.11	1286.6	3.8	.8	1.7	+1.2	+0.5	1280.2	-4.96	+5
May 26, 1916.....	2	+1.45	2.17	4.38	.07	1327.8	3.0	2.8	3.4	+2.1	+0.3	1333.1	-21.83	+8
June 1, 1916.....	3	+1.45	2.17	4.38	.07	1340.1	3.0	.4	3.1	+2.1	+0.1	1333.2	-22.09	+7
June 2, 1916.....	2	+1.45	2.17	4.38	.07	1332.7	2.9	.8	1.5	-5.7	-0.0	1335.2	-22.48	+7
Do.....	3	+1.45	2.17	4.38	.07	1339.2	2.9	1.9	2.8	-0.1	-0.0	1335.5	-22.58	+7
June 3, 1916.....	2	+1.63	2.05	4.38	.04	1378.8	1.6	1.5	2.2	+0.7	+0.0	1375.8	-35.75	+9
June 5, 1916.....	2	+1.69	2.63	4.35	.20	1195.8	14.2	.5	1.5	+3.9	+0.3	1179.3	+22.22	+8
June 8, 1916.....	2	+1.72	2.60	4.35	.20	1194.6	13.7	.6	1.6	-3.9	-0.4	1184.1	+20.46	+8
June 10, 1916.....	2	+1.72	2.60	4.35	.20	1200.2	13.5	.2	2.1	-1.4	-0.2	1186.2	+20.32	+7
Do.....	3	+1.72	2.60	4.35	.20	1200.3	13.5	.2	2.1	-1.4	-0.2	1186.5	+20.35	+7
June 13, 1916.....	2	+1.97	2.60	4.35	.20	1201.7	13.2	.3	1.4	-0.7	-0.1	1188.2	+19.74	+2
June 15, 1916.....	2	+1.97	2.46	4.38	.15	1243.3	8.9	.5	1.5	-1.5	-0.3	1235.2	+7.60	+1
Do.....	3	+1.97	2.46	4.38	.15	1246.0	8.9	.6	1.7	+0.4	+0.1	1235.5	+7.53	+1
June 16, 1916.....	2	+1.97	2.46	4.38	.15	1246.3	8.9	.3	1.3	-0.0	-0.0	1236.3	+7.38	+3
June 19, 1916.....	2	+1.17	2.34	4.36	.12	1284.2	6.0	.4	1.5	+0.2	+0.2	1276.7	-4.09	+1
Do.....	3	+1.17	2.34	4.36	.12	1284.0	6.0	.6	1.7	+0.1	+0.1	1276.7	-4.08	+2
June 21, 1916.....	2	+1.69	2.00	4.38	.03	1394.8	1.2	1.0	1.6	-0.7	-0.0	1393.7	-42.01	+6
Do.....	3	+1.69	2.00	4.38	.03	1376.3	1.2	.7	1.3	-15.4	-0.3	1393.2	-42.01	+3
June 22, 1916.....	2	+1.53	2.10	4.38	.06	1355.0	2.3	.6	1.3	-2.4	-0.1	1354.5	-28.28	+7
Do.....	3	+1.53	2.10	4.38	.06	1355.5	2.3	.7	1.4	-2.4	-0.0	1353.9	-28.31	+6
June 23, 1916.....	2	+1.35	2.25	4.38	.09	1310.1	4.0	.6	1.7	-7.1	-0.4	1312.5	-34.98	+9
Do.....	3	+1.35	2.25	4.38	.09	1310.8	4.0	.8	1.8	-5.2	-0.3	1311.3	-34.70	+4
June 24, 1916.....	2	+1.34	2.62	4.35	.27	1197.0	21.5	1	2.5	-3.4	-0.8	1117.5	+36.48	+3
June 27, 1916.....	2	-1.11	3.10	4.27	.36	1072.4	32.2	+1	1.5	-2.4	-0.9	1041.9	+31.60	+6
June 28, 1916.....	2	-1.07	3.10	4.27	.35	1078.9	31.0	.0	1.5	-1.5	-0.6	1048.5	+30.61	+1
Do.....	3	-1.07	3.10	4.27	.35	1079.4	31.0	.0	1.5	-1.1	-0.6	1048.6	+30.56	+3
Do.....	4	-1.07	3.10	4.27	.35	1082.0	31.0	.0	1.8	-0.7	-0.1	1049.0	+30.53	+1
June 30, 1916.....	2	+1.36	2.62	4.33	.27	1140.8	21.1	-1	2.0	+0.3	+0.2	1117.3	+36.21	+7
Do.....	3	+1.36	2.62	4.33	.27	1140.8	21.1	-1	2.0	+0.3	+0.2	1118.2	+36.22	+1

2. COMPUTATION OF RESULTS

In Table 2 is given the computation of the latent heat of vaporization, L , from the experimental data in Table 1, and accessory data from other sources, suitably combined in Table 3. The computation in Table 2 is made according to equation (22). This equation involves in addition to the data of the present experiments only the specific volumes u and u' of the two phases and the specific heat C'_p of the vapor at constant pressure.

The following abbreviations are employed in the computations:

$$\begin{aligned}\frac{dH'}{d\theta} &= \frac{dL}{d\theta} + \sigma + \frac{L}{\theta} \frac{u}{u' - u} \\ Y &= \theta \frac{d}{d\theta} \left[\frac{L}{\theta} \frac{1}{u' - u} \right] \\ Z &= \sigma - \theta \frac{d}{d\theta} \left[\frac{L}{\theta} \frac{u}{u' - u} \right]\end{aligned}$$

The values used for the specific volumes of the liquid and vapor phases of ammonia are preliminary values from the measurements made at this Bureau by Messrs. Harper, Cragoe, and O'Connor, the final results of which will be published in a separate paper.

The values of C'_p used are provisional values computed from Nernst's¹⁵ empirical formula for the specific heat of ammonia vapor. This computation was made, using a previously described¹⁶ form of characteristic equation for a gas, by Dr. Thadee Peczalski, formerly of this Bureau. The constants in the equation were determined by means of Holst's¹⁷ data on ammonia. The limiting values of C'_p at saturation are included in Table 2. Approximate values only are needed in the computation of the small correction term and consequently the change due to variation in temperature in the actual experiment is disregarded.

In the computation of the coefficients in Table 3 and the interpolation for the values used in Table 2, graphical methods were used where convenient.

¹⁵ Nernst, *Zeitschr. f. Elektrochemie*, 17, p. 49; 1910.

¹⁶ *Ann. de Phys. (9)*, 1, p. 457; 1914.

¹⁷ Holst, *Bull. Assn. Internationale du Froid* (51); 1915.

TABLE 3.—Supplementary Computation of Coefficients $\frac{dH'}{d\theta}$, Y, and Z, from Accessory Data

Temperature, θ	Specific volume, saturated liquid, v_s	Specific volume, saturated vapor, v_s'	$\frac{v_s}{v_s' - v_s}$	Latent heat of vaporization, L	$L \frac{v_s}{v_s' - v_s}$	$\frac{L}{\theta} \frac{v_s}{v_s' - v_s}$	Specific heat saturated liquid, c_s
$^{\circ}\text{C}$	$\frac{\text{cm}^3}{\text{g.}}$	$\frac{\text{cm}^3}{\text{g.}}$		Joules g.	Joules g.	Joules g. deg.	Joules g. deg.
-50	1.424	2550	0.00056	1413	0.79	0.0035	4.41
-40	1.449	1514	.00096	1387	1.33	.0057	4.44
-30	1.475	941	.00157	1358	2.13	.0088	4.47
-20	1.503	612.3	.00246	1329	3.27	.0129	4.51
-10	1.533	414.0	.00372	1298	4.83	.0184	4.55
0	1.566	288.7	.00545	1263	6.88	.0252	4.60
+10	1.601	205.0	.00787	1228	9.66	.0341	4.65
+20	1.638	148.0	.01119	1190	13.32	.0455	4.71
+30	1.679	110.1	.01548	1147	17.76	.0586	4.78
+40	1.725	82.8	.02128	1100	23.41	.0748	4.86
+50	1.776	63.0	.02901	1051	30.49	.0944	4.96

Temperature, θ	$\frac{dL}{d\theta}$	$\frac{dH'}{d\theta}$	$\frac{d}{d\theta} \left[\frac{L v_s}{\theta(v_s' - v_s)} \right]$	$\frac{d}{d\theta} \left[\frac{L v_s}{\theta(v_s' - v_s)} \right]$	$\frac{d}{d\theta} \left[\frac{L v_s}{\theta(v_s' - v_s)} \right]$	$\frac{L}{\theta(v_s' - v_s)}$	$\frac{d}{d\theta} \left[\frac{L}{\theta(v_s' - v_s)} \right]$	$\frac{d}{d\theta} \left[\frac{L}{\theta(v_s' - v_s)} \right]$
$^{\circ}\text{C}$	Joules g. deg.	Joules g. deg.	Joules g. deg. ²	Joules g. deg.	Joules g. deg.	Joules cm ² . deg.	Joules cm ² . deg. ²	Joules cm ² . deg.
-50	-2.63	+1.78	0.00019	0.04	4.37	0.0025	0.00011	0.02
-40	-2.77	+1.68	.00026	.06	4.38	.0039	.00017	.04
-30	-2.93	+1.55	.00036	.09	4.38	.0060	.00023	.06
-20	-3.11	+1.41	.00047	.12	4.39	.0086	.00030	.08
-10	-3.30	+1.27	.00060	.16	4.39	.0120	.00037	.10
0	-3.51	+1.12	.00077	.21	4.39	.0161	.00045	.12
+10	-3.75	+0.93	.00097	.27	4.36	.0213	.00054	.15
+20	-4.02	+0.74	.00120	.35	4.36	.0278	.00065	.19
+30	-4.33	+0.51	.00149	.45	4.33	.0349	.00078	.24
+40	-4.68	+0.25	.00180	.56	4.30	.0433	.00092	.29
+50	-5.10	-0.05	.00215	.69	4.27	.0532	.00107	.35

For the experiments where $\theta_0 - \theta_1$ was not observed on account of the temperature being higher than would allow the use of the manometer, the value of $\theta_0 - \theta_1$ was taken equal to 0.5° , this being the mean value in previous experiments. The greatest deviation from this value in the preceding experiments was not more than 0.3° , and this amount would not change the computed value of L more than 1 part in 1000. The final results probably are affected less than one-third this amount.

Except for 23 experiments, which were excluded on account of liability to error due to precondensation (see Sec. VI) and a very few others in which incomplete records were obtained, all the experiments made are included in Tables 1 and 2. All experiments discredited for reasons previously given may be found in Table 4.

TABLE 4.—Experiments Discredited on Account of Precondensation.

Date	Ex- peri- ment No.	Tem- pera- ture, θ	Heat of vapor- ization, L	De- via- tion from equa- tion	Date	Ex- peri- ment No.	Tem- pera- ture, θ	Heat of vapor- ization, L	De- via- tion from equa- tion
		$^{\circ}\text{C}$	Joules gram	Parts in 10 000			$^{\circ}\text{C}$	Joules gram	Parts in 10 000
Apr. 28, 1916.....	1	- 5.37	1278.8	- 16	June 10, 1916.....	1	+20.38	1183.0	- 24
Apr. 29, 1916.....	1	- 5.33	1279.5	- 10	June 13, 1916.....	1	+19.84	1185.3	- 23
May 8, 1916.....	1	- 4.96	1279.5	$\pm$ 0	Do.....	2	+19.82	1173.9	-120
May 22, 1916.....	1	- 4.97	1278.4	- 9					
May 23, 1916.....	1	- 4.96	1279.2	- 2	June 16, 1916.....	1	+ 7.32	1235.6	- 4
May 29, 1916.....	1	-21.27	1332.8	+ 2	June 19, 1916.....	1	- 4.24	1275.7	- 11
June 1, 1916.....	1	-22.05	1335.4	+ 4	June 22, 1916.....	1	-28.24	1352.5	- 8
June 2, 1916.....	1	-22.51	1335.6	- 5	June 23, 1916.....	1	-14.73	1310.4	- 11
June 3, 1916.....	1	-35.80	1375.8	+ 2	June 24, 1916.....	1	+36.49	1115.8	- 10
June 3, 1916.....	1	+22.28	1176.0	- 18	June 27, 1916.....	1	+51.92	1039.9	- 19
June 6, 1916.....	1	+22.24	1178.2	- 1	June 29, 1916.....	1	+50.61	1043.5	- 49
June 9, 1916.....	1	+20.86	1180.4	- 30	June 30, 1916.....	1	+36.18	1117.2	- 10

* Precondensation induced.

VIII. FORM OF EMPIRICAL EQUATION

An empirical equation the form of which is consistent with the behavior of substances at the critical point should give a value of zero at the critical point for L , and for the derivative $\frac{dL}{d\theta}$ should give a value of $-\infty$. An equation of the form

$$L = A(\theta_0 - \theta) + B(\theta_0 - \theta)^{1/2}$$

in which θ_0 = the critical temperature, was found to meet these requirements and to represent the experimental results closely. The equation has the further advantage of giving no real roots above the critical temperature.

IX. DISCUSSION OF RESULTS

Table 5 contains the results of all the reliable experiments and the additional data regarding variation in the experimental conditions.

This table is given for the purpose of showing the manner in which the conditions were varied and the effect produced upon the results. The following conditions which might conceivably independently affect the results are shown: (a) Rate of evaporation; (b) thermal leakage; (c) average deviation of evaporation temperature from initial temperature; (d) maximum deviation of evaporation temperature from initial temperature; (e) portion of heating coil used; (f) amount removed; (g) amount in calorimeter (mean during experiment); and (h) temperature of experiment.

TABLE 5.—Effect of Variation in Experimental Conditions

Date	Experiment No.	Tem- perature, θ ° C	Heat of vaporiza- tion, L	Deviation from equation	Rate of evapora- tion	Mean value, $\theta_s - \theta_v$ ° C	Maximum value, $\theta_s - \theta_v$ ° C	Mean value, $\theta_s - \theta_v$ ° C	Thermal leakage	Section of heating coil used	Mean mass in calorimeter	Mass removed
June 21, 1916.....	2	-42.01	1353.7	Parts in 10 000	Grams min.	-0.63	-1.0	+0.79	Joules	Lower.....	Grams	9.3
Do.....	3	-42.01	1353.2	+6	0.29	-0.43	-1.3	+0.65	+12	Upper.....	231	9.5
June 3, 1916.....	2	-35.75	1375.8	+3	.47	-0.93	-1.2	+1.05	+45	Lower.....	222	19.3
Do.....	2	-28.28	1354.5	+7	.36	-0.39	-0.7	+0.60	+5	Upper.....	200	11.0
June 22, 1916.....	3	-22.31	1352.9	-6	.37	-0.44	-1.1	+0.68	± 0	do.....	189	11.0
June 2, 1916.....	2	-22.48	1335.2	-7	.23	-0.50	-1.1	+0.68	-8	Lower.....	142	10.1
Do.....	2	-22.48	1333.2	-7	.24	-0.52	-1.0	+0.73	-68	do.....	162	10.5
May 1, 1916.....	2	-21.68	1333.1	-9	.47	-1.25	-2.2	+1.35	+210	do.....	220	18.2
May 2, 1916.....	2	-22.48	1333.3	-7	.63	-1.32	-2.0	+1.30	+34	do.....	127	20.8
June 23, 1916.....	2	-14.36	1312.5	± 0	.45	-0.44	-0.9	+0.76	-24	Upper.....	170	9.4
Do.....	3	-14.70	1311.6	-4	.45	-0.60	-0.6	+0.80	± 0	do.....	161	9.4
May 22, 1916.....	3	-4.96	1280.2	+5	.47	-0.69	-0.9	+0.75	+84	Lower.....	45	19.2
Do.....	2	-4.96	1279.1	-4	.46	-1.17	-1.3	+0.99	+191	do.....	64	19.3
May 8, 1916.....	2	-4.84	1280.1	+8	.70	-2.23	-3.7	+1.61	+225	do.....	115	13.9
Apr. 23, 1916.....	2	-5.33	1281.2	+3	.62	-1.10	-0.9	+0.66	-232	do.....	186	19.6
Do.....	3	-5.40	1281.4	+2	.87	-1.23	-1.9	+1.11	-15	do.....	167	19.6
Do.....	4	-5.27	1280.0	-5	.87	-1.10	-1.6	+1.01	-5	do.....	147	19.6
June 19, 1916.....	3	-4.08	1276.7	+2	.39	-0.48	-1.4	+0.74	+10	do.....	111	17.4
Do.....	2	-4.08	1276.7	+1	.43	-0.35	-0.6	+0.64	+4	do.....	25	17.4
June 15, 1916.....	2	+7.60	1235.2	+1	.50	-0.54	-0.6	+0.59	-14	do.....	114	20.1
Do.....	3	+7.23	1235.5	+1	.50	-0.63	-0.8	+0.69	-5	do.....	94	19.6
June 16, 1916.....	2	+7.38	1236.3	+3	.50	-0.25	-0.4	+0.52	-44	do.....	57	13.0
June 5, 1916.....	2	+22.23	1179.3	+8	.44	-0.61	-2.0	+0.85	+131	do.....	37	17.0
June 9, 1916.....	2	+20.84	1184.1	+1	.47	-0.77	-1.2	+0.60	+91	Upper.....	220	19.0
June 13, 1916.....	3	+19.74	1183.2	-2	.47	-0.44	-0.9	+0.53	-78	Lower.....	136	18.9
June 10, 1916.....	2	+20.38	1186.2	+3	.49	-0.85	-1.1	+0.96	+34	Upper.....	181	19.7
Do.....	3	+20.35	1186.5	+7	.49	-0.33	-0.4	+0.79	-112	do.....	161	19.5
June 30, 1916.....	3	+36.21	1117.3	-7	.49				-20	Lower.....	50	10.2
Do.....	4	+36.22	1118.2	+1	.57				-25	do.....	41	10.2
June 24, 1916.....	2	+36.43	1117.5	+3	.62				+9	do.....	140	10.5
June 27, 1916.....	2	+51.80	1041.9	-6	.54				± 0	do.....	118	10.8
June 28, 1916.....	2	+50.61	1048.5	-1	.55				+10	do.....	96	11.1
Do.....	3	+50.56	1048.6	-3	.53				± 0	do.....	85	11.0
Do.....	4	+50.53	1049.0	-1	.55				-3	do.....	74	10.9

These various conditions were not all systematically changed one at a time, keeping all the others constant, as would have been necessary if large variations had been found to result from the changes. Except for the actual temperature of the determination, which is the primary independent variable, effects of variations in the other conditions are presumed to have been either eliminated experimentally or taken into account quantitatively in the computation of the results, supposing, of course, that such variations were kept within reasonable limits. If the results were dependent to a significant extent on these incidental conditions, it would scarcely be possible to combine them as in Table 5 without some effect being observed. By examination of the deviations from the value given by the empirical equation it has not been possible to trace any systematic deviation to any of the experimental conditions other than the temperature of the experiment. It may, therefore, be concluded that no systematic error resulted from these conditions which was not obscured by accidental errors.

Particular mention might be made of the rate of evaporation. If this rate were increased to the point where active boiling occurred, it is possible that the system of baffle plates might have failed to intercept all of the fine spray thrown up and the evaporation of the withdrawn material would have been incomplete, or, in other words, the vapor would have been wet. It is, therefore, of especial interest to note the considerable variation in rate which occurred without any consequent effect upon the resulting value for latent heat, indicating that within the range of variation used the vapor was dry. Additional confirmation on this point is furnished by the fact that the temperature of the vapor withdrawn was, with rare exceptions, always higher than the temperature at which evaporation was proceeding within the calorimeter.

The results of the determinations may be expressed by means of the following empirical equation:

$$L = 137.91 \sqrt{133 - \theta} - 2.466 (133 - \theta)$$

By assigning average and maximum fortuitous errors to each element that enters into a determination it is possible to estimate the magnitude of the corresponding resultant errors in the heat of vaporization. The *average* fortuitous error estimated in this way should, of course, agree with the observed average deviation from the mean curve, and the observed *maximum* error should not exceed the estimated. It is also possible to estimate the systematic errors

introduced by the uncertainty of the accessory data used in the calculation of the latent heat, and in so doing get some idea of the accuracy of the experiments.

In Table 6 will be found such a tabulation of estimated errors, both fortuitous and systematic, together with the average deviations of the experiments from the mean curve. It will be seen that the agreement between the observed and estimated average fortuitous error is very satisfactory, which means that by a good fortune unusual in calorimetry the precision attained in the results was that to be expected from the precision of the various factors. The observed maximum fortuitous error is much less than the calculated, as might very well happen in a limited number of experiments where there is little chance of all the errors occurring in any one. It will also be noted that the total estimated errors, both fortuitous and systematic, are practically the same as those due to accident alone.

TABLE 6.—Latent Heat of Vaporization NH_3

[Discussion of Errors. Parts in 10 000 produced in L]

Quantity involved	Kind of error	Average error	Maximum error	Quantity involved	Kind of error	Average error	Maximum error
Energy.....	Fortuitous.	± 2	± 4	C'p.....	Systematic.	2	4
Thermal leakage.....	do.....	2	5	Y.....	do.....	0	1
M'.....	do.....	3	6	Z.....	do.....	0	1
$\theta_1 - \theta_2$	do.....	1	4	Estimated error in L..	Fortuitous.	± 3	± 39
$\theta_1 - \theta_3$	do.....	2	10	Observed deviations from empirical equation for L.		± 4	± 8
$\theta_2 - \theta_3$	do.....	3	10	Estimated total error..	Fortuitous and systematic.	± 3	± 49
$L \frac{u}{u' - u}$	Systematic.	1	2				
$\frac{dH'}{ds}$	do.....	1	2				

X. SUMMARY

Using a calorimeter of the aneroid type specially designed for the peculiar conditions, the latent heat of vaporization of ammonia has been determined throughout the temperature interval -42 to $+52^\circ \text{C}$.

A detailed description of the design and construction of the instrument having been given in a separate paper, only a brief description of the apparatus is included here, the theory of the method and the experimental details, however, being discussed. The instrument contains no special liquid for a calorimetric medium, the ammonia being the only liquid in the calorimeter.

Heat developed and measured electrically in a coil is transmitted by conduction and convection to the surrounding ammonia and is utilized to effect the evaporation of a determined amount of the ammonia which is withdrawn as superheated vapor at a determined temperature and pressure. Heat from other sources is minimized by reducing as far as practicable the avenues for heat transfer between the calorimeter and its environment, and by so manipulating as to keep the temperatures of the calorimeter and jacket surfaces nearly equal, means being provided for determining the amount of thermal leakage which is not thus annulled. Initial and final temperatures are measured by a platinum resistance thermometer.

An analysis of the process occurring in the calorimeter during an experiment leads to a method of calculation of the results whereby data from other sources than the direct calorimetric observations enter only in the computation of correction terms which can by careful manipulation be made small.

Variations in manipulation were introduced as a means of detecting possible systematic errors, particularly in regard to the matter of dryness of the vapor withdrawn.

The result of each of the 34 determinations agrees with the mean result as expressed by means of an empirical equation within 1 part in 1000. An empirical equation was found which in addition to representing closely the results in the range of temperature covered experimentally, also conforms to what is known about the behavior of substances in general when approaching the critical point.

As a final result the latent heat of vaporization of ammonia—that is, the heat in joules per gram required to convert saturated liquid into saturated vapor at constant temperature—is expressed in the range -42° to $+52^{\circ}$ C by the equation:

$$L = 137.91\sqrt{133 - \theta} - 2.466(133 - \theta).$$

If the latent heat of vaporization be expressed in calories₂₀ per gram, taking 1 calorie₂₀ = 4.183 joules, the equation becomes

$$L = 32.968\sqrt{133 - \theta} - 0.5895(133 - \theta).$$

The values computed from this equation every 5 degrees are as follows:

Temperature	Latent heat of vaporization	Temperature	Latent heat of vaporization
°C	calories/g	°C	calories/g
-45	334.9	+ 5	297.5
-40	331.7	+10	295.1
-35	328.3	+15	288.6
-30	324.8	+20	283.8
-25	321.3	+25	278.9
-20	317.6	+30	273.9
-15	313.8	+35	268.6
-10	309.9	+40	263.1
- 5	305.9	+45	257.4
0	301.8	+50	251.4
		+55	245.1

WASHINGTON, May 1, 1917.

APPENDIXES

APPENDIX 1.—LATENT HEAT OF VAPORIZATION OF AMMONIA

CALORIES PER GRAM

Temperature	0	1	2	3	4	5	6	7	8	9
°C										
-40	331.7	332.3	333.0	333.6	334.3	334.9	335.5	336.2	336.8	337.5
-30	324.8	325.5	326.2	326.9	327.6	328.3	329.0	329.7	330.3	331.0
-20	317.6	318.3	319.1	319.8	320.6	321.3	322.0	322.7	323.4	324.1
-10	309.9	310.7	311.5	312.2	313.0	313.8	314.6	315.3	316.1	316.8
- 0	301.8	302.6	303.4	304.3	305.1	305.9	306.7	307.5	308.3	309.1
+ 0	301.8	300.9	300.1	299.2	298.4	297.5	296.6	295.7	294.9	294.0
+10	293.1	292.2	291.3	290.4	289.5	288.6	287.6	286.7	285.7	284.8
+20	283.8	282.8	281.8	280.9	279.9	278.9	277.9	276.9	275.9	274.9
+30	273.9	272.8	271.8	270.7	269.7	268.6	267.5	266.4	265.3	264.2
+40	263.1	262.0	260.8	259.7	258.5	257.4	256.2	255.0	253.8	252.6

BTU PER POUND

°F										
- 40	597.0	597.7	598.3	599.0	599.6	600.3	600.9	601.6	602.2	602.9
- 30	590.2	590.9	591.6	592.3	592.9	593.6	594.3	595.0	595.6	596.3
- 20	583.3	584.0	584.7	585.4	586.1	586.8	587.5	588.1	588.8	589.5
- 10	576.1	576.8	577.6	578.3	579.0	579.7	580.4	581.1	581.9	582.6
- 0	568.7	569.4	570.2	570.9	571.7	572.4	573.2	573.9	574.6	575.4
+ 0	568.7	567.9	567.2	566.4	565.7	564.9	564.1	563.3	562.6	561.8
+10	561.0	560.2	559.5	558.7	557.9	557.1	556.3	555.5	554.7	553.9
+20	553.1	552.3	551.5	550.7	549.9	549.1	548.2	547.4	546.6	545.8
+30	544.9	544.1	543.3	542.4	541.6	540.7	539.9	539.0	538.2	537.3
+40	536.5	535.6	534.7	533.8	533.0	532.1	531.2	530.3	529.5	528.6
+50	527.7	526.8	525.9	524.9	524.0	523.1	522.2	521.2	520.3	519.4
+60	518.5	517.5	516.6	515.6	514.7	513.7	512.8	511.8	510.9	509.9
+70	508.9	508.0	507.0	506.0	505.0	504.1	503.1	502.1	501.1	500.1
+80	499.1	498.1	497.0	496.0	495.0	494.0	493.0	491.9	490.9	489.8
+90	488.8	487.7	486.7	485.6	484.6	483.5	482.4	481.3	480.2	479.1
+100	478.0	476.9	475.8	474.7	473.6	472.5	471.3	470.2	469.0	467.9
+110	466.7	465.6	464.4	463.3	462.1	460.9	459.7	458.5	457.3	456.1

APPENDIX 2.—SPECIFIC HEAT OF SATURATED AMMONIA VAPOR

If the latent heat of vaporization and the specific heat of the saturated liquid are known as functions of the temperature, it is possible to compute the specific heat of the saturated vapor by using the general relation

$$\sigma' - \sigma = \frac{dL}{d\theta} - \frac{L}{\theta} = \theta \frac{d}{d\theta} \left(\frac{L}{\theta} \right)$$

The data for the specific heat of saturated liquid ammonia are available from a previous investigation¹⁸ the results of which are expressed by the empirical equation

$$\sigma = 3.1365 - 0.00057\theta + \frac{16.842}{\sqrt{133 - \theta}}$$

in which σ is the specific heat in joules per gram degree at temperature θ of the saturated liquid. The rate of change $\frac{dL}{d\theta}$ of the heat of vaporization of ammonia with temperature and the quantity $\frac{L}{\theta}$ can be calculated from the results of the present work. The computed values of the specific heat σ' in joules per gram degree of saturated ammonia vapor are given in the following table:

Temperature	Specific heat of saturated vapor	Temperature	Specific heat of saturated vapor
°C	Joules/g deg.	°C	Joules/g deg.
-45	-4.42	0	-3.54
-40	-4.29	+5	-3.48
-35	-4.17	+10	-3.43
-30	-4.05	+15	-3.40
-25	-3.95	+20	-3.36
-20	-3.85	+25	-3.34
-15	-3.76	+30	-3.33
-10	-3.68	+35	-3.33
-5	-3.61	+40	-3.34
0	-3.54	+45	-3.36

¹⁸ This Bulletin, 14, p. 397; 1917.

**APPENDIX 3.—HEAT OF VAPORIZATION OF AMMONIA IN CALORIES,
PER GRAM COMPUTED BY VARIOUS WRITERS AND GIVEN IN THEIR
AMMONIA TABLES**

[The results of the present work are included for comparison]

Temperature		Leducx, 1878	Peabody, 1889	Wood, 1889	Zemmer, 1890	Mollier, 1895	Dieterici, 1904	Wobsa, 1908
°F	°C							
- 40	- 40	335.2	332	322.0	333.0	332.7		
- 22	- 30	330.5	324	316.0	329.9	330.6		324.3
- 4	- 20	325.3	316	309.9	325.8	327.2		317.0
+ 14	- 10	319.7	308	303.8	320.8	322.3		309.0
+ 32	0	313.6	300	297.6	314.9	316.1	308.7	300.4
.....								
+ 50	+ 10	307.2	292	291.3	308.0	308.6	298.4	290.9
+ 68	+ 20	300.3	284	284.8	300.1	299.9	285.4	280.6
+ 86	+ 30	293.0	276	278.4	291.3	289.7	272.2	269.4
+ 104	+ 40	285.3		271.9	281.6	278.0	258.3	257.4
+ 122	+ 50			265.3			243.6	244.6
.....								
+ 140	+ 60			258.6			227.9	
+ 200	+ 93.3						165.2	
+ 250	+ 121.1							
+ 270	+ 132.2							

Temperature		Hybl, 1911	Macintire, 1911	Lucke, 1912	Mosher, 1913	Holst, 1915	Keyes, 1916	Osborne and Van Dusen, 1917
°F	°C							
- 40	- 40			335.3	334.4	328.5	342.0	331.7
- 22	- 30	325.2	327.9	328.1	327.1	322.5	333.6	324.8
- 4	- 20	318.2	320.8	320.9	319.6	316.0	324.9	317.6
+ 14	- 10	310.7	313.0	313.1	311.8	309.0	315.7	309.9
+ 32	0	302.6	304.4	304.6	303.6	301.4	306.0	301.8
.....								
+ 50	+ 10	293.7	295.0	294.8	295.0	293.2	296.0	293.1
+ 68	+ 20	284.2	284.7	284.6	285.9	284.4	285.5	283.8
+ 86	+ 30	274.0	273.5	273.5	276.4	274.8	274.4	273.9
+ 104	+ 40	263.0		261.4	266.2	264.2	262.7	263.1
+ 122	+ 50			248.3	255.4		250.2	251.4
.....								
+ 140	+ 60			234.7	243.7		236.8	
+ 200	+ 93.3			176.7	195.3		181.9	
+ 250	+ 121.1				127.6			
+ 270	+ 132.2				61.2			

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GAS INTERFEROMETER CALIBRATION

By Junius David Edwards

The Rayleigh-Zeiss gas interferometer has found numerous applications in high-precision gas analysis. However, it requires preliminary calibration, and its accuracy is limited by the accuracy of the calibration, even though the precision of reading be considerably greater. This interferometer and its uses have been described in detail by Haber and Lowe,¹ L. Stuckert,² L. H. Adams,³ and others.

This type of gas interferometer measures the difference in refractivity of two samples of gas contained in two gas chambers which, in the laboratory type of apparatus, are 100 cm long. Light from an illuminated slit passes through both chambers, after which the two beams combine to produce interference fringes which are observed through an eyepiece. The optical path of the two beams can be brought to equality by tilting a glass compensator plate which is placed in the path of one of the beams. If the temperature, pressure, or composition of the gas in one of the chambers is changed, the optical paths are different and the interference fringes are shifted. The fringes are brought back to their original position by tilting the compensator plate. The angle through which the compensator plate has been tilted, which is measured by means of a drum attached to the micrometer screw, is a measure of the change in refractive index of the two gases.

The customary method of calibration is to place a standard gas in one chamber and to determine the number of scale

¹ Haber and Lowe, *Zs. f. angew. Chem.*, **23**, p. 1393; 1910.

² Stuckert, *Zs. f. Elektrochemie*, **16**, p. 37; 1910.

³ L. H. Adams, *J. Am. Chem. Soc.*, **37**, p. 1181; 1915.

divisions the fringes are shifted when the second chamber contains the standard gas plus various known percentages of the gas for which the instrument is being calibrated; the gases in the two chambers are kept at the same temperature and pressure. The results are, of course, no more accurate than the method of analysis of the mixtures, although the precision of reading may be considerably greater.

L. H. Adams has shown how the sensitivity of the interferometer (water or gas) can be calculated from certain dimensions and constants of the interferometer parts. This is a rather involved procedure, however, and one practicable only in well-equipped physical laboratories.

The method here proposed requires no special apparatus and only a knowledge of the refractive indices of the gases which are to be used. The calibration is accomplished by filling both chambers of the interferometer with dry air free from carbon dioxide and determining the scale reading when the pressure in one chamber is reduced by known amounts. The pressure can be most accurately determined by use of a water gage; a drying tube should be inserted, however, between the gage and the gas chamber to prevent diffusion of water vapor into the chamber. For the portable instrument, which is less sensitive than the laboratory type, a mercury gage is sufficiently sensitive. The scale limit of the laboratory type interferometer is reached when the pressure difference has reached about 100 mm of mercury. The theoretical considerations on which the method is based are as follows:

The refractivity (R) of a gas may be defined as the function $(n-1)$, in which n is the refractive index of the gas. The refractivity of a gas is proportional to its density,⁴ or

$$\frac{n-1}{d} = \text{constant}$$

From this relation the refractivity of a gas can be calculated for different temperatures and pressures by application of the gas laws.

⁴ Mathews (J. Frank. Inst., 177, p. 673; 1914) discusses the various formulas proposed to show the relation between refractive index and density. The Lorenz-Lorentz formula $\left(\frac{n^2-1}{n^2+2}\right)$ and the Gladstone-Dale formula $(n-1)$ are equivalent for gases at pressures up to a few atmospheres. Gale (Phys. Rev., 14, p. 1; 1902) concludes from his results that if there is any variation from the formula $\frac{n-1}{d} = k$ for air up to 20 atmospheres, it is less than 1 part per 1000.

The interferometer serves to measure the difference in refractivity of the gases in the two chambers. Hence, the interferometer scale may be calibrated in terms of refractivity differences. To do this it is necessary to know the scale reading corresponding to a given difference in refractivity. This is determined by filling both chambers of the interferometer with dry air at a definite temperature and pressure and then progressively varying the pressure on one chamber and noting the scale reading corresponding to each pressure. The difference in refractivity given

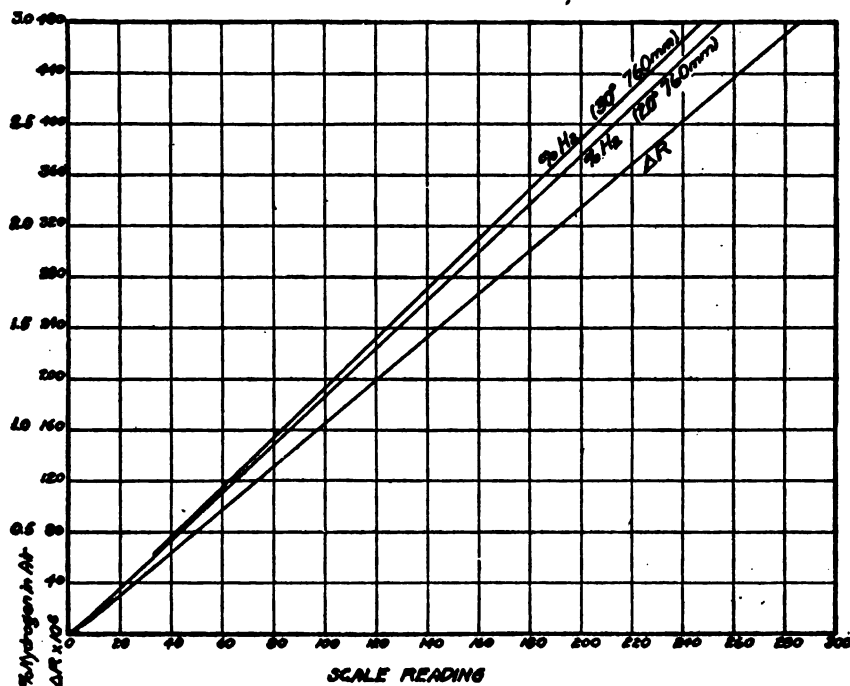


FIG. 1.—Interferometer calibration curves.

by each scale reading (observed reading minus "zero" reading) is then calculated from the following formula:

$$\Delta R = \frac{273 \times 0.0002926 (p_1 - p_2)}{760 T} \quad (1)$$

The pressures in the two chambers are p_1 and p_2 and the temperature T remains constant. The refractive index ⁵ (n_D) of air at 0°, 760 mm pressure is taken as 1.0002926.

The relation between scale reading and refractivity differences for part of the range of one of these instruments (100 cm gas

⁵ Jones and Partington, Phil. Mag., 29, p. 28; 1915.

chambers) is shown in Fig. 1. By means of such a refractivity curve the calibration of the instrument can be calculated for any gas, as follows:

If R_1 is the refractivity of the standard gas at 0° and 760 mm and R_2 is the refractivity under the same conditions of the second gas for which the calibration is sought, the difference in refractivity for any percentage of the latter is given by the following equation, in which a is the percentage of the second gas present in the mixture:

$$\Delta R = \frac{273 p}{T_{760}} \cdot \frac{a}{100} \cdot (R_1 - R_2) \quad (2)$$

The validity of this equation depends upon the fact that the refractivity of a mixture of gases is equal to the sum of the refractivities of each constituent times the percentage of it present in the mixture. It has been shown⁶ that although this does not hold strictly, it is a very close approximation. For example, the refractivity of air calculated according to this mixture rule from the refractivity values given by Jones and Partington⁷ for oxygen, nitrogen, and argon and the percentage composition of air as given by Sir William Ramsey is as follows:

$$(0.0002711 \times 0.2094) + (0.0002976 \times 0.7812) \\ + (0.0002802 \times 0.0094) = 0.0002919.$$

This is only 0.24 per cent lower than the observed value, 0.0002926.

Since the refractivity of a gas is proportional to its density, it is necessary to consider the deviation from Boyle's law in the case of gases at pressures much different from the normal pressure (760 mm). For example, in a mixture of air with 1 per cent of carbon dioxide, the partial pressure of the carbon dioxide will be 7.6 mm under standard conditions; but its density, and therefore its refractivity, will be approximately 0.67 per cent lower than the value calculated by means of Boyle's law, because of the fact that it is too compressible to correspond to an "ideal gas." The values for the "ideal refractivity" calculated by Jones and Partington by means of Berthelot's equation of state can be conveniently used in this connection.

It should be noted from the form of equations 1 and 2 that the readings are influenced by the temperature and pressure even

⁶ Ramsey and Travers, *Proc. Roy. Soc. London*, **68**, p. 225, 1897-98; Cumseus, *Zs. physik. Chem.*, **86**, p. 232, 1901.

⁷ *Loc. cit.*

though the gases in both chambers are under like conditions. This effect is the smallest when the refractivities of the two gases are most nearly alike.

From these relations of scale reading and refractivity and of percentage composition and refractivity it is evident that the relation of scale reading to percentage composition can be computed. The curve marked "hydrogen" in Fig. 1 gives the scale readings for different percentages of hydrogen in air using air as the comparison gas.

To check this method of calibration by analysis a mixture of air and carbon dioxide was made up and stored in a 14-liter, mercury-sealed gas holder. Volumetric analysis of this mixture showed the presence of 2.52 per cent CO_2 (± 0.05 per cent). Two different interferometers calibrated by the method described each indicated the presence of 2.52 per cent of carbon dioxide in this mixture, showing agreement well within the limits of error of the volumetric analysis. If the interferometer calibration had not been corrected for the deviation of carbon dioxide from Boyle's law, it would have indicated 2.47 per cent CO_2 , a difference which is, however, negligible for a great many purposes.

No general statement can be made as to the accuracy of the method proposed, for this depends upon the gases in question; however, the accuracy is usually greater than is possible by volumetric gas-analytical methods, especially in the case of small percentages. The greatest advantage of the method is the speed and ease with which results for the entire range of the instrument can be obtained. This advantage is especially great in case the calibration curves are not straight lines, for in such cases a series of mixtures of known composition must be prepared if employing the usual calibration procedure.

WASHINGTON, July 12, 1917.

RESONANCE AND IONIZATION POTENTIALS FOR ELECTRONS IN CADMIUM VAPOR

By John T. Tate, Assistant Physicist, and Paul D. Foote, Associate Physicist

The phenomena connected with the passage of electrons through mercury vapor have been fairly completely investigated. Franck and Hertz¹ have shown that when the velocity of the electrons is less than that attained in falling through 4.9 volts there is no loss of energy at impact with a mercury atom. After falling through a difference in potential of 4.9 volts, however, the electrons are capable of transferring all of their kinetic energy to the mercury atom. The energy absorbed by the atom under these conditions is emitted as radiant energy of the single wave length, $\lambda = 2536.72 \text{ \AA}$. It was at first thought that this radiation of light was the result of a recombination with the atom of an electron which had been removed from it by the energy of impact. That this is not the case was proven by one of the writers,² who showed that no appreciable ionization takes place in mercury vapor until the electrons have fallen through a difference in potential of 10.3 volts. Later Gaucher,³ using a method which was sensitive to secondary photo-electric effects obtained results which were thought to prove conclusively the setting in of ionization at 4.9 volts. The more recent work of Davis and Gaucher⁴ eliminates these secondary effects, and the value obtained for the ionizing potential, 10.4 volts, is in good agreement with that already experimentally determined by Tate.

¹ Franck and Hertz, *Verh. d. Phys. Ges.*, 16, pp. 457-467, 1914.

² Tate, *Phys. R.*, 7, p. 686, 1916; *Phys. R.*, 10, p. 81, 1917.

³ Gaucher, *Phys. R.*, 8, p. 561, 1916.

⁴ Davis and Gaucher, *Phys. R.*, 10, p. 101, 1917.

It seems evident, therefore, that the energy lost by an electron having a velocity corresponding to 4.9 volts on impact with a mercury atom goes over into energy of agitation of one of the electrons bound in the atom and that the natural frequency of vibration of this electron corresponds to a wave length of 2536.72 Å. To explain the large transfer of energy at such a collision, we may assume that the electron is traveling with a velocity such that it remains within reaction distance of the atom during a time which bears some simple relation to the natural period of vibration of one of the bound electrons. The velocity of the impinging electron for which this condition is true may be termed a resonance velocity of the electron in mercury vapor. That there may be other resonance velocities than that corresponding to 4.9 volts has been recently shown by Davis and Gaucher.⁵

It is now known that a real ionization of the mercury atom occurs at about 10.3 or 10.4 volts. This ionization is accompanied by the emission of the complete mercury spectrum.⁶

Franck and Hertz were the first to point out that the frequency of the radiation emitted at 4.9 volts is that which should follow from the well-known relation $h\nu = eV$, if we substitute 4.9 volts for V . McLennan has pointed out that the ionizing potential might be predicted on the basis of Bohr's theory by substituting for ν in the above relation the frequency $\nu = (1.5, S)$ of the mercury spectrum. The value so calculated is 10.41 volts (putting $h = 6.56 \times 10^{-27}$).⁷ This value is in excellent agreement with the experimental values of 10.3 volts, obtained by Tate, and, more recently, by Davis and Gaucher, of 10.4 volts. From observations on the single and many lined spectra of other metallic vapors McLennan has been able to predict the resonance and ionization potentials for electrons in these vapors.

An experimental determination of the resonance and ionization potentials for electrons in cadmium vapor is the purpose of the present investigation.

The general method employed is the same as that used by one of the writers⁸ for determining the ionizing potential in mercury vapor. The arrangement of apparatus is shown in Fig. 1. Cadmium was vaporized at the bottom of the pyrex glass tube, and the vapor, after passing through the superheated ionization chamber, condensed in the upper half of the tube. The ionization

⁵ Davis and Gaucher, *loc. cit.*

⁶ Tate, *Phys. Rev.*, *loc. cit.*

⁷ Coblenz, *B. S. Bulletin*, 18; 1916.

⁸ Tate, *loc. cit.*

chamber was supported entirely by steel rods running from the top of the tube. The source of electrons was the hot platinum wire cathode *A*, of low resistance, coated with lime. Surrounding the cathode was a cylinder of nickel net, *B*, and outside and coaxial with this a cylinder of nickel foil, *C*. The apparatus was evacuated by means of a Langmuir condensation pump or by a Stimson^{*} two-stage condensation pump. In general, the pressure as registered by a McLeod gage was about 10^{-6} cm Hg. Sufficient vapor was obtained by heating the cadmium to its melting point.

The experimental procedure consisted in applying a constant retarding potential of about 2 volts between the net and the outside cylinder and measuring both the total current from the hot wire and that portion of it which reached the outside cylinder against the retarding field, as functions of a varying accelerating potential applied between the hot wire and net.

The results of the present work are represented graphically by Figs. 2 and 3. Reference numbers from 1 to 8 are assigned to the curves in these figures, and a complete analysis of the curves is given in Table I. Curves 1 to 7 represent the current (in arbitrary units) flowing between the net and outside cylinder of the ionization chamber as a function of the accelerating potential applied between the hot-wire cathode and the net. Curve 8 represents the total current from the hot wire as a function of the same accelerating potential.

The curves 1 to 7 are similar to those obtained by Franck and Hertz and by Tate for mercury vapor, in that they are characterized by successive maxima occurring at definite intervals. The potential difference between the successive corresponding parts of these maxima represents the potential through which an electron must fall in order that, on collision, it be capable of trans-

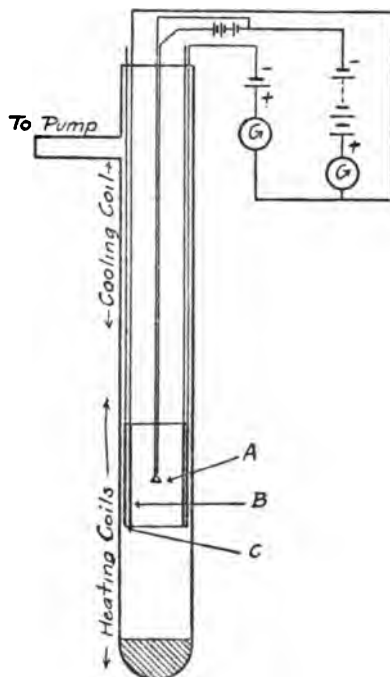


FIG. 1.—Diagram of apparatus

^{*} Stimson, J. Washington Acad. Sci. 7, p. 477, 1917.

ferring its kinetic energy to the cadmium atom. Curve 8 shows the characteristic variation with potential of the current from a hot wire in the presence of a gas or vapor at low pressure. The sudden increase in current at about 9 volts is due to ionization of the cadmium atom by impact with electrons having a velocity corresponding to that potential.

In the curves 1 to 6, Fig. 2, the points *a*, *b*, *c* represent the voltages at which occurs a pronounced falling off in the rate of increase of current with potential. These points must be interpreted as determining the applied voltages at which the electrons

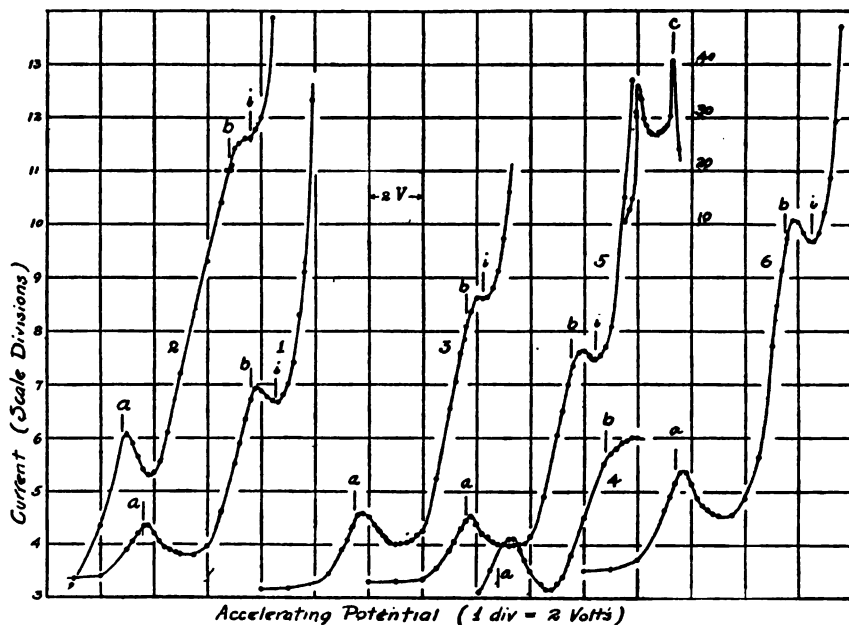


FIG. 2.—Variation with accelerating potential of current of outside cylinder

begin to suffer inelastic collision. If the electrons were emitted from the hot cathode with zero velocity the potential of the points *a* would represent the actual value of the resonance potential. Since, however, the electrons are ejected with some initial velocity, all potentials read from the curve must be corrected for the potential corresponding to the initial velocity. The value of the initial velocity is readily obtained from the curves. Points *a* represent the resonance potential minus the initial potential; points *b* represent twice the resonance potential minus the initial potential. Whence, the difference between *b* and *a* is the correct resonance potential, and the latter value together with the value *a* gives the initial potential. A similar procedure may be carried through for

the points of higher order, such as *c*, curve 5, and also in curve 7. Curve 7 is especially interesting. Resonance collision is observed to take place at five successive multiples of the fundamental potential interval, viz, at *a*, *b*, *c*, *d*, and *e*.

In addition to these points, all the curves show an abrupt change in curvature at *i*, followed by a very rapid increase in current. This fact, taken in connection with the abrupt change in total-current curve 8 at *P*, coincident with *i*, indicates that the setting in of ionization, shown by *P*, produces a redistribution of potential

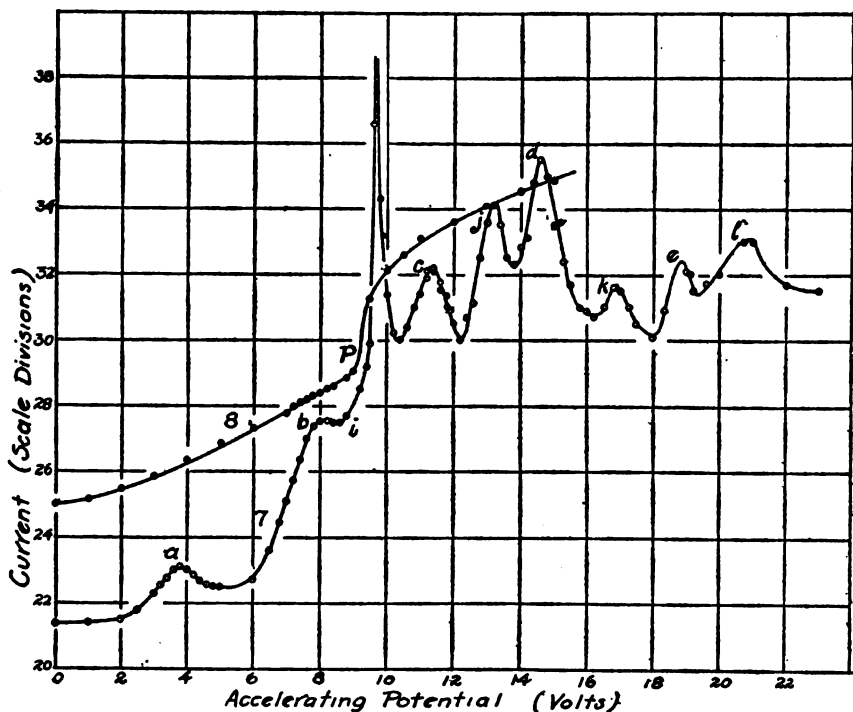


FIG. 3.—Variation with accelerating potential of current to outside cylinder and of total current from hot wire

between the wire and net, which is such as to result in a large increase in the number of electrons reaching the outside cylinder. The effect is rapidly counterbalanced, however, by the penetration and formation of positive ions in the space between the net and cylinder. The presence of inelastic impact at *P* is indicated by the occurrence of three successive maxima (more accurately, pronounced changes in curvature as discussed above) at potentials differing from *P* by multiples of the fundamental resonance potential, viz, at *j*, *k*, and *l*. These three points represent not

only resonance collision of the electrons liberated from the atom with zero velocity at ionization, but also resonance collision of those electrons whose kinetic energy was lost in producing the ionization.

In determining the applied potential at which inelastic impact occurs, a , b , c , etc., were chosen at the points where the curves show a marked falling off in the rate of increase of current. Were there

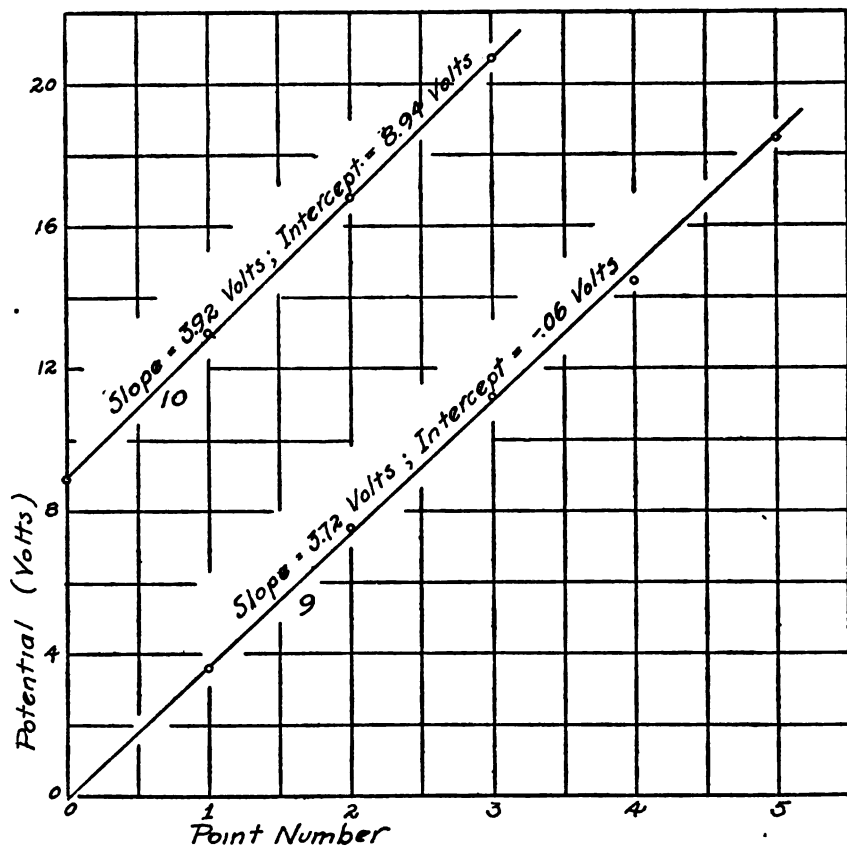


FIG. 4.—Graphical representation of data from curves 7 and 8

no inelastic collision at a , for example, the lower portion of the curve would continue to increase approximately exponentially. The curves were carefully plotted on a large scale and this portion of the curve graphically extrapolated. The point a was chosen where the deviation of the extrapolated curve from the actual curve reached a predetermined very small amount. A similar procedure was adopted for the other points. Since one is here concerned only with differences between successive, similarly chosen points,

there can be little question in regard to the somewhat arbitrary method of choice.

The method of least squares was applied to the data obtained from curves 7 and 8. If y is the initial potential of the electrons, x the resonance potential, and Q the ionization potential, we have:

$$\left. \begin{aligned} a &= x - y \\ b &= 2x - y \\ c &= 3x - y \\ d &= 4x - y \\ e &= 5x - y \end{aligned} \right\} \text{I.} \quad \left. \begin{aligned} Q &= P + y \\ j &= P + x \\ k &= P + 2x \\ l &= P + 3x \end{aligned} \right\} \text{II.}$$

The solution of these equations may be represented graphically by Fig. 4. The slope of the curve 9 (group I, above) determines the resonance potential, and the intercept on the potential axis the initial potential. The slope of curve 10 (group II, above) also determines the resonance potential and its intercept the most probable value of the point P on the basis of equal weights to the observed points P, j, k, l . It is of interest to note that the new value for P differs by a very small amount from the value obtained by graphical means, using only curve 8.

The true value of the ionizing potential Q is obtained from the relation $Q = P + y$, where y is the initial velocity as determined by group I, above. The results of the present work are detailed in Table 1. Weighted averages give the most probable value of the resonance potential as 3.88 volts and the ionizing potential 8.92 volts, with an estimated probable error slightly less than 0.1 volts.

TABLE 1.—Resonance and Ionization Potentials for Cadmium

Curve	Resonance points					Ioniza- tion ob- served.	Re- son- ance po- ten- tial	Weight	Initial velocity	Ioniza- tion po- ten- tial	Weight
	1st	2d	3d	4th	5th						
1.....	3.6	7.6				8.5	4.0	1	0.9	8.9	1
2.....	2.8	6.8				7.6	4.0	1	.8	8.8	1
3.....	3.5	7.6				8.2	4.1	1	.6	8.8	1
4.....	2.8	6.8					4.0	1			
5.....	3.6	7.5	11.3			8.4	$\left\{ \begin{array}{l} 3.9 \\ 3.8 \end{array} \right.$	$\left\{ \begin{array}{l} 1 \\ 1 \end{array} \right.$	.9	8.6	1
6.....	3.4	7.5				8.5	4.1	1	1.1	9.2	1
7(A).....	3.6	7.7	11.2	14.4	18.4		a 3.72	5	a .06		
7(B).....	8.6	13.0	16.8	20.7		b 8.97	b 3.92	2	a .06	b 9.03	3
8.....	Total current curve.....					8.9			a .06	c 8.96	
Mean							3.88 volts.			8.92 volts.	

^a Least square reduction of 5 points.

^b Least square reduction of 4 points.

^c Not used in average because this point was involved in obtaining the value immediately above.

SUMMARY AND CONCLUSION

The two types of electronic impact already mentioned in connection with mercury vapor are thus clearly shown to take place in cadmium vapor. The sudden decrease in current through the retarding field at an effective potential of 3.88 volts indicates a loss in energy of electrons having this critical velocity on impact with a cadmium atom. The fact that there is no corresponding increase in the total current at 3.88 volts shows that no appreciable ionization takes place at this potential. When the velocity of the electrons corresponds to 8.92 volts, the collisions again become inelastic, and the sudden increase in the total current indicates that the loss in energy of the colliding electrons has gone over into separating one or more of the electrons bound in the atom from the nucleus.

Under conditions such that only the first type of collision could take place McLennan and Henderson¹⁰ have shown that light of the single wave length 3260.17 Å is emitted. That the entire spectrum of cadmium is probably radiated at potentials above 8.92 volts is indicated by the appearance of a visible glow around the hot wire at 9.2 volts. McLennan and Henderson were able to get the complete spectrum at about 15 volts.

If we substitute the frequency corresponding to a wave length 3260.17 Å in the relation $h\nu = eV$, we find the theoretical value for the resonance potential (putting $h = 6.56 \cdot 10^{-27}$ erg. sec.) to be 3.79 volts, as compared with the experimentally observed value 3.88 volts. If, following McLennan, we calculate the ionizing potential on the basis of Bohr's theory, using the frequency $\nu = (1.5, S)$ of the cadmium spectrum ($\lambda = 1378.69$ Å) in the above relation, we obtain a theoretical value of 8.97 volts, as compared with 8.92 volts observed. The agreement is within the probable error of the experiment.

The results obtained for the resonance and ionization potentials for electrons in mercury and cadmium vapors must be regarded as another very striking example of the fundamental accuracy of conclusions based upon Bohr's theory of atomic structure. Especially is this true of the results for mercury vapor, since for a long time the view that ionization took place at 4.9 volts, together with the known fact that the radiation emitted at that potential was monochromatic, formed a very serious obstacle to the complete acceptance of the theory.

WASHINGTON, August 27, 1917.

¹⁰ McLennan and Henderson, Proc. Roy. Soc. Lond., 91, pp. 485-491; 1915.

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APPLICATION OF DICYANIN* TO THE PHOTOGRAPHY OF STELLAR SPECTRA

By Paul W. Merrill

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I. INTRODUCTION

1. ADVANTAGES OF PHOTOGRAPHY IN SPECTROSCOPY

It has become almost axiomatic in modern spectroscopy that photography should be applied wherever possible. This is particularly true in astrophysics, since the observational conditions give photography especially great advantages. It is a mistake to suppose that these advantages are limited to the more refrangible portions of the spectrum. In the spectral region to which the eye is most sensitive, namely, near wave length $.56\mu$ (yellow-green), a rather long exposure is required to show photographically the faintest details seen by a skillful observer, but even here it is only under special conditions that visual results are of great value as compared with a photographic record.

The visibility of radiation¹ falls rapidly on either side of λ_{56} ,² being equal to 1 per cent of the maximum at λ_{40} (violet) and at

* Dicyanin, a synthetic dye used for sensitizing photographic plates.

¹ Coblentz and Emerson, this Bulletin, 14, p. 167; 1917.

² Throughout this paper λ will be used as an abbreviation for wave length. In referring to wave lengths between 0.3 and 1.0μ (3000 and 10 000 angstroms) two or more significant figures will be given without decimal points or units. Thus, λ_{56} is to be understood as wave length 0.56μ (5600 angstroms); λ_{560} or λ_{5600} would refer to the same wave length, but would indicate greater accuracy.

$\lambda 69$ (red). From $\lambda 69$ on toward longer waves this value is halved every 100 angstroms, so that to produce a given sensation of vision 5000 times as much energy is required at $\lambda 75$ as at $\lambda 56$. The inefficiency of the eye for the red rays is, perhaps, not generally recognized, since it has not been emphasized so greatly as in the case of the violet rays by the success of photography.

To the eye a continuous spectrum is usually terminated at the red end, not by the lack of energy but by the ineffectiveness of the longer waves in producing vision.

This fact has been known, as far as the solar spectrum is concerned, since the days of the elder Herschel, and has for many years been inferred to hold true in stellar spectra from the assumption that the spectral-energy curve is comparable with that of a black body. Nichols's observations in 1900 showed the existence of considerable infra-red energy, at least for the star Arcturus. "The ratio greater than 2 to 1, of the total radiation of Arcturus to that of Vega (stars which by most observers are estimated to be of nearly equal photometric magnitude) indicates a proportionately more intense infra-red spectrum for the former than for the latter star."³ By observations of stars with and without a water screen Coblentz⁴ has recently proved that a large fraction of the energy in stellar spectra is of greater wave length than 1.4μ (14 000 angstroms).

The results obtained in the infra-red by the use of radiometric instruments are of the utmost value, and these methods are indispensable for energy measurements, but for mapping spectral detail there is no question that photography, where it can be applied, is far superior both in refinement of observation and in ease of manipulation.

From general considerations, therefore, it would seem to be important to extend the application of photography toward the infra-red, as well as toward the ultra-violet.

2. HISTORICAL

Modern commercial photographic plates are comparatively insensitive beyond about $\lambda 50$. For longer wave lengths specially prepared emulsions have been used with some success, but in order to obtain any considerable extension into the red there has been more activity of recent years, in sensitizing ordinary dry plates by means of certain dyes. The literature on this sub-

³ Nichols, *Astrophysical Journal*, 18, p. 135; 1901.

⁴ This Bulletin, 11, p. 646; 1915.

ject has become so extensive that it can not be reviewed here.⁵ The dyes which have been most used in astronomical spectroscopy seem to be erythrosin, pinacyanol, and a combination of homocol, pinaverdol, and pinacyanol recommended by Wallace.⁶ By the use of pinacyanol, stellar spectra have been quite widely observed in the $H\alpha$ ($\lambda 6563$) region, and have been pushed as far as $\lambda 71$, but are poorly observed at this extreme wave length because of the rapid decrease in sensitivity. Dicyanin has been tried by several astronomers,⁷ but except in a few instances the limit has not been greatly extended beyond $\lambda 71$. Dr. Slipher has kindly furnished photographic prints of the originals of the illustrations of stellar spectra in the *Encyclopedia Britannica*. These spectra extend well into the red; the A line appears in several, being most distinctly shown in α Orionis.

In view of the favorable results at wave lengths greater than $\lambda 71$ obtained at the Bureau of Standards with dicyanin, it seemed that the full value of the dye had never been realized in astrophysics, and that further trials might be profitable. In 1912 Dr. Burns had expressed the opinion⁸ that with existing instruments it would be possible to photograph the spectra of bright stars as far as $\lambda 80$.

Lack of complete success heretofore can probably be traced to one or more of the following causes: (a) Deterioration of the dye; (b) failure of the staining process employed to give sufficient sensitivity; or (c) ignorance of the spectral region in which dicyanin is most valuable.

(a) It is an unfortunate fact that dicyanin can not be depended upon to preserve its useful properties indefinitely under ordinary conditions; some chemical change takes place which destroys wholly or in part its sensitizing value. It appears quite probable that numerous tests of dicyanin have been made using samples which were no longer in good condition. The deterioration can be retarded and perhaps nearly eliminated by proper precautions.

(b) The sensitivity of the dyed plate depends on the composition of the staining bath and on the details of the manipulation. Various processes have been recommended by the makers of the

⁵ Eder and Valenta, *Beiträge Zur Photochemie und Spectralanalyse*, Wien, 1904; Wallace, *Astrophysical Journal*, 26, p. 311, 1907; Baly, *Spectroscopy*, Chap. XI, London, 1912; and Eder, *Photographische Korrespondenz*, September, 1915.

⁶ *Astrophysical Journal*, 26, p. 311; 1907.

⁷ V. M. Slipher, *Astrophysical Journal*, 28, p. 400, 1908; *Enc. Brit.*, 11th ed., 21, p. 717, 1910-11; Bosler, *Comptes Rendus*, 160, p. 124, 1915.

⁸ *Publications of the Astronomical Society of the Pacific*, 24, p. 251; 1912.

dye and others; some of these processes probably would not give sufficient speed for exposures on faint sources.

(c) Some experimenters have possibly not realized at what wave lengths dicyanin has its greatest value, namely, those *greater than* 68. Pinacyanol is superior in speed up to about $\lambda 68$, but for continuous and absorption spectra, dicyanin is probably to be preferred even at this point on account of its flatter sensitivity curve.

3. PURPOSE OF THIS INVESTIGATION.

The general aim of the investigation reported upon in this paper was to ascertain whether stellar spectra are sufficiently intense in the region of wave length 0.80μ ($8000\text{\AA}$) to allow these wave lengths to be photographed with moderate exposure times by means of dicyanin. If this proved to be the case, it was hoped that the observations might indicate something of the value of this part of the spectrum to astronomers and physicists.

II. OBSERVATIONS

1. THE INSTRUMENT

By an arrangement with Director Pickering, of the Harvard College Observatory, the 24-inch reflector of the observatory was used

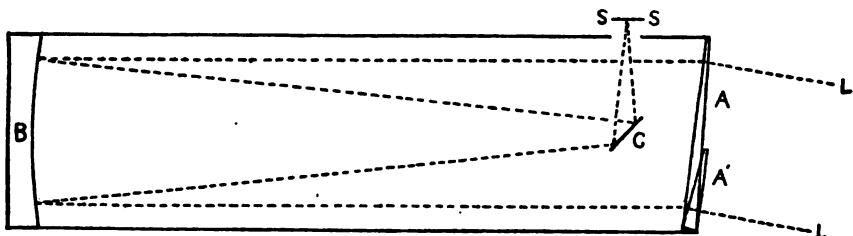


FIG. 1.—Schematic diagram of optical system of 24-inch reflector with objective prism

L, L, Light rays from star; *A*, prism *P*, refracting angle 6° , aperture 24 inches (circular); or *A'*, prism *D*, refracting angle 15° , aperture 9 by 11 inches; *B*, concave mirror, aperture 24 inches, focal length 11 feet; *C*, plane mirror; *S-S*, spectrum.

during the month of February, 1917, to make the tests referred to in the above paragraph. The reflector is of the Newtonian type, with an aperture of 24 inches and a focal length of 11 feet. The spectra were formed by an objective prism placed at the upper end of the tube above the secondary mirror. This arrangement, due to Prof. Pickering, combining as it does great light-gathering power with achromatism, was obviously well suited to the problem in hand. Two prisms were used: Prism *P*, circular, aperture 24 inches, angle about 6° ; Prism *D*, nearly rectangular, 9 by 11 inches, angle about 15° . The latter prism was placed eccentrically

at the mouth of the tube in order that the secondary mirror might not occult too much of the light. Thus, practically the whole aperture of the prism was made use of. (See Fig. 1.) The linear dispersions yielded by the prisms are as follows:

Linear Dispersion

	H ϵ to H β	H β to H α
Prism P.....	mm 1.7	mm 1.4
Prism D.....	15.8	12.3

The telescope is equipped with electric drive and pendulum control which afforded great convenience in obtaining any desired departure from sidereal rate.

2. PHOTOGRAPHIC PROCESSES

The exact action of the dye in sensitizing plates is so little understood, and the subject is so complex, that in spite of the large amount of research upon it, any statements made at this time in regard to the best procedure must be regarded as provisional, as they are more than likely to be revised by subsequent experience. Similarly, certain precautions mentioned below may later be shown to be unnecessary.

Both dicyanin and a more recent product, dicyanin A, were tried at Harvard. The difference in the action of the two dyes is not very marked, dicyanin A, perhaps, being better for the very long waves. The dyes should be obtained as soon after manufacture as possible and should be kept cool and free from moisture. The stock solution should be made in the ratio 1 : 1000 or 1 : 2000 in the highest grade of absolute ethyl alcohol. The best results obtained by the writer have been with the following procedure:

Staining Solution

	Cubic centimeters
Distilled water	140
Ethyl alcohol.....	120
Dicyanin A (1 : 2000).....	18
Or dicyanin 1 : 1000.....	7
Ammonia, 26 degrees.....	9

Directions for Sensitizing.—Mix the water and alcohol thoroughly and allow to stand. After adding the dye allow several minutes before the ammonia is put in and again an interval before staining. Bathe the plates $4\frac{1}{2}$ minutes at 20° C (68° F). Place for about 40 seconds in a bath of pure alcohol at the same temperature, remove,

and dry in a current of air.⁹ Avoid high temperatures at any point of the process.

The bath can be used repeatedly if 3 or 4 c c of ammonia is added each time. The dye and all solutions should be protected from light. In packing the plates after staining do not allow the film to come in contact with any solid object, even glass or another film. The stained plates deteriorate rather quickly; if placed in an ice box, they will keep fairly well for a few days, but it is advisable to sensitize them as shortly before using as convenient. Use any clear working developer¹⁰ a few degrees cooler than usual.

The fact that dicyanin does not sensitize the plates to green and yellow light is made apparent by the appearance of the photographed spectra. As the illustrations show, the blue and red portions of the spectra are usually separated by a gap representing the insensitive region. A more nearly uniform spectrum can undoubtedly be obtained by the addition of pinaverdol¹¹ to the staining bath.

3. SPECTROSCOPIC RESULTS

Between February 4 and March 1, 1917, 67 stellar spectrograms were secured on 14 nights. Those of interest are included in the following list: Columns one to seven require no explanation. In the eighth column, "Exposure," c indicates that the observation was interfered with by clouds; in the ninth column, "Prism," P denotes the 24-inch 6° prism; D denotes the 9 by 11 inch 15° prism; in the tenth column, "Emulsion," HS indicates Hammer Special, 27 indicates Seed 27; in the last column, "Dye," D and DA signify dicyanin and dicyanin A, respectively.

Journal of Observations

Star	α	δ	Mag.	Class	Plate H	Date, 1917	Exposure	Prism	Emulsion	Dye
	h m						min.			
-12° 1172.....	5 23	-12°.8	9.1	Pa	1794	Feb. 11	62	P	HS	DA
					1800	Feb. 12	72	P	HS	DA
-23° 4553.....	6 50	-23.8	6.6	Ob	1790	Feb. 10	36	P	HS	DA
					1795	Feb. 11	62	P	HS	D
					1796	...do....	42	P	HS	D
Orionis.....	5 30	-6.0	2.9	Oe	1821	Feb. 18	74	D	HS	DA
Pleiades.....	3 40	+24.		B	1783	Feb. 6	10	P	HS	D
					1784	...do....	10	P	HS	D

⁹ The above procedure is much the same as that recommended by Wallace for his three-dye stain, *Astro-physical Journal*, 26, p. 311; 1907.

¹⁰ In this investigation paramidophenol (1 part to 16 parts of water), as recommended by Prof. King, was used and seemed satisfactory for the dicyanin plates.

¹¹ Possibly also erythrosin or orthochrome T or other dyes could be used, but they have not been actually tried in combination with dicyanin by the writer. Dicyanin and pinaverdol have been used together many times at the Bureau of Standards in photographing laboratory spectra.

Journal of Observations—Continued

Star	α	δ	Mag.	Class	Plate H	Date, 1917	Ex- posure	Prism	Emul- sion	Dye
	h m						min.			
β Orionis.....	5 10	-8° 3	0.3	B8	1787	Feb. 10	14	P	HS	D
α Draconis.....	12 29	+70.3	3.9	B5P	1813	Feb. 16	54c	D	27	DA
β Lyrae.....	18 46	+33.2	()	Bp	1829	Feb. 18	25c	D	HS	D
					1844	Feb. 24	38	D	HS	DA
ρ Cygni.....	20 15	+37.7	4.9	Bip	1845	...do....	20c	D	HS	DA
α Can. Maj.....	6 41	-16.6	-1.6	Ao	1791	Feb. 10	16	D	HS	D
					1810	Feb. 16	42c	P	27	DA
					1822	Feb. 18	39	P	HS	DA
					1839	Feb. 24	60	P	HS	D
α Boötis.....	14 11	+19.7	0.2	Ko	1815	Feb. 16	56c	P	HS	DA
α Urs. Maj.....	10 58	+62.3	2.0	Ko	1825	Feb. 18	37	P	HS	DA
α Tauri.....	4 30	+16.3	1.1	K5	1785	Feb. 6	5	D	HS	D
					1807	Feb. 14	35c	P	HS	DA
α Orionis.....	5 50	+ 7.4	0.9	Ma	1789	Feb. 10	17	P	HS	DA
					1809	Feb. 16	24c	D	27	DA
					1832	Feb. 21	75	D	HS	D
λ Draconis.....	11 26	+69.9	4.1	Ma	1793	Feb. 10	24c	P	HS	DA
					1834	Feb. 21	65	D	HS	D
α Herculis.....	17 10	+14.5	3.5	Mb	1817	Feb. 16	7	D	HS	DA
					1828	Feb. 18	38c	D	HS	D
					1837	Feb. 22	22c	D	HS	D
					1843	Feb. 24	52	D	HS	DA
					1849	Feb. 28	23c	P	27	DA
ψ Virginis.....	12 51	- 9.0	4.9	M(b)	1814	Feb. 16	66c	D	27	DA
HR 5589.....	14 56	+66.3	4.9	Mb	1842	Feb. 24	30	D	HS	DA
					1847	Feb. 28	41c	P	27	DA
40 Com. Ber.....	13 2	+23.2	5.9	Mbp	1841	Feb. 24	72	D	HS	DA
ϵ Ceti.....	2 14	- 3.4	(5.7)	Md9	1831	Feb. 21	68	D	HS	D
			(5.8)		1838	Feb. 24	72	D	HS	D
R Leo. Min.....	9 39	+35.0	(7.8)	Md8	1797	Feb. 11	32	P	HS	D
			(7.8)		1804	Feb. 12	62	P	HS	DA
			(7.9)		1824	Feb. 18	43	D	HS	D
			(7.9)		1830	Feb. 20	112c	D	HS	D
U Hydrae.....	10 33	-12.8	(5.4)	N	1792	Feb. 10	22c?	P	HS	D
			(5.4)		1798	Feb. 11	24	P	HS	D
			(5.4)		1812	Feb. 16	50c	D	27	DA
DM+38° 1539.....	6 30	+38.5	6.3	N	1802	Feb. 12	71	P	HS	DA
					1811	Feb. 16	49c	D	27	DA
					1833	Feb. 21	72	D	HS	D
Y Hydrae.....	9 46	-22.6	(6.7)	Np	1840	Feb. 24	78c?	D	HS	DA
			(6.8)		1846	Feb. 28	46c	P	27	DA
DM+61° 667.....	3 57	+61.5	7.9	N	1799	Feb. 12	16	P	HS	DA
DM+14° 1283.....	6 20	+14.8	6.5	N	1801	...do....	72	P	HS	DA
DM+46° 817.....	12 40	+46.0	()	N	1827	Feb. 18	54c	D	HS	D
DM+68° 617.....	10 38	+67.9	()	N	1826	...do....	70	D	HS	D
DM-24° 12084.....	15 22	-24.8	7.4	R	1805	Feb. 12	16	P	HS	DA
DM+42° 2811.....	17 13	+42.2	7.3	R	1848	Feb. 28	41c	P	HS	DA

Description of the spectra obtained will now be taken up in the order of the Harvard classification.¹² Wave lengths were deter-

¹² *Annals Harvard College Observatory*, 28, p. 140; 56, p. 66.

mined or identified by micrometer measurement of the photographs. Those given in italics are the results as measured. The accuracy of the various measurements is indicated by the number of figures given in stating the wave lengths. The units will be omitted, since it will be understood that all wave lengths referred to lie between 3000 and 10 000 angstroms (0.3 and 1.0 μ). The accuracy attained is not great, and the plates probably fail to show many of the finer details of the spectra. This brief study is therefore in the nature of a reconnoissance. The results indicate the desirability of future observations in the extreme red and infra-red regions with both objective-prism and slit spectroscopes.

DM-12° 1172—Spectral Class Pa.—In the spectrum of this interesting planetary nebula the following emission lines, all previously observed, are shown: 3728 (nebular), H δ , H γ , H β , H α . H α is the strongest line, 3728 next. There is a suggestion of an emission line of somewhat shorter wave length than H α , and on one plate a faint trace of an emission line at about 714. H α has been observed visually by Campbell¹³ and photographically by Bosler.¹⁴ Campbell remarked that H α was "very difficult." This is in striking contrast with my photographs on which H α is the strongest line of all.

DM-23° 4553.—Emission lines only were observed in this Wolf-Rayet star as follows:

348
4059 Wolf-Rayet line
H δ
H δ'
H γ
• 453 H γ' ?
4686 Wolf-Rayet line
H β
Line near D₃
H α
(702)

The italicized numbers above, as also those shown on succeeding pages of this paper, are wave lengths as measured. The identifications assumed for the other lines depend also upon measurements. The strongest line in the spectrum is 4686; 348 is a distinct, well-marked line on two plates. It may have the

¹³ Astronomy and Astrophysics, 8, p. 494; 1894.

¹⁴ Comptes Rendus, 100, p. 124; 1915.

same origin as a line in about this position observed by Palmer¹⁵ in Nova Persei and in two planetary nebulae N. G. C. 6741 and N. G. C. 6886, but the identification is doubtful. The wave lengths of 702 from different plates are not accordant, but the line looks real. Lines at 691 and 711 have been observed by Bosler¹⁶ in other Wolf-Rayet stars. He did not observe this star. $H\alpha$ was the only line in the red seen by Campbell.¹⁷

♄Orionis—Spectral Class Oe.—The terrestrial atmospheric absorption lines *B*, *a*, and *A* are the only details observed in the red. $H\alpha$ is lost by overexposure. In the microscope of power 10 the spectrum can be seen to about 796. With the naked eye it can be traced, perhaps, 200 angstroms farther. Unless otherwise specified, the limits mentioned for other stars refer to the appearance of the spectrum in the microscope. With dicyanin the spectrum shades off gradually, so that these limits are indefinite, but care has been taken not to overestimate them.

Stars of Class B.—On account of the limited time at my disposal only a few spectra of the earlier classes were obtained. In Class B no new features were discovered. The characteristic intensity curve is well shown by the plate of the Pleiades, Fig. 6. It is interesting to compare the relative intensities of the two portions of the spectrum with the intensities shown by stars of other classes. The differences are so great that the class could be determined fairly well from an image so weak or so poorly defined that no lines are visible.

Bright-Line Spectra of Class B.—Several spectra of this type were photographed, including κ Draconis, β Lyrae, and P Cygni. $H\alpha$ is a strong emission line in all these spectra. The photographs of β Lyrae and of P Cygni were taken under poor conditions and hence are not so valuable as they might otherwise be. Besides D_2 and $H\alpha$ several emission lines of greater wave length are seen with low power. These probably include 6678 and 7065 of helium. The spectra extend to about 75.

α Canis Majoris—Spectral Class A0.—The plate of low dispersion is overexposed in the stronger portions of the spectrum. The following features have been measured in the less refrangible part: 627, slight maximum in continuous spectrum; 650, minimum in continuous spectrum; 693, center of strong broad maximum; 727, drop in intensity; 87–89, end.

¹⁵ Lick Observatory Bulletins, 2, p. 46; 1902.

¹⁶ Comptes Rendus, 160, p. 124; 1915.

¹⁷ Astronomy and Astrophysics, 8, p. 456; 1894.

The longer spectra show the absorption lines $H\alpha$, B , a , A . A is strong and conspicuous. There is a decided suggestion of an absorption line at about 81 . The spectrum extends faintly to 84 . (See Fig. 2.)

α Boötis—Spectral Class Ko.—The features measured include K , H , g , absorption band head at 5166 , A , band head (?) at 808 , end of spectrum 854 .

α Ursae Majoris—Spectral Class Ko.—This spectrum, which is not so strongly exposed as that of α Boötis, shows K , H , g , band head at 5166 , D , B , A . In both spectra there is apparently some faint detail beyond A , but it is too poorly shown to be interpreted.

α Tauri—Spectral Class K5.—The spectrum taken with the 6° prism is too short to allow many features in the red to be distinguished. There is a broad maximum at 70 , while the spectrum ends about 82 . An absorption line (probably A) beyond the maximum is seen with low power. (See Fig. 7.)

The longer spectrum, which is not strongly exposed in the red, shows D , $H\alpha$, B , a , A . There is a broad maximum of the continuous spectrum between B and a .

α Orionis—Spectral Class Ma.—On the small dispersion plate K , H , g , A , and a slight maximum at 79 , were measured in addition to heads of absorption bands at 4760 , 4955 , 5166 , 5862 , 6156 , and 7054 . These bands all head toward shorter wave lengths, and are well known in Class M spectra. The spectrum extends to 85 .

The following features were measured in the longer spectra: K , H , g , D , B , A , and band heads at 4955 , 5166 , 6156 , 7054 . The end of the spectrum as measured on the two plates falls at 833 and 874 . (See Fig. 3.)

On all the plates of α Orionis there is apparently an absorption band with its head (toward the violet) nearly coincident with A . It probably represents the early stage of a band which develops to great prominence in Mb and Md spectra. In common with many other bands in the spectrum of α Orionis, this new band is almost undoubtedly to be ascribed to titanium oxide. The identification will be discussed later in this paper.

λ Draconis—Spectral Class Ma.—The bands are not so pronounced as in α Orionis. A is well shown on the high dispersion plate, but the band having its head at that point is not in evidence.

α Herculis—Spectral Class Mb.—This star has a spectrum which is rendered very striking by the strong flutings found throughout a long range of wave lengths. The plate of small dispersion shows

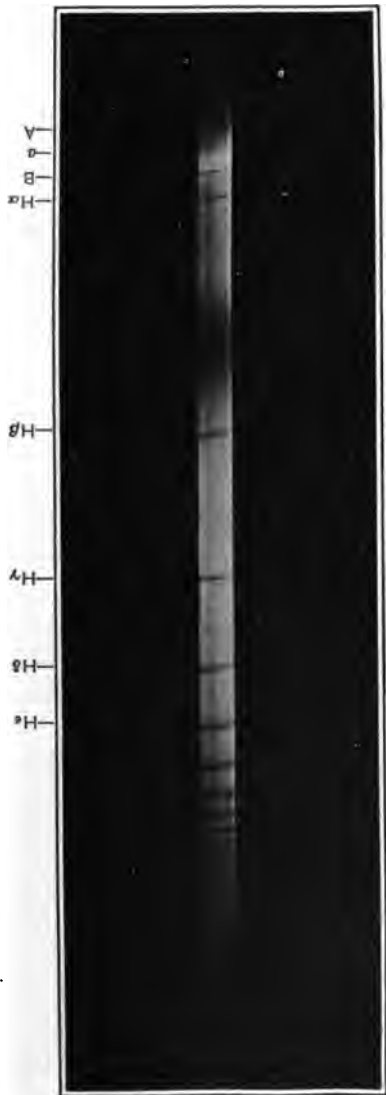


FIG. 2.— α Canis Majoris. Class A₀. Plate H1810

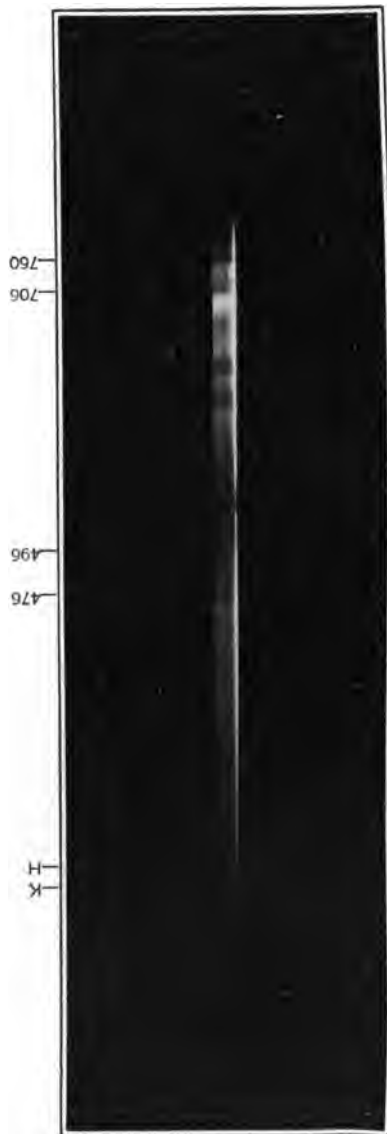
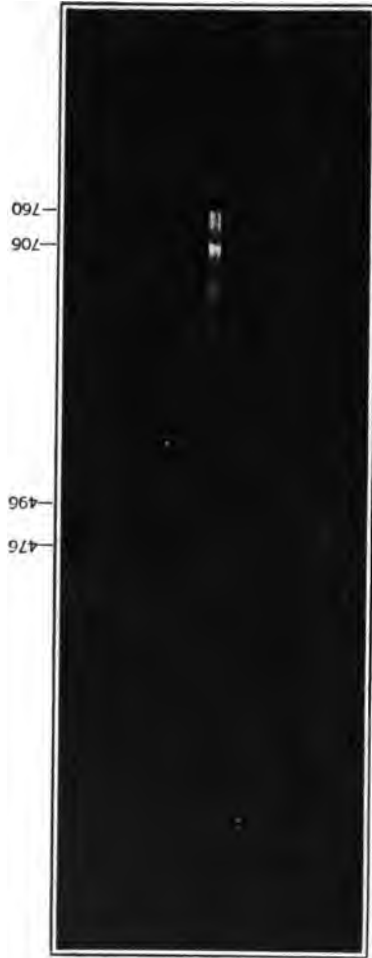


FIG. 3.— α Orionis. Class Ma. Plate H1832



(496-2)

FIG. 4.— α Herculis. Class Mb. Plate H1843

the following features: *K*, *H*, *g*, *A*, and band heads at 4760, 4955, 5166, 5862, 6156, 6652, 706, 760. There is a distinct maximum at 808, perhaps indicating an absorption band beginning at a somewhat greater wave length. The spectrum is observed to 83.

With higher dispersion the spectrum contains so much detail that an adequate description is difficult. The chief features are shown in Fig. 4. The measurements include settings on *g*, *B*, *A*, and band heads at 476, 4955, 5763, 5815, 585, 6156, 706, 760.

On the best plate the band head at 706 is distinctly triple. Assuming the first component to lie at 7054, the wave lengths of the second and third were measured as 709 and 713. These are well-known titanium oxide bands observed in the laboratory by Fowler,¹⁸ and by Hale and Adams¹⁹ in both laboratory and sun-spot spectra. Referring to stellar spectra, these bands have been observed in α Orionis by Newall and Cookson,²⁰ and by Slipher.²¹ The latter observed them, partially, in the spectrum of σ Ceti.²² On two plates there is seen what may be the head of a rather weak absorption band at 811. This agrees with the appearance of the short spectrum as described above. The settings on the end of the spectrum range on different plates from 825 to 836. A faint continuation is visible without magnification to a considerably greater wave length. As this is true in the spectra of other stars also, it is worth noting because it indicates that the sensitivity curve of the dicyanin-plate does not drop abruptly, and that greater effective exposures will record stellar spectra to even longer wave lengths than those obtained in this investigation.

The absorption band beginning sharply near *A* and shading off toward longer waves is strongly developed, and with the fluting at 706 gives the extreme red portion of the spectrum a remarkable appearance. The spectrum here might conveniently be described as consisting of two emission bands sharply cut off at 706 and 760, and gradually weakening toward the violet. As these features of the spectrum seem to vary considerably from star to star, it is not improbable that they may be useful in distinguishing the subdivisions of Class M. The band of greater wave length seems to become intensified with advance in spectral type.

ψ *Virginis*—Spectral Class M(b).—Although classified as Mb this spectrum resembles that of λ Draconis more than that of α Herculis. It does not appear to be more advanced than α Orionis. A single

¹⁸ Monthly Notices, R. A. S., 69, p. 508; 1909.

¹⁹ Astrophysical Journal, 25, p. 77; 1907.

²⁰ Monthly Notices, R. A. S., 67, p. 482; 1907.

²¹ Monthly Notices, R. A. S., 69, p. 510; 1909.

²² Astrophysical Journal, 25, p. 236; 1907.

spectrum of ψ Virginis was secured and its quality is only fair, but the appearance of the bands in the red favors the classification Ma rather than Mb.

HR 5589—Spectral Class Mb.—This appears to be a typical Mb spectrum. On Plate H1847 (small dispersion) the following features were measured: *K*, *H*, *g*, *A*, band heads at 4760, 4955, 5166, 586, 6156, 6652, 706, 76. The spectrum ends more abruptly than usual at 83. (See Fig. 8.)

40 Comæ Bereniciæ—Spectral Class Mb.—This spectrum is similar to HR 5589.

o Ceti—Spectral Class Md 9.—The characteristics distinguishing Class Mb from Class Ma are still more pronounced in this Md spectrum. The absorption in the flutings is more nearly complete, giving the spectrum a curiously disconnected appearance. By far the most conspicuous features of the whole spectrum, as seen on these plates, Fig. 5, are the two (apparently) emission flutings at 706 and 760. While these are probably the unshaded portions of continuous spectrum between absorption bands, the maxima are much stronger than anything else seen in the spectrum. The measurements include *B*, *A*, end of spectrum at 837, and band heads at 4505, 4626, 476, 4955, 5843, 6156, 656, 665 $\pm$, 706, 760, and possibly one between 81 and 82. *H* δ and *H* γ are seen as emission lines, the former being the stronger.

R Leonis Minoris—Spectral Class Md 8.—In the short spectra the following were observed: *H* (and *K*), emission *H* δ , trace of emission *H* γ , maximum at 814 $\pm$, end 83, bands at 4760, 4955, 5862, 6156, 6652, 706, 760 (reinforced by *A*).

In the longer spectra, which are both underexposed, the stronger bands and emission *H* δ are seen. As in *o Ceti* the portions of continuous spectrum about 69 and about 74 are the most striking features of the spectrum. Bright *H* δ is very strong.

Stars of Class N.—The spectra of Class N as shown on these plates contain considerable detail, but most of it is of such a character that it can not be measured with accuracy. While no precision can be claimed for the wave-length determinations, they are sufficient to identify the main features and to furnish a fairly definite description of what is seen on the photographs. (See Fig. 9.)

In the following tabulations of measurements the terms "rise" and "drop" refer to changes in the intensity curve when proceeding toward longer waves. "pr" indicates present.

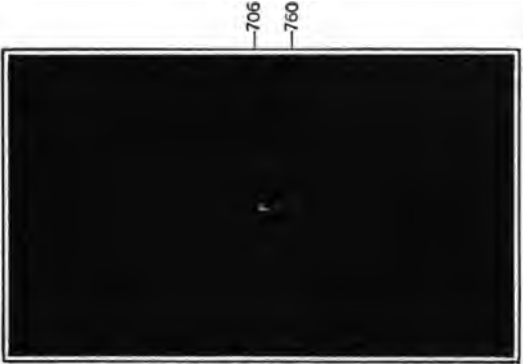


FIG. 5.—*Ceti*. Class *Md9*. Plate *H1831*

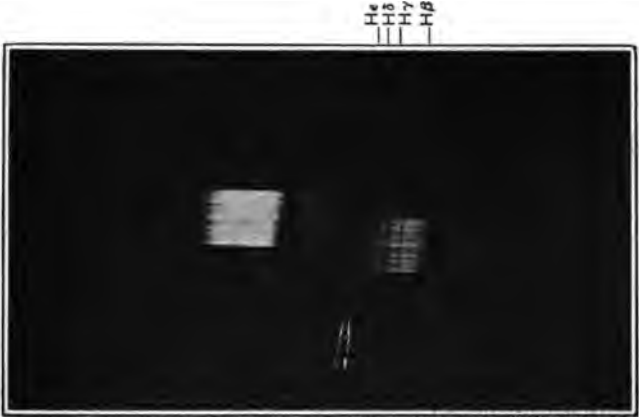


FIG. 6.—*Atlas and Pleione*. Class *B*. Plate *H1783*

(498-1)

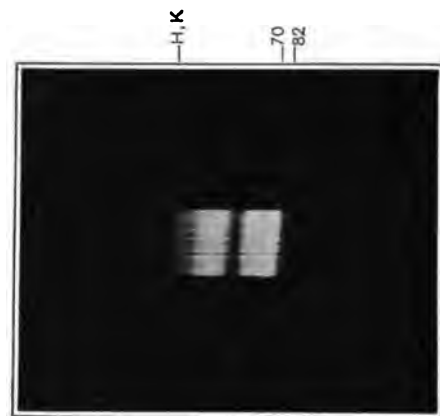


FIG. 7.— α Tauri. Class K5. Plate H1785



FIG. 8.—H. R. 5589. Class Mb. Plate H1847
(498-2)

Stars of Class N

OBSERVATIONS WITH PRISM P

Description	U Hydrae		+38° 1539, H 1802	+14° 1283, H 1801	Identification
	H 1792	H 1798			
Absorption line.....	457		456 ^a		Swan band?
Absorption 464-474.....	pr	pr	pr	pr	Swan band
Rise.....	517				Do
Rise 564.....	pr	pr	pr	pr	Do
Absorption line 591.....	pr	pr	pr	pr	Cyanogen band?
Broad maximum.....	681	683	683	683	
Absorption line?.....		729	731		Do
Narrow absorption line.....			758		Probably A
Drop.....			786		
Narrow absorption line?.....			(802)		
Absorption line?.....	(80)	(91)	(90)		

OBSERVATIONS WITH PRISM D

Description	U Hydrae, H 1812	D M +38° 1539		D M +46° 817, H 1827	Identification
		H 1811	H 1833		
Rise.....			5597		Detached portion of Swan band
Do.....	5845		5845	5830	Swan band
Drop.....			5730		Cyanogen band?
Rise.....		5805	5810		
Absorption line.....	5876		5809		Do
Maximum.....			687	687	
Drop.....	691	691	693 ^a	692	Do
Do.....	707	708	709 ^a	707	Do
Do.....	722	724			Do
A.....	pr	pr	pr		
End.....	79+		821	795	

Low dispersion—Prism P.—At 682 there is a strong broad maximum in the continuous spectrum. It rises so abruptly on the side of shorter wave length, at about 67, as to give the impression that an absorption band terminates here, but the higher dispersion does not bear out that interpretation. This maximum may arise from a combination of the intensity curve of the star and of the dispersion of the prism, which condenses a considerable amount of energy in a comparatively narrow space. The change with wave length of the sensitivity of the emulsion of course adds its effect. These considerations will be briefly discussed later in this paper.

The absorption line measured at 73, as shown by the longer spectra, is in reality an absorption band with its head at 72.

Beyond 80 the spectra are very weak—in fact almost invisible—but an indication of an absorption line at a great wave length, approximately 90, was measured on three plates. The reality of this line on any one plate is so doubtful that the measurements would not be recorded here except for the circumstance that all three yield the same wave length as closely as could be expected in the case of an actual line. This, however, does not prove the existence of the line (or band), which must be left to future observations to decide. If the line is real, it should be clearly shown by effective exposures two or three times as long as those received by these plates. Such exposures are not out of the question.

The stars Y Hydrae and D M + 61° 667 appear with small dispersion to be similar to U Hydrae.

High dispersion—Prism D.—Very little is seen in the ordinary photographic region in these high-dispersion spectra owing to under-exposure, and the other portions are not so strong as is desirable. Nevertheless, several new features are clearly indicated. The most important of these are the absorption bands beginning at 692, 708, 723 and degenerating toward the red. They face in the opposite direction from the strong bands of shorter wave length characteristic of Class N. Several of the new features seem to correspond with cyanogen bands as listed by Kayser,²³ and since other cyanogen bands have previously been found in similar stars²⁴ this identification is suggested for them. Unfortunately, this identification is little more than formal, as we are not certain of the origin of these bands. Certain “cyanogen” bands have been observed in the spectrum of a copper arc burning in nitrogen.²⁵ The writer is not informed as to whether or not this applies to all of what is ordinarily called the cyanogen spectrum, but we can no longer accept the designation cyanogen at its face value. If we reject the connection between “cyanogen” flutings and a carbon source, it seems a remarkable thing that two sets of bands ordinarily associated with carbon, namely, the Swan spectrum and the cyanogen spectrum should appear together in a stellar source, as they do in the spectrum of a carbon arc. It is impossible, however, to press this argument to a conclusion because of incomplete data. The need for further laboratory research upon banded spectra, particu-

²³ Handbuch der Spectroscopie, 5, p. 228.

²⁴ Hale, Ellerman, and Parkhurst, Publications of the Yerkes Observatory, 2, p. 115; 1903.

²⁵ Uhler and Patterson, Astrophysical Journal, 42, p. 435, 1915; also Grotian and Runge, Physikalische Zeitschrift, 15, p. 545, 1914.

larly in the red, is emphasized; and it is not unlikely that stellar spectra of Class N, when more completely observed, will be of value to the laboratory physicist.

Y Hydrae and D M + 68° 617, as judged by one underexposed plate of each, possess spectra which seem to be essentially similar to U Hydrae and D M + 38° 1539.

Stars of Class R.—Two spectra of Class R, D M - 24° 12084 and D M + 42° 2811, were photographed. The plates are not of the best quality, the details not being well shown, but the great departure from Class N in the matter of spectral distribution of energy is evident at a glance. In the Class R spectra the blue and red portions are of approximately equal intensity, more or less as in Class K, and decidedly at variance with Class N. Taking the central portion of the spectrum, say about 48, as a standard of comparison, the violet rays are stronger and the red rays weaker than in Class N. These observations are in harmony with and extend the Harvard descriptions of Class R and support in an interesting way the analogy between the sequences *G*, *K*, *M*, and *G*, *R*, *N*, as set forth by Dr. Rufus.²⁶ The spectral distribution of intensity in stars of various classes will be further discussed later in this paper.

III. GENERAL DISCUSSION

1. IDENTIFICATION OF BAND AT WAVE LENGTH .760 μ

As previously described, an absorption band with its head at $\lambda 760$, about coincident with Fraunhofer A, and extending toward longer wave lengths, has been observed in spectra belonging to various subdivisions of class M. This band appears to increase in strength with advancing spectral type. Since it occurs in spectra in which other bands, also heading toward shorter wave lengths, are known to be due to titanium oxide, it was natural to suppose that this might have the same origin. There are several published descriptions and illustrations of the titanium oxide spectrum, but none that the writer has seen refers to wave lengths as great as 760. Hence this portion of the spectrum was photographed by him since returning to Washington. The arc formed between electrodes of metallic titanium was projected upon the slit of the spectroscope. The arc burned brilliantly at 2 or 3 amperes, but displayed an annoying tendency to wander about over the elec-

²⁶ Publications of the Astronomical Observatory of the University of Michigan, 2, p. 103; 1916.

trodes.²⁷ Since the banded spectrum is stronger in the outer envelope of the arc an attempt was made to observe this part of it, but the position of the image of the arc upon the slit could not be satisfactorily controlled for reasons just referred to. Nevertheless, photographs were obtained in which the flutings stand out quite prominently, and as was anticipated, a band at about $\lambda 760$ is present. An illustration (Fig. 10) shows this new band as it appears in a one-prism spectroscope. Photographs with a 21-foot grating (Fig. 11) show that it is multiple, the first head being near $\lambda 7590$ with others toward longer waves following at intervals of 30 to 40 angstroms, one of the most pronounced being at $\lambda 766$. Traces of the multiple character are seen on a few of the stellar spectrograms.

2. DISTRIBUTION OF ENERGY IN STELLAR SPECTRA

When solar or stellar spectra are photographed with a prism spectroscope on red sensitive plates there is usually found to be a decided maximum of intensity at some point in the red. With pinacyanol it occurs at $\lambda 66-67$, with dicyanin near $\lambda 70$, but this does not necessarily imply real maxima either in the stellar spectra or in the sensitizing power of the dyes. The opacity of the photographic emulsion at different points in a spectrum depends upon many factors, the three most important being (1) spectral intensity of the source (energy per angstrom); (2) dispersion of the spectroscope (angstroms per millimeter); and (3) sensitivity of the photographic emulsion (darkening per unit of incident energy). Neglecting many complications, the ordinates of the opacity curve are found by multiplying together the corresponding ordinates of the curves (1), (2), and (3). Now, even though the intensity and the sensitivity steadily decrease toward the red, it may happen that at a certain point the condensation of energy due to the dispersion curve will cause an increase in the photographic action, while a little farther along the spectrum this is overbalanced again by smaller values of intensity and sensitivity. These considerations may explain certain maxima so commonly shown in stellar spectra, particularly in the red. If so, no great importance is to be attached to them.

While actual spectral-energy curves can not be drawn from the data at hand, the great differences between those of various types

²⁷ Bars or rods of metallic titanium were not obtainable at this time, hence it was necessary to use pieces of irregular shape.



FIG. 9.— $D M + 38^{\circ} 1539$. Class N. Plate H1802



FIG. 10.—Titanium arc; one-prism spectrograph
(502-1)

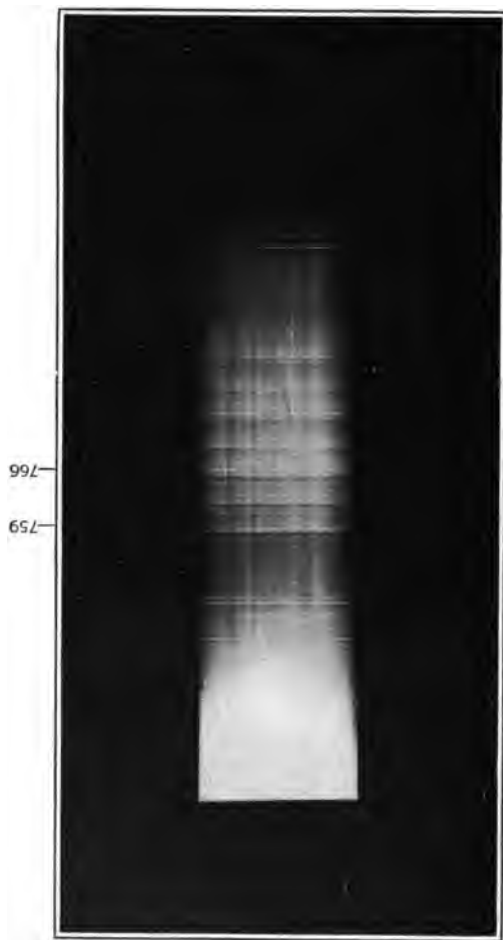


FIG. 11.—Titanium arc; grating spectrograph

of stars have been emphasized by the photographs taken during this investigation.

Nearly all stellar spectra taken with moderate exposures on dicyanin plates appear in two parts, there being a gap in the yellow-green, due to the insensitiveness of the emulsion. It is instructive to observe the steady shift of energy from the blue to the red side of this gap in passing from early to late types of spectra. For classes B and A the blue portion is much the stronger; at class K the two parts are about equal; while for classes M and N the red portion is stronger. The great contrast in energy distribution in the different classes is clearly shown in the illustrations. It has been determined, roughly, in a numerical way by estimating the relative amounts of photographic action in the two portions of nearly 100 spectra appearing on the plates of small dispersion. The table shows the ratio of the red to the blue portion. At my request Dr. Burns made independent estimates of the same stars. The two series agree well where the ratio is near unity, and in other cases do not differ more than might be expected from the uncertainties in estimates of this kind.

Ratio of Red to Blue Light—Average Values

Class	Burns	Merrill
B, A.....	0.1	0.2
F, G.....	.4	.5
K.....	1.2	1.2
M.....	1.3	1.3
N.....	4.1	3.6
R.....	.9	.8

It is somewhat surprising to find so small a difference between Classes K and M. This does not necessarily contradict the belief that stars of Class M are considerably redder than those of Class K, for these spectra refer in part to waves too long to affect the eye, and it is probably at great wave lengths that the strong absorption bands of titanium oxide partly compensate the increasing redness.

The ratio B and A to N is 41 according to Burns's estimates and 18 according to the writer's,²⁸ corresponding to magnitude differences of 4.0 and 3.1, respectively. This would undoubtedly be considerably increased if we could compare wave lengths shorter than 40 with those longer than 65.

²⁸ Burns's estimates are evidently on a more extended scale than the writer's and probably more nearly correspond to the actual energy ratios.

Only two Class R stars are included, but the results are of special interest. The estimates are as follows:

Ratio of Red to Blue Light—Class R

	Burns	Merrill
—24° 12084.....	0.95	1.0
+42° 2811.....	.85	.7
Mean.....	.90	.85

These numbers show in a definite way the wide divergence from Class N and the similarity to Class K, as referred to previously in this article. The correspondence of these two stars is apparently to an early rather than to a late subdivision of Class K.

IV. SUMMARY

1. Fraunhofer's A band (wave length 0.760μ) and a considerable region of greater wave length have been photographed in numerous stellar spectra by means of plates sensitized with dicyanin.

2. A strong band at 0.760μ , nearly coincident with A, has been discovered in spectra of Class M. It is very marked in Mb and Md spectra, and may be useful for purposes of classification.

3. The titanium arc shows flutings near this wave length, which probably correspond to the stellar bands.

4. In spectra of Class N, bands at 0.692μ , 0.708μ (0.723μ) have been observed, which may be identical in origin with bands in the so-called cyanogen spectrum.

5. On dicyanin plates, spectra of Class R are strikingly dissimilar to those of Class N in spectral distribution of energy.

6. The greatest wave lengths observed appear by extrapolation to be about 0.87μ .

7. General conclusions may be stated as follows: (a) Many stellar spectra possess sufficient intensity in the region of wave length 0.80μ (infra-red) to enable this portion of the spectrum to be photographed on plates sensitized with dicyanin; (b) in favorable instances stellar spectra can be well observed to 0.85μ , or possibly to even greater wave lengths; (c) the region of stellar spectra of greater wave length than 0.70μ contains features of importance, especially in the case of the later types.

Acknowledgment is hereby made of the courtesy of Director Pickering, of the Harvard College Observatory, in making this investigation possible, and of the aid rendered by him, by Prof. King, Miss Cannon, and other members of the staff during the progress of the observations.

WASHINGTON, April 12, 1917.

INSTRUMENTS AND METHODS USED IN RADIOMETRY, III

THE PHOTO-ELECTRIC CELL AND OTHER SELECTIVE RADIOMETERS

By W. W. Coblenz

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I. INTRODUCTORY STATEMENT

During the past 10 years investigations have been in progress in this Bureau to determine the characteristics, comparative sensitivity, and applicability of various types of instruments for measuring radiant energy.

The various types of instruments which may be used for measuring radiant energy may be divided into three groups.

Group 1 consists of radiometers which are quite nonselective in their response to stimuli of radiant energy of all wave lengths yet measured, extending from the extreme ultra-violet (Schumann region $\lambda = 0.1\mu$) to the longest infra-red rays yet isolated, viz, $\lambda = 340\mu$ ($\mu = 0.001$ mm).

In these instruments the radiant energy is absorbed by a blackened receiver and converted into heat.

To this group of radiation meters belong (1) the Nichols radiometer, which functions as the result of gas repulsion against a blackened vane, which is suspended in a partial vacuum and exposed to radiation; (2) the thermocouple (and auxiliary galvanometer) and the Boys radiomicrometer (which consists of a thermocouple attached to the suspended coil of a d'Arsonval galvanometer), which function as the result of generation of an electric current by heating the juncture of the two metals by absorption of radiant energy; and (3) the Langley bolometer, which functions as a result of a change in electrical resistance of a very thin, blackened strip of metal, caused by the heat produced by absorbing radiant energy, to which it is exposed in the form of an arm of a Wheatstone bridge.

This group includes also calorimetric devices. Other devices such as expansion of a gas or metal due to warming by absorption of radiant energy are not sufficiently sensitive to be considered.

There is no marked difference in the limiting sensitivity of these instruments. It is primarily a question of adaptability and speed. If it requires two seconds to obtain a galvanometer reading when using a bolometer, it will require three to four seconds when using a thermopile and one to two minutes, or even as high as five minutes, when using a Nichols radiometer.

By marked difference in sensitivity is meant a factor of 10 to 100. The factor, 2, counts but little when we are interested in measuring the radiations from an eighth to tenth magnitude star, or the light reflected from one of Jupiter's satellites.

In previous communications attention was directed principally to these nonselective instruments—that is, instruments which function independently of the (frequency) wave length of the stimulus—because they have a wider application than instruments which respond only to certain frequencies, whether “visible” or “ultra-violet” radiations.

Group II includes substances which have the property of decreasing in electrical resistance when exposed to radiant energy of short wave lengths, especially the visible and ultra-violet rays. The character of this phenomenon depends entirely upon the wave length (frequency) of the radiant energy stimulus. It does not depend upon thermal conditions; that is, it does not depend upon the amount of radiant energy absorbed and the consequent rise in temperature of the substance exposed to radiation. In fact,

at least some of these substances undergo the greatest change in resistance (are the most sensitive) for radiations of wave lengths which are the least absorbed. The resistance of selenium increases with decrease in temperature, which is opposite to the effect produced by "light" rays. For example, experiments show that a selenium cell having a resistance of 1 000 000 ohms at 20° C increases to three times that value (3 000 000 ohms) at 0° C. When such a cell was exposed to diffuse daylight, its resistance decreased to one-tenth to one-fiftieth of its value when unexposed to light.

Many substances are "light sensitive," for example, copper oxide, sulphides of antimony, silver, etc. The best-known example is crystalline selenium, which will be discussed presently.

Group III includes substances which, when charged to a negative potential (in an evacuated chamber), lose their charge when exposed to light, especially violet and ultra-violet rays. When used in this manner they are commonly called "photoelectric cells." The emission of electrons is a surface phenomenon which is easily affected by oxidation of the surface.

The effect produced by exposing the electrode to light does not disappear instantaneously. It is claimed that if light is applied to the cell without a potential difference, and the light is then interrupted and the field is applied, there is found a considerable photoelectric current.

The photoelectric cell seems to become fatigued and, unless it is of special construction, its response is not directly proportional to the intensity of the stimulus; but differing from the selenium cell, the writer has not found this lack of proportionality of response to depend upon the wave length of the exciting light.¹

Many substances (not necessarily metal) are sensitive photoelectrically. Silver iodide is said to be ten times as sensitive as pure aluminum.

The sensitivity of potassium is increased one hundred times by changing the material into a hydride.

For the alkali metals the wave length of maximum sensitivity shifts to the long wave lengths with increase in molecular weight; for Na, $\lambda_m = 0.4\mu$; for K, $\lambda_m = 0.436\mu$; for Rb, $\lambda_m = 0.5\mu$.² In the nonselective radiometers the high local sensitivity of the

¹ In one of the tests on a photoelectric cell of potassium it was found that on doubling the intensity of the light the correction for lack of proportionality of response (galvanometer deflection) was 10 per cent, while for an increase in intensity of ten times the original value this correction was 15.3 per cent.

² Ladenburg, *Ann. der Phys.* (4), 12, p. 558; 1903. For a complete summary, see Chr. Ries, *Das Licht in Seiner Elektrischen und Magnetischen Wirkungen*, Leipzig, 1909.

selective instruments seems to be spread over the whole spectrum, with a corresponding reduction to a uniform and much lower value.

In previous communications³ the characteristics and comparative sensitiveness of the various types of nonselective radiometers (Group I) were described.

In the first communication⁴ attention was called to the fact that, at that time, but little was known concerning the selective radiometers, and that their application was very limited. In the meantime extensive researches by various observers have thoroughly proven that the selenium cell, in spite of its very high sensitivity, is useless as an instrument for precise measurements.

On the other hand, the photoelectric cell, Group III, is being developed into a precision instrument which will be useful for special kinds of measurements in the region of the spectrum extending from the yellow throughout the extreme ultra-violet. In view of the fact, as will be shown presently, that in the violet and ultra-violet, a photoelectric cell of potassium hydride is more sensitive than a thermopile, it can be used to supplement the latter in certain radiometric investigations; for example, in transmission and reflection measurements.

There seems to be considerable misconception concerning the functioning of selective and nonselective radiometers. Hence, before discussing the various types of selective radiometers in detail, it is relevant to add a few comments on the physical conditions which control the selectivity and nonselectivity, also the proportionality of the response, when the radiometer is irradiated⁵ by radiant energy of different wave lengths.

A radiometer, giving responses which are directly proportional to the incident radiation is not necessarily equally sensitive to all wave lengths.⁶ The thermopile is a typical example. The surface of the receiver is equally sensitive to all wave lengths because it is covered with lampblack, which has the property of absorbing equally (within experimental errors of observation) all wave lengths. If the thermopile receiver were covered with chromium oxide, it would absorb 97 per cent in the visible spectrum and 85

³ Instruments and Methods, I, this Bulletin, 4, p. 392, 1908; Instruments and Methods, II, this Bulletin, 9, p. 7, 1911; Various Modifications of Thermopiles, this Bulletin, 11, p. 132, 1914; and Energy Measurements in Absolute Value, this Bulletin, 12, p. 504, 1916.

⁴ This Bulletin, 4, p. 454, 1908; 9, p. 45, 1912.

⁵ It is logical to use the term "irradiate" instead of "illuminate" in view of the fact that both visible and invisible radiant power is being discussed. For a logical use of the term "irradiate" in connection with X rays and crystal structure, see a paper by W. H. Bragg, *Phil. Mag.*, 27, p. 835, 1914.

⁶ In a recent paper on radiometric apparatus for use in psychological optics (Ferre and Rand, *Psychological Monograph No. 103*, p. 1, Princeton University Press) the statement is made that an instrument giving direct proportionality of response is equally sensitive to all wave lengths.

to 95 per cent, depending upon the wave length, in the infra-red.⁷ If the surface of the receiver were covered with aluminum oxide, it would absorb only 15 per cent in the visible spectrum, while at 8.8μ it would be as "black" as lampblack, absorbing 98 per cent of the incident radiation. Evidently the question of equality of sensitivity for different wave lengths depends entirely on the kind of material used as an absorbing surface of the thermopile (or bolometer) receiver.

The response of the thermopile is directly proportional to the intensity of the incident radiation because, for the small rise in temperature involved, the thermal emf generated is directly proportional to the heat generated, which in turn is directly proportional to the energy absorbed. However, in order to utilize this direct proportionality of response of the thermocouple it is necessary to design the galvanometer so that its deflection is directly proportional to the current passing through the circuit.

On the other hand, the distinguishing characteristics of the photoelectric cell is its selective sensitivity to various wave lengths, being the most sensitive to the ultra-violet. Thus far it has not been possible to modify appreciably this inequality of sensitivity for different parts of the spectrum. However, while most photoelectric cells do not give responses which are directly proportional to the intensity of the incident radiation, it is possible to design the cell so that it gives direct proportionality of response.

In concluding this part of the discussion of selective radiometers it is relevant to add that the eye is one of the most sensitive detectors of radiant energy. The unaided eye can detect a seventh to eighth magnitude star. It required a 3-foot reflecting telescope to detect a seventh-magnitude star by means of its total radiation. On the basis of the "light" emitted from a star, assuming a luminous efficiency of 20 per cent, it would require a 7-foot telescope combined with a stellar thermocouple in order to attain the sensitivity of the human eye. Calculations show that the eye responds to light having an intensity less than 1×10^{-9} erg—a value so small that it would require 60 million years to raise 1 g of water 1° C. Since the thermocouple (or bolometer) is non-selective, it gives an accurate register of the radiation from an incandescent body; as, for example, a star. The stars that appear the brightest to the eye do not necessarily appear the "brightest" radiometrically, because they may have dark companions. In fact, a very useful application of the thermocouple will be a search

⁷ This Bulletin, 9, p. 283; 1913.

for binary stars having companion stars which are not sufficiently luminous to be visible to the eye.

II. THE PHOTOELECTRIC CELL

Recent advances in the construction of photoelectric cells, especially the highly sensitive cells of Kunz, in which the active material is potassium hydride, seem to warrant their use as a precision radiometer for a limited class of investigations, such as, for example, transmission and reflection spectra in that part of the spectrum extending from the yellowish green into the extreme ultra-violet. Such investigations have already been made.⁸ In stellar photometry important results have been obtained by Stebbins⁹ in the measurement of the variation in brightness of variable stars, while numerous other applications are being made. However, before undertaking such applications the cell must be thoroughly tested for direct proportionality between the intensity of the incident light and the resulting photoelectric current. This

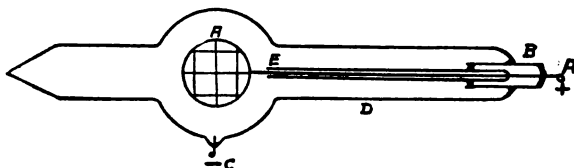


FIG. 1.—Photoelectric cell (Kunz)

may be accomplished by changing the distance of the light, by using Nicol prisms, or by means of a sectored disk, as described elsewhere in this paper.

In stellar radiometry its greatest usefulness will be in studying variable stars which do not change in color.

(a) *Description of Cell.*—Various designs of photoelectric cells have been suggested. Kunz¹⁰ has described the latest developments which appear to give a cell which is quite free from "dark currents"; that is, leakage currents flowing when the cell is not exposed to light. The design is illustrated in Fig. 1. It may be constructed of quartz or of glass or of glass with a quartz window. The central bulb is about 3.5 cm in diameter. The cathode, C, is of small platinum wire, and this part of the bulb is silvered to

⁸ Hulburt, *Astrophys. Jour.*, 42, p. 205, 1915; and Nathanson, *Phys. Rev.*, 7, p. 403, 1916.

It is beyond the scope of this paper to give a complete history of the development of the photoelectric cell. The recent developments are easily accessible, and the early work is described by Ries, "Das Licht in Seiner Elektrischen Und Magnetischen Wirkungen," Leipzig, 1909.

⁹ Stebbins, *Bull. Lick Observatory*, 8, pp. 186 and 192; 1916.

¹⁰ Kunz, *Phys. Rev.*, 7, p. 62, 1916; *Astrophys. Jour.*, 46, p. 69, 1917.

insure good contact with the platinum and the glass. The anode, *A*, is a ring of platinum about 2 cm in diameter. Fine silver or platinum wire is stretched across this ring, thus forming a net which produces a more uniform electric field. The anode is surrounded by a long glass tube to prevent leakage. The use of a special nonconducting glass for this part of the tube has been suggested. It appears feasible to make the supporting tube, *E*, of quartz (attaching it to the glass by means of the glass fluxes now obtainable), which would reduce the leakage over the inner surface. For the most refined work in stellar photometry, and for measuring weak radiations, the anode, *A*, is surrounded by a platinum cylinder, *B*, by means of which the surface and the electrolytic currents are lead to earth. The dark currents may be further suppressed by wrapping tin foil around the cylindrical part, *D*, and the cathode, *C*; and it can be compensated by the application of a lower potential ¹¹ at the platinum cylinder, *B*, as described elsewhere in this paper.

(b) *Preparation of Cell.*—The manner of construction of a photoelectric cell is intricate, and readers are referred to original sources for details.¹² The cells most commonly used are made of potassium hydride or of pure sodium.

In order to produce good contact ¹³ at the cathode a layer of silver is deposited on and around the platinum terminals on the inside of the bulb, and the potassium is distilled upon the silver which is kept cool by ice or cold water. The cylindrical part of the tube, *D*, Fig. 1, is heated (160° to 240° C, according to the alkali metal used), by means of a heating coil while the distillation is in progress. Hydrogen is introduced by heating a strip of palladium in a side tube. The potassium hydride is formed (by Schulz, loc. cit.) by applying a potential of about 550 volts direct current to the cell (Kunz, loc. cit., mentions 280 volts for rubidium), the potassium electrode being negative, with about 3000 ohms resistance in series with the cell. On closing the circuit for a few seconds the potassium assumes a brilliant violet-blue color. The circuit is then broken and the hydrogen pumped out. Helium or argon, free from oxygen, is then introduced, the pressure being regulated to give the maximum galvanometer deflection when the cell is exposed to light.

¹¹ Nathanson, *Astrophys. Jour.*, 44, p. 137; 1916.

¹² The pioneers in this work are Elster and Geitel, *Ann. der Physik*, 48, p. 225, 1891; 48, pp. 338 and 625, 1893; *Phys. ZS.*, 10, p. 457, 1909; 11, pp. 257 and 1082, 1910; 18, p. 468, 1912 (formation of hydride of alkali metals); 14, p. 741, 1913.

¹³ Schulz, *Astrophys. Jour.*, 88, p. 187; 1913. Instead of the platinum cylinder, *B*, Fig. 1, a coating of silver, touching a sealed-in platinum wire, is used; *Astrophys. Jour.*, 46, p. 241; 1917.

The successful distillation of the alkali metal, for example, potassium) into the bulb has been described by Ives¹⁴ and by Hulburt,¹⁵ who prepared sodium cells.

The preparation of cells of caesium and rubidium from the chlorides of these metals has been described by Cornelius.¹⁶

The conditions controlling the sensitivity of the photoelectric cells with alkali metals in hydrogen have been investigated by Kemp.¹⁷ He varied the conditions as regards gas pressure, electrode distance, potential difference applied to the electrodes, area illuminated, intensity of illumination, and temperature of the cell. Kemp found that the sensitivity was increased more than 100 times by formation of the hydride. The sensitivity did not appear to vary much with temperature. For hydrogen he found a pressure of 2 to 3 mm. of mercury to produce the highest sensitivity, the optimum electrode distance being 5 to 6 mm.

One objectionable characteristic of a photoelectric cell is its lack of direct proportionality of response (photoelectric current) with variation in intensity of illumination. Conflicting reports have been published; some experimenters having cells which gave direct proportionality while others did not find proportionality of response.

Kunz¹⁸ seems to have overcome this difficulty to some extent, and seems to be able to make cells which are quite reproducible in their characteristics. This is accomplished by distilling the potassium over the whole bulb, except a small opening 5 to 8 mm. in diameter, for admitting radiation into the bulb. A further improvement is in producing a more uniform electric field by using a net of wire instead of the single loop of platinum previously used. In this manner he is able to operate a spherical form of cell so that, for low illuminations, it gives exact proportionality of photoelectric current for a variation of 4 to perhaps 10 times the intensity of illumination. Using a cylindrical cell, with parallel electrodes, the current was found to be proportional to the illumination, for a variation of 100 times in intensity if the cell was filled with argon.

Recently he has described experiments on amplification of the photoelectric current by means of the audion amplifier.¹⁹

¹⁴ Ives, *Astrophys. Jour.*, **39**, p. 428; 1914.

¹⁵ Hulburt, *Astrophys. Jour.*, **41**, p. 400; 1915.

¹⁶ Cornelius, *Phys. Rev.*, **1**, p. 16; 1913.

¹⁷ Kemp, *Phys. Rev. (2)*, **1**, p. 274; 1913. See also paper by Ives, *Astrophys. Jour.*, **39**, p. 428; 1914.

¹⁸ Kunz, *Astrophys. Jour.*, **45**, p. 69; 1917.

¹⁹ Kunz, *Phys. Rev. (2)*, **10**, p. 205; 1917.

An outstanding difficulty that needs investigation is the variation (decrease) in sensitivity with age. While this does not appear to be serious, it is nevertheless important to determine whether it is due to release of gases from the walls of the cell, and whether the cell can be aged at the time of construction.

(c) *Voltage-Current Characteristics*.—It has just been noticed that the sensitivity of a photoelectric cell is a complicated function of gas pressure, electrode distance, etc. Under the present caption it is of interest to describe the electrical characteristics of the cell.

Applying a potential difference to the electrodes of an unilluminated photoelectric cell it is found that the voltage can be increased to a certain definite maximum value before ionization occurs. If this voltage is exceeded by an amount which is barely perceptible on a voltmeter, a deflection of the galvanometer results, indicating a flow of current through the cell. This is the critical voltage (and critical current) above which the cell can not be operated without introducing a permanent galvanometer deflection. This is to be avoided, as is also the "dark current" which is due to leakage over the cell. This leakage can be annulled as described under "Methods of operation."

Photoelectric cells, which are purchased, should be accompanied by a statement concerning the maximum voltage that can be applied. Even then the cell should be operated with caution. For example, a certain cell which was labeled "maximum 150 volts" could not be operated above 80 to 90 volts on account of excessive leakage. It would be of interest to determine whether this unstable condition varies with the age of the cell.

(d) *Voltage-Radiation Sensitivity*.—The variation of the radiation sensitivity of a potassium-hydride photoelectric cell with the voltage applied to the electrodes is shown in Fig. 2. In making this test the cell was exposed to light from a monochromatic illuminator. The intensity of the light was kept constant while the voltage which was applied to the cell was varied. The critical voltage of this cell was about 230 volts. It is of interest to note the rapid change in radiation sensitivity near the critical voltage. Hence, if high sensitivity is not required, it is advisable to use a lower voltage since the effects of a slight variation in voltage will be less perceptible at a lower voltage.

(e) *Spectral-Radiation Sensitivity*.—The highly selective character of the spectral-radiation sensitivity (the wave-length

sensitivity) of the photoelectric cell is shown in Fig. 3, which gives the apparent distribution of energy in the visible spectrum of a gas-filled tungsten lamp as observed, curve *A*, with a photoelectric cell, and the true energy distribution as observed, curve *B*, with a thermopile which is a nonselective radiometer. An ironclad Thomson galvanometer was used with both of these instruments.

From these curves it will be noticed that the photoelectric cell is not sensitive to the deep red. In the blue it is very sensitive, so that as a radiometer in this region of the spectrum it surpasses the thermopile or bolometer in sensitivity. Curve *C* gives the true sensitivity of the photoelectric cell of potassium hydride resulting from stimulating the cell with equal energies throughout the spectrum. It is the ratio of curve *A* to curve *B*, in view of

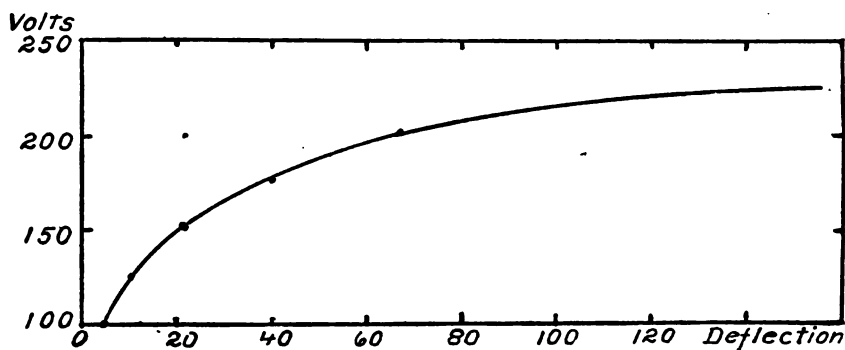


FIG. 2.—Variation in radiation sensitivity of a photoelectric cell with change in applied voltage

the fact that for this range of intensities the photoelectric current is proportional to the energy stimulus; or it is the wave-length sensitivity curve.

As already mentioned, the maximum sensitivity shifts toward the short wave lengths with decrease in atomic weight of the alkali metal. The data published by Ives²⁰ show a great variation in the wave-length sensitivity curves of potassium cells. In Fig. 4, curve *A* is shown the wave length, photoelectric sensitivity of calcium²¹ and curve *B* of rubidium,²² which has its maximum sensitivity at 0.508μ . The visibility curve of the average eye²³ (125 observers) is illustrated in curve *C*. Con-

²⁰ Ives, *Astrophys. Jour.*, 40, p. 182; 1914. In a recent paper (*Astrophys. Jour.*, 46, p. 241, 1917) Ives shows that the spectral sensitivity changes with time, becoming relatively more sensitive in the red.

²¹ Pohl and Pringsheim, *Verh. Phys. Gesell.*, 15, p. 111; 1913.

²² Braun, *Dissertation*, Bonn; 1906.

²³ This Bulletin, 14, p. 168; 1917.

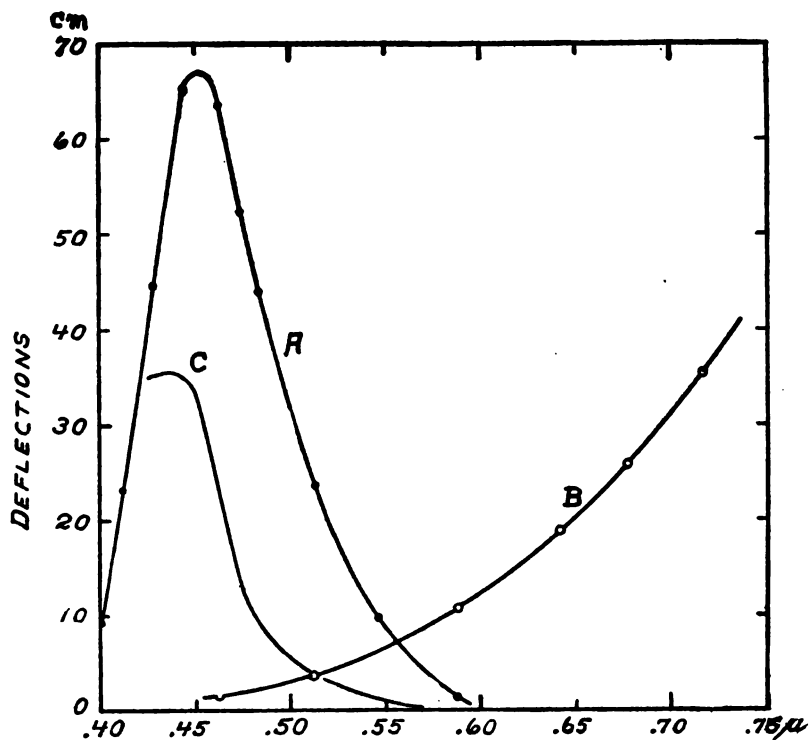


FIG. 3.—Distribution of energy of a gas-filled tungsten lamp as observed with: A, photoelectric cell; and B, with a thermopile. Curve C is the true sensitivity of the photoelectric cell for an equal energy spectrum.

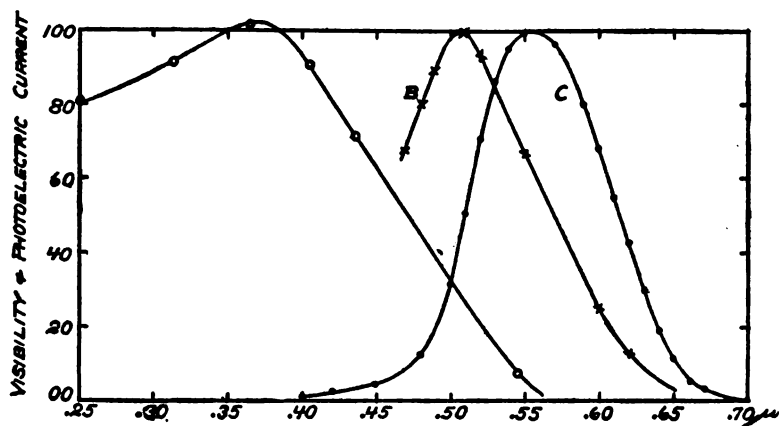


FIG. 4.—Photoelectric sensitivity: A=calcium; B=rubidium. Sensibility of the average eye: C

trary to published statements,²⁴ there is no marked similarity between the photoelectric sensitivity curves of these metals and the wave-length sensibility curve of the eye.

(f) *The Auxiliary Galvanometer.*—At best a sensitive electrometer is slow to act, and it is difficult to operate, on account of leakage, etc. Attention has already been called²⁵ to the usefulness of a high-resistance ironclad Thomson galvanometer to measure photoelectric currents. There is, of course, nothing new in the use of a high-resistance galvanometer in a circuit of this type which has a very high external resistance. However, an examination of the older data²⁶ concerning high-resistance galvanometers (up to 300 000 ohms) shows that the sensitivity obtained is far less than is attainable at the present day, by using even low-resistance galvanometers. During recent years high-resistance galvanometers have been used but little, the Einthoven string galvanometer being used instead.

It is, of course, possible to investigate (and precise work has already been done in investigating) the violet and ultra-violet spectrum by means of a thermopile and a suitable galvanometer. However, it requires an experienced observer to operate this form of radiometer. On the other hand, the photoelectric cell is not affected by thermal changes and stray infra-red rays. Hence, one need not wait until thermal conditions are constant, and the device can be operated by a less experienced observer.

In Fig. 3 is given the distribution of energy in the visible spectrum of a gas-filled tungsten lamp as observed with a photoelectric cell, curve *A*, and with a thermopile, curve *B*. These curves represent actual observations drawn to scale, showing what deflections one would obtain in practice when using these instruments under average conditions. From these curves it is evident that for investigating transmission spectra in the visible and the ultra-violet parts of the spectrum, the most efficient procedure is to use a thermopile for measurements in the red and yellow and a photoelectric cell for measurements in the blue and ultra-violet. The same galvanometer can be used with both instruments. For this purpose an ironclad galvanometer²⁷ of 20-ohm coils has all its coils joined in parallel (resistance = 5 ohms) when connected with the thermopile, and joined all in series (resistance = 80 ohms) when used with the photoelectric cell.

²⁴ Ferree and Rand, *Psychological Monograph* No. 103, p. 45; 1917.

²⁵ Coblenz, *Phys. Rev.*, 10, p. 97; 1917.

²⁶ Ayrton, Mather, and Sumpner, *Phil. Mag.*, 80, p. 58; 1890.

²⁷ Coblenz, *this Bulletin*, 18, p. 423; 1916.

With such a galvanometer (having its coils all in series) a current sensitivity of $i = 1 \times 10^{-10}$ ampere is easily attained on a single swing of only two seconds.²⁸ Tests were made upon a two-coil instrument having a total resistance of 1300 ohms. Using a single swing of only two seconds and scale at 2 m the current sensitivity was $i = 2.7 \times 10^{-11}$ ampere. A four-coil instrument, having a total resistance of 5300 ohms and a heavy suspension under similar conditions, had a current sensitivity of $i = 6.2 \times 10^{-11}$ amperes, or 8×10^{-12} ampere for a resistance of 1 ohm.²⁹

From this it is evident that a current sensitivity of $i = 1 \times 10^{-12}$ ampere is easily attained, which from personal experience is far greater than will be required for spectral transmission and reflection investigations in the blue, violet, and ultra-violet.

A test was made of the radiation sensitivity of a potassium hydride cell (No. 113, made by Dr. Kunz) when combined with a low-resistance galvanometer (84 ohms) and the above-mentioned high-resistance galvanometer of 5300 ohms. The test showed that when using the high-resistance instrument the radiation sensitivity was at least 10 times as great as when using the low-resistance galvanometer. The computed sensitivity was only 8 times that of the low-resistance galvanometer.

In these tests a 500-watt tungsten stereopticon lamp and also a Nernst glower were used as a source of light. The spectrum was produced by a short-focus, high-intensity illuminator,³⁰ having a light flint-glass prism and interchangeable plano-convex lenses of quartz and triple achromatic lenses of glass.

The arrangement of the apparatus when using a galvanometer for measuring the photoelectric current is shown in Fig. 5. In order to eliminate leakage currents in the circuit, Nathanson grounded the terminal of the galvanometer which ordinarily would be connected to the negative pole of the battery. The writer did not experience this difficulty when using a galvanometer sensitivity of $i = 1 \times 10^{-10}$ ampere, and hence did not resort to

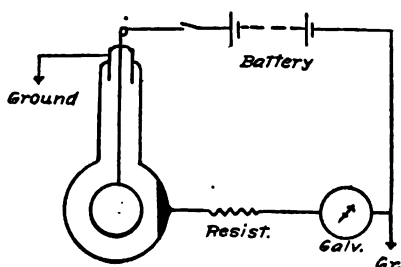


FIG. 5.—Connections of apparatus when using a galvanometer with a photoelectric cell

²⁸ This applies to a galvanometer which has been in use for two years without remagnetizing the needles. A freshly magnetized suspension would be twice as sensitive.

²⁹ These coils were wound upon mandrel No. 1, Fig. 1, this Bulletin, 18, p. 426. The coils were wound in two sections of equal resistance, using Nos. 40 and 38 enameled copper wire.

³⁰ This Bulletin, 7, p. 245, 1911; 18, pp. 358 and 360, 1916.

grounding. However, when using the high-resistance coils the highest current sensitivity would necessitate grounding the instrument.

(g) *The Auxiliary Electrometer.*—The electrometer most commonly used with a photoelectric cell is the Dolezalek³¹ instrument; Müllly's electrometer³² has also been used. A new quadrant electrometer by Compton similar to the Dolezalek instrument is just appearing on the market.³³

In order to shield the electrometer from leakage, static charges, etc., the whole apparatus including the photoelectric cell, is usually inclosed in a tight metal box and the moisture is absorbed by drying material; for example, calcium chloride or better still phosphorus pentoxide. The metal box is grounded.

The needle is usually charged to a potential of 80 to 100 volts.³⁴ One pair of quadrants is grounded. The other pair of quadrants,

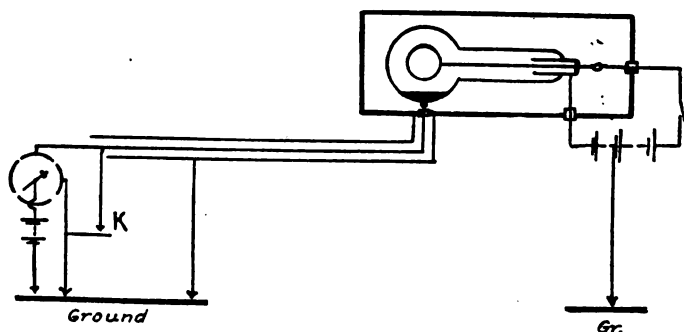


FIG. 6.—Arrangement of apparatus used by Nathanson

which can also be grounded, is connected to the cathode of the photoelectric cell.

Nathanson³⁵ having the electrometer and the photoelectric cell in separate boxes found it necessary to protect the connecting wires, Fig. 6, between these two instruments by passing them through a glass tube which was covered with tin foil and was well grounded. To overcome the drift of the needle of the electrometer

³¹ Made by the Cambridge Sci. Inst. Co., Cambridge, England; Dolezalek, Z. S. für Instrumentenkunde 21, p. 345, 1901.

³² Müllly, Phys. Zeitsch., 14, p. 237; 1913.

³³ Compton, Phys. Rev. (2), 7, p. 646; 1916. Made by the Pyroelectric Instrument Co., Trenton, N. J.

³⁴ The quartz-fiber suspension is from 6 to 9 μ thick; giving a free period of swing of 15 to 20 seconds, and a sensitivity of 300 to 3000 mm per volt difference in potential between the quadrants, depending upon the scale distance.

The quartz fibers are prepared by blowing or shooting (Boys London Electrician, p. 220, Dec. 11, 1896; see also Publication No. 65, p. 70; Carnegie Institution of Washington, 1906). They are rendered conducting by covering them with platinum, deposited by cathode disintegration (Williams, Phys. Rev., 4, p. 517, 1914; see also bibliography by Kohlschütter, Jahrb. Radioaktivität, 9, p. 355, 1912.)

³⁵ Nathanson, Astrophys. Jour., 44, p. 137; 1916.

due to the dark current or leakage current across the glass of the photoelectric cell, he put the metal ring (*B*, Fig. 1) at a potential of about 200 volts below that of the anode. In this manner the drift was completely eliminated.

(*h*) *High Resistance*.—In order to avoid injury to the photoelectric cell by the accidental passage of an excessive current which will result when an excessive voltage is applied to the cell, or when the cell is exposed to light of too great intensity, it is necessary to use a high resistance in series with the galvanometer or electrometer. Various forms of resistances have been used; for example, a solution of mannite with nonpolarizable electrodes,³⁶ alcohol in a capillary tube,³⁷ and various forms of carbon resistances.³⁸ The liquid resistances have not been so satisfactory as is desired, owing to polarization, evaporation, etc. The carbon resistances appear to be satisfactory if care is taken to have good contact at the electrodes. According to Kunz (*loc. cit.*), when the applied potential difference is greater than 1 volt deviations from Ohm's law are observed.

A carbon resistance is easily prepared by attaching terminals of copper wire, say No. 26, to the ends of a short piece of porcelain tubing (3 cm in length and 3 mm in diameter, such as is used for insulating thermocouples) by melting the wire in a blast lamp. A fine line of graphite (ordinary lead pencil) is then drawn upon the surface of the tube connecting the copper terminals, which are then painted with lampblack in shellac to insure good contact. The resistance is then adjusted by rubbing the pencil line. The whole is then placed in a glass tube about 5 cm long and 8 to 10 mm bore, one end of which is closed with cork, which is perforated to allow the copper-wire terminal to pass through. This tube is filled with melted paraffin and the ends sealed with sealing wax. In most of the resistance units constructed in this manner the resistance increased about four times on applying the molten paraffin. It is, of course, not necessary to embed the resistance in paraffin, but the latter acts as a protection from moisture and mechanical injury. A resistance of one to ten million ohms appears to be the most useful.

(*i*) *Source of Potential*.—A high voltage is easily provided by using a battery of dry cells. The cells used in the smallest flash lamps seem to deteriorate rapidly as a result of corrosion of the

³⁶ Pohl and Fringsheim, *Ber. d. Deutsch. Phys. Ges.*, 6, p. 174; 1913.

³⁷ Nichols and Merritt, *Phys. Rev.*, 84, p. 475; 1902.

³⁸ Stewart, *Phys. Rev.*, 26, p. 302, 1908; Ives, *Astrophys. Jour.*, 89, p. 428, 1914; Kunz, *Astrophys. Jour.*, 45, p. 69, 1917.

zinc. It is therefore more economical to use the medium-sized cells. These can be arranged in trays or covered boxes and secured by a thin layer of paraffin. They are connected by soldered wires. A battery of 30 cells consisting of 6 rows and 5 cells in a row and giving about 42 volts is a convenient unit to handle. By means of binding posts one can easily arrange to obtain 10 or 20 volts and smaller voltages may be obtained by clamping to the connecting wires.

When the current through the cell increases, there is an appreciable drop in potential at the electrodes of the cell. By means of an additional storage cell and a variable resistance Kunz (*loc. cit.*) maintained a constant potential as he varied the illumination and hence the current.

(j) *Methods of Operation.*—There are several methods for measuring the photoelectric current. In cases where the light intensities are fairly high, the current may be measured directly by means of the deflection of a sensitive galvanometer, provided the question of proportionality is considered. For this purpose the high-resistance ironclad Thomson galvanometer described on page 519 is applicable and is to be recommended.

A second method of operation is to use an electrometer or a sensitive galvanometer as a detector or indicator, and to balance the photoelectric current with a current which can be varied in a known manner."³⁹

A third method is to observe the rate of drift of an electrometer needle. Contrary to other observers, Ives ⁴⁰ found that the needle did not "move at a uniform rate."

A fourth method (which experimenters seem to prefer to the one just described) is the "ballistic throw method." In this method the photoelectric cell is exposed to light for a convenient length of time, say 10 seconds, and the charge acquired by the electrometer needle is noted. The "natural drift" of the needle is determined by noting the drift in 10 seconds when the cell is not exposed to light, and this is subtracted from the observed deflection.⁴¹ To accomplish this, Hulburt broke the ground connection *K*, Fig. 6, and, after conditions had become quiet, recorded the reading of the electrometer needle. He then exposed the cell to light for a definite period, say 10 seconds, and observed the throw of the electrometer needle. This is much quicker than waiting for the steady deflection.

³⁹ Griffith, *Phil. Mag.*, 14, p. 297, 1907; Richtmyer, *Phys. Rev.*, 29, pp. 71 and 204, 1909.

⁴⁰ Ives, *Astrophys. Jour.*, 89, p. 432; 1914.

⁴¹ Hulburt, *Astrophys. Jour.*, 42, p. 210; 1915.

Hulburt's grounding key was a pointed brass rod touching a small brass plate, made from the same piece of brass. It gave no trouble with contact difference of potential. Sulphur was used for supporting the wires leading to the electrometer.⁴²

Nichols and Merritt⁴³ have described a method of using the photoelectric cell, whereby the deflection rather than the rate of change of deflection is read. The electrical connections are shown in A, Fig. 7. The cathode of the photoelectric cell is connected to one pair of quadrants of an electrometer and also through a resistance, R (capillary tube of absolute alcohol with adjustable wire immersion), to earth. The anode (platinum wire) is connected to the plus side of a 110-volt battery. To balance the "dark current" of the cell they connected the other pair of quadrants through a variable resistance to the battery and to earth.

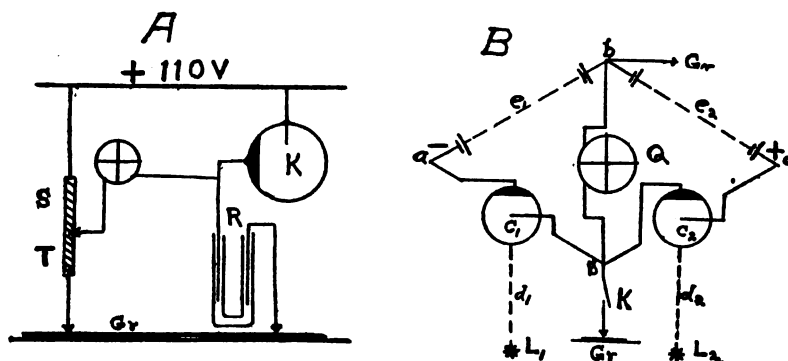


FIG. 7.—Arrangement of apparatus used by: A=Nichols and Merritt; B=Richtmyer

By adjusting S and T the two pairs of quadrants were brought to the same potential when the cell was unilluminated. The deflection of the electrometer upon illuminating the cell was a measure of the intensity of the incident light.

Richtmyer⁴⁴ has described a null method of making measurements with photoelectric cells and an electrometer. This evades the question of proportionality of response (photoelectric current) in the cell. The arrangement of the apparatus is shown in B, Fig. 7. This requires two photoelectric cells, C_1 and C_2 , and two sources of potential, e_1 and e_2 , which can be varied from 0 to 120 volts. The minus terminal of e_1 and the plus terminal of e_2 are connected in sort of a Wheatstone Bridge to the photoelectric

⁴² For further information concerning keys for connecting and disconnecting the photoelectric cell, the electrometer, and the known emf used for calibration purposes, see McClung, *Conductivity of Electricity Through Gases and Radioactivity*. See also papers by Cornelius, *Phys. Rev.*, 1, p. 16, 1913, and by Tugman, *Astrophys. Jour.*, 43, p. 321, 1915, for interesting applications of the photoelectric cell.

⁴³ Nichols and Merritt, *Phys. Rev.*, 34, p. 475; 1912.

⁴⁴ Richtmyer, *Phys. Rev.*, 6, p. 66; 1915.

cells. The points b and b' are connected to an electrometer, and are grounded; b' through a suitable make-and-break key, k . By properly choosing the relative values of e_1 and e_2 , the dark current through C_1 equals that through C_2 , which is determined by the reading of the electrometer when the key k is open.

On illuminating the cells C_1 and C_2 by sources L_1 and L_2 at the distances d_1 and d_2 , and adjusting these distances, the photoelectric currents can be made equal, which is indicated by the null reading of the electrometer. For various positions of L_1 there are corresponding positions of L_2 for a balance. By plotting these distances a calibration curve is easily established.

In a recent paper by Kunz⁴⁵ various arrangements of apparatus were used. In one case he used an illuminated photoelectric cell as a resistance. The system was found to be very sensitive, but requires great care in handling. In another arrangement, in which the rate of drift method of observation was used, the quadrant electrometer was replaced by a string electrometer in order to eliminate the inertia of the needle. Kunz states that this arrangement seems to be preferred in stellar photometry.

In investigations of transmission and reflection spectra, or in using the photoelectric cell as a photometer, when the source of radiation is of sufficient intensity, an equal-deflection method⁴⁶ may be employed instead of the null-deflection method. In this method the radiation passes through a pair of Nicol prisms, a sector disk of variable opening or an absorption wedge; or the source of radiation is moved upon an optical bench in order to vary the intensity. This intensity is measured after transmission through, or reflection from, the substance under investigation, by noting the galvanometer, or electrometer, deflection. On removing the substance from the path of the rays, the intensity of the direct radiation is reduced, by one of the above-mentioned devices, so that the deflection equals that just observed with the substance in place. The decrease in intensity is easily deduced from the constants of either one of these devices. In this manner the question of the proportionality of the photoelectric current with variation in intensity of illumination is easily evaded. It is easily determined whether the response is directly proportional to the stimulus by using crossed Nicols, a sector disk,⁴⁷ or a plane-parallel plate of colored glass, whose spectral transmission has been determined by a thermopile or some other accurate radiometer.

⁴⁵ Kunz, "The law of photoelectric photometry," *Astrophys. Jour.*, 45, p. 69; 1917.

⁴⁶ This Bulletin, 12, p. 504; 1916.

⁴⁷ Nathanson, *Astrophys Jour.*, 44, p. 137, 1916; Kunz, *Astrophys Jour.*, 45, p. 76, 1917.

After investigating at least a dozen photoelectric cells the writer has come to the conclusion that, if the deflection method is to be used in making the measurements, the proportionality test should be made upon every cell in the apparatus in which it is being used even when the maker's tests indicate that the cell functions in direct proportionality.

For example, a certain potassium-hydride photoelectric cell which was supposed to give direct proportionality was tested by determining the spectral transmission of a certain piece of colored glass by the direct-deflection method. Using a certain intensity of incident light the transmission at a certain wave length was 58.2 per cent. Reducing the intensity of the incident light (gas-filled tungsten lamp, variable current) to about one-half, the transmission was decreased to 55.9 per cent; and on reducing the intensity to one-fourth its initial value, the transmission was reduced to 53.3 per cent. By making corrections for lack of direct proportionality of deflection with intensity of illumination when using this photoelectric cell, the transmission curve so obtained coincided exactly with that observed by means of a thermopile. This cell happened to have a straight line proportionality of current with illumination.

Similar tests were made on other cells, in one of which the current (galvanometer deflection) increased more rapidly than the illumination. Using crossed Nicol prisms and the equal-deflection method, at a certain wave length in the spectrum the rotation of the Nicol prism, N_2 ⁴⁸ (when the test glass was not in the path of the rays), was 50.3° , corresponding to a transmission of 59.2 per cent. The direct-deflection method gave a transmission of 61.7 per cent. When corrected for lack of proportionality of current with illumination, the true transmission is 60 per cent, which is in agreement with the equal-deflection method.

Using another cell, in which the current did not increase as rapidly as the illumination, the apparent transmission of this same piece of glass at the same wave length was 53.2 per cent. Correcting this value for lack of proportionality of deflection (current) with illumination gave a transmission of 59.4 per cent, which is in agreement with the transmission (59.3 per cent) previously observed by the equal-deflection method.

When using such a device as a radiometer by the deflection method, it would evidently be necessary to maintain a constant light intensity and galvanometer sensitivity and calibrate the

⁴⁸ This Bulletin, 14 (Fig. 2), p. 173; 1917.

photoelectric cell for lack of proportionality of response with variation in intensity of illumination, such as would be observed in making transmission measurements. In view of the numerous factors that must be taken into consideration in operating a photoelectric cell, the most reliable procedure is to use the null method or the less complicated equal-deflection method just described.

III. THE SELENIUM CELL

The selenium cell is extraordinarily sensitive to light. Unfortunately it has several objectionable characteristics which render it unsuitable for precise quantitative measurements of radiant energy. As will be shown presently, its sensitivity depends upon heat treatment and varies not only with the wave length, but also with the intensity of the light stimulus.⁴⁹ In order to use it as a precise radiometer, the sensitivity of the selenium cell must therefore be calibrated for intensity as well as for the wave length of the incident light. This involves comparison measurements with some form of nonselective instrument; for example, a thermopile.

In addition to its variation in sensitivity with intensity and wave length of the stimulus, another objectionable characteristic is its great inertia or slowness to recover its normal "dark" resistance after exposure to light. For example, in experiments made by the writer on various selenium cells obtained in the market (imported and domestic), and also on cells of his own construction,⁵⁰ the cell under test was exposed for 5 seconds. After exposure to low intensities it required 30 seconds for a certain cell to recover its normal resistance. Increasing the intensity 20 times as measured with a thermopile the response (galvanometer deflection) as indicated by the selenium cell was only 8 times that of the low intensity, while the delay (2 minutes) in recovery to normal resistance was increased 4 times. Exposure to daylight required more than 10 minutes for recovery.

Hence, if an attempt is made to use the selenium cell as a photometer, it must be operated in a special manner as regards intensity, time of exposure,⁵¹ etc.

⁴⁹ The literature bearing upon this subject is very extensive, and handbooks have been written on it, e. g., C. Ries; "Electrical properties of selenium." Also "Das licht in seiner elektrischen und magnetischen wirkungen, Leipzig, 1909." See also papers by Pfund, *Phys. Rev.*, 28, p. 324, 1909, and by Brown and Sieg, *Phys. Rev.* (2), 2, p. 487, 1913; (2), 4, p. 48, 1914.

⁵⁰ Two of these cells were made nine years ago.

⁵¹ Pfund, *Phys. Rev.*, 24, p. 370; 1912.

The single crystals of selenium grown by Brown⁵³ have an extraordinarily high sensitivity as compared with a selenium cell; but they also have the characteristic slow recovery after exposure to light. From published data it appears that a single crystal of selenium, 1 mm² in area, is 100 times as sensitive as the best selenium cell. In connection with a 36-inch telescope such a crystal receiver could detect the light from a candle at a distance of 350 miles.

The wave-length sensitivity curve of a selenium crystal depends upon the temperature at which the crystal was formed. A crystal which was formed in the cooler part of the furnace had its maximum sensitivity in the violet.⁵³ A crystal formed in the hottest part of the furnace had its maximum sensitivity in the extreme red, just as is true of an ordinary selenium cell which is, no doubt, composed of mixed crystals.

Brown⁵⁴ and his collaborators have recorded many types of sensitivity curves, and Dietrich,⁵⁵ has shown that the character of the wave-length sensitivity curve of selenium can be controlled by heat treatment. Annealing the cell at 200° C. produces a maximum sensitivity in the extreme red, while annealing at 150° C. shifts the maximum sensitivity to 0.55 μ .

From the foregoing brief summary of experimental data now available it appears that selenium as such does not have a characteristic wave-length sensitivity curve; that the magnitude and position of the maximum of sensitivity in the spectrum can be controlled by heat treatment of the selenium.

It is of interest to compare the distribution of energy in the visible spectrum of the Nernst glower as registered by means of a selenium cell and by a thermopile. The former was used in connection with a d'Arsonval galvanometer and the latter with a Thomson galvanometer. Fig. 8, curve *A*, gives the spectral energy distribution obtained with a bismuth-silver thermopile (a nonselective radiometer), while curve *B* gives the response of a selenium cell when similarly exposed (for 10 seconds) in different parts of the spectrum. After exposure to the low intensities, in the blue violet, it requires 20 to 30 seconds for the galvanometer reading to return to zero; and after exposure to the highest inten-

⁵³ Brown, *Phys. Rev. (2)*, 4, p. 85, 1914; *Electrical Experimenter*, 8, p. 677, 1916. In the present investigation two cells of single crystals were tested.

⁵⁴ See further data by Sieg and Brown, *Phys. Rev. (2)*, 5, p. 65; 1915.

⁵⁵ Brown and Sieg, *Phys. Rev. (2)*, 4, p. 48; 1914.

⁵⁶ Dietrich, *Phys. Rev. (2)*, 4, p. 467, 1914; 8, p. 191, 1916.

sities it required over 2 minutes for the cell to return to its initial resistance.

The curve obtained with the selenium cell is entirely erroneous as regards the actual spectral energy distribution. Similarly, erroneous results would be obtained if one attempted to measure the radiation from red and from blue stars.

Curve *C* gives⁶⁶ the response that is obtained by exposing the selenium cell to equal amounts of energy in different parts of the spectrum using a high intensity, while curve *D* represents the response at one-sixteenth the intensity used to obtain curve *C*. According to the measurements of Pfund, for wave lengths shorter than 0.65μ the deflections of the galvanometer (used in connection with the selenium cell) are approximately proportional to the square root of the incident energy, while for wave lengths greater than 0.7μ the deflections are approximately directly proportional to the intensity of the incident light. (Exposures were 12.5 seconds.)

The present discussion of selenium as a radiometer relates to the application of the device as a high-precision instrument. In view of the adverse facts just reported it is but fair to add that the device is very sensitive, and in the early work on the photometry of variable stars some far-reaching results have been obtained by Stebbins.⁶⁷ He recognizes, however, that the photoelectric cell (of rubidium which from tests made in this laboratory is only one-sixth as sensitive as potassium hydride) is five to six times as sensitive as the selenium cell.⁶⁸

It is relevant to add that, concerning the peculiar electrical properties of selenium, some hold the view that the increased conductivity of selenium is caused by (a resonance) freeing of electrons on exposure to light. Others consider it a modification of the crystal structure, assuming that selenium occurs in several allotropic forms of widely different electrical conductivity. The absence of polarization indicates that the conduction is not electrolytic. Experiments at liquid-air temperatures, where the light sensitivity is retained, seem to be evidence supporting the electronic hypothesis.⁶⁹

In conclusion it may be added that the selenium cell may be used as an indicator in the null and the equal-deflection methods of obtaining ratios of intensities which were considered in the

⁶⁶ Pfund, *Phys. Rev.*, **84**, p. 370; 1912.

⁶⁷ Stebbins, *Astrophys. Jour.*, **89**, p. 459, 1914; **42**, p. 133, 1915.

⁶⁸ Stebbins, *Observatory*, No. 501, p. 257; 1916.

⁶⁹ Elliot, *Phys. Rev. (2)*, **5**, p. 53; 1915.

operation of the photoelectric cell. For example, the galvanometer deflection might be observed when the cell is exposed for, say, five seconds to the lower intensity. Then the higher intensity is reduced, by means of (calibrated) crossed Nicol prisms, a wire grating, an absorption wedge, a variable-sector disk, or some

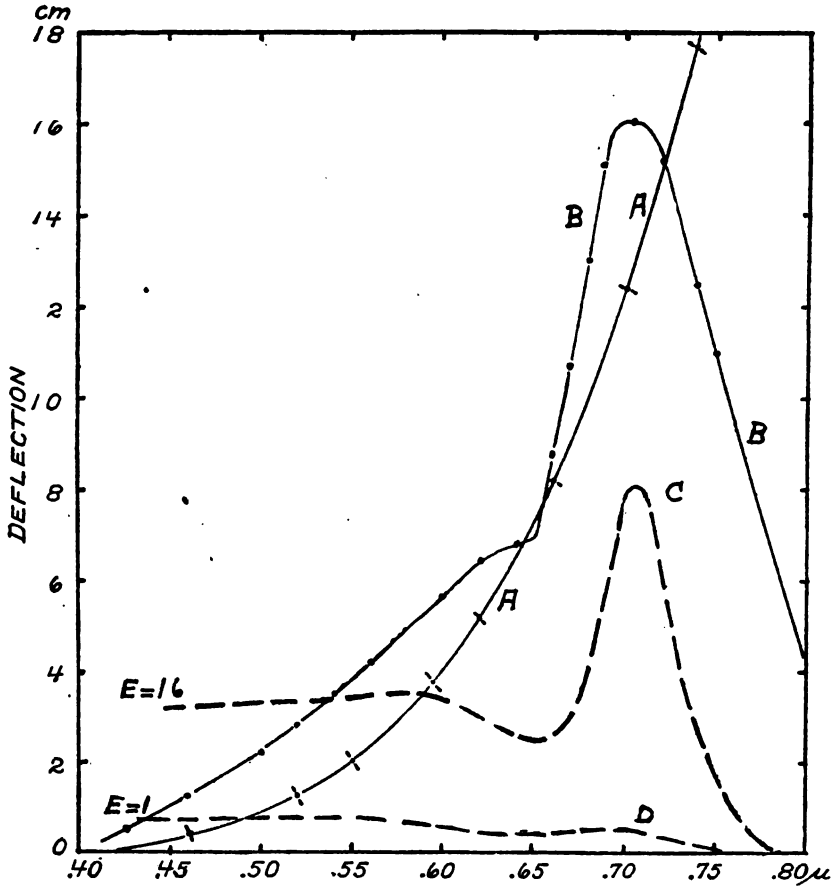


FIG. 8.—Distribution of energy in the spectrum of a Nernst glower as observed with a nonselective radiometer, A, and with a selenium cell, B. Curves C and D (from Pfund) give the responses of a selenium cell in an equal energy spectrum

other device, to give the same deflection. In this manner the ratios of intensities of monochromatic light of the same wave length might be observed; but the selenium cell can not be used in this manner to compare accurately the intensities of light of two sources differing in color. This restriction applies, of course, to all of the instruments discussed in this paper.

IV. PHOTOGRAPHIC RADIOMETRY

The use of the photographic plate is a roundabout method for making radiometric measurements. It has been used in measuring the distribution of energy in the spectrum of the light of the firefly,⁶⁰ and, more recently, in the determination of stellar magnitudes.⁶¹ In the latter method the blackening of the photographic plate by the star images is measured by photometering the plate by means of a bismuth-silver thermopile of special design.⁶²

Probably the most useful application of the photographic plate is in determining transmission spectra in the ultra-violet part of the spectrum, especially of substances having sharp absorption bands. Various arrangements of apparatus and methods of exposure have been used,⁶³ the most recent being by Howe.⁶⁴

In view of the fact that so much can be accomplished by using the photoelectric cell, which is rapidly approaching perfection as a radiometer, it seems unnecessary to fully discuss the applicability of photographic radiometry. It may be added, however, that a series of exposures can be quickly made upon a photographic plate and in this manner a permanent record obtained of transmission and reflection spectra. The plates can be examined at leisure. By using spectral lines the effect of diffuse light can be eliminated by comparing the image of the line with the effect of the diffuse light upon the adjacent part of the plate. A plate holder is a rather simple instrument as compared with a photoelectric installation, and it remains to be determined whether the latter will prove more accurate than the photographic method. One criticism that has been made against the photographic method is that the plate is not very sensitive to small variations in intensity, so that in the "flat" part of a transmission curve great variations occur in the photometric observations of the blackening of the photographic plate.

V. SUMMARY

The present paper deals with the application of certain special physical and chemical properties of matter as a means of quantitatively measuring radiant energy.

⁶⁰ Ives and Coblentz, this Bulletin, 6, p. 321, 1909; Coblentz, Publication No. 164, Carnegie Institution of Washington, 1912.

⁶¹ Stetson, *Astrophys. Jour.*, 43, pp. 253 and 325; 1916.

⁶² This Bulletin, 11, p. 168; 1915.

⁶³ Ham, Fehr, and Bitner, *Jour. Franklin Inst.*, 178, p. 299; 1914.

⁶⁴ Howe, *Phys. Rev. (2)*, 8, p. 674; 1916.

Certain substances have the property of decreasing in electrical resistance when exposed to radiant energy of short wave lengths, especially the visible and ultra-violet rays. Crystalline selenium belongs to this class of substances, and because of its very marked response to light, its application as a radiometer has been thoroughly investigated by many observers. The sensitivity of the selenium cell varies not only with the wave length but also with the intensity of the light stimulus, and it recovers but slowly from the effects of the light stimulus. It therefore fails to meet the requirements of a radiometer, except that of high sensitivity.

The application of the photochemical action upon a photographic plate, as a means of making quantitative radiometric measurements, is considered. While this method of radiometry has been used successfully, its applications seem to be rather limited.

A third application to radiometry considered in this paper is based upon the fact that some substances, when charged to a negative potential, lose their charge when exposed to light, especially violet and ultra-violet rays. In this respect the alkali metals, and especially their hydrides, are very sensitive to light stimuli. The photoelectric cells made of these substances can be constructed and operated so that the response (photoelectric current) is directly proportional to the intensity of the stimulus. This meets one of the principal requirements of a satisfactory radiometer. Details of the construction, operation, and characteristics of the photoelectric cell are given.

Attention is called to the importance of making the proportionality test of the photoelectric outfit as used, especially of cells with spherical bulbs.

A satisfactory, high-resistance, ironclad Thomson galvanometer is described which may be used successfully with the photoelectric cell.

The advantages of the photoelectric cell over the thermopile are considered and the application of the former is suggested for measurements of radiant energy (especially ratios of intensities) in the violet and ultra-violet parts of the spectrum, where the photoelectric cell greatly exceeds the thermopile and the bolometer in sensitivity.

In conclusion, special acknowledgement is due W. B. Emerson for assistance in this investigation.

WASHINGTON, September 10, 1917.

APPENDICES

Appendix 1.—NEW DESIGNS OF RADIOMETERS

In continuing the improvement of stellar radiometers several new designs of instruments were considered and some of the preliminary tests of their efficiency appear to be of sufficient importance to warrant publication.

When a very thin strip of blackened metal—for example, a bolometer strip—is exposed to radiation it becomes warmed and it, in turn, emits radiation. In previous investigations of the diffuse reflecting power of various substances⁶⁵ and of the behavior of an absolute thermopile,⁶⁶ it was found that this warming of the receiver is quite appreciable, and that this receiver can be a very efficient secondary source of radiation which, in turn, can be used to stimulate a radiometer. The utilization of this secondary source of radiation can be accomplished by placing the receiver at the center of an accurately ground hollow sphere having an opening to admit radiation. In this case one would utilize the "reradiation" which has to be very carefully excluded in diffuse-reflection measurements.⁶⁷

One method of increasing the radiation sensitivity of a bolometer is to place a plane—or cylindrical mirror close back of it, to reflect the radiation emitted by the bolometer strip back upon itself.

Another method for utilizing this radiation is by the employment of multiple receivers, one being placed back of another; for example, a thermopile receiver back of a bolometer strip, or one bolometer strip or thermopile receiver back of another. It is with this method that the present paper is chiefly concerned.

The efficiency of such a device was tested in the following manner. Two strips of very thin platinum, such as is used in bolometers (thickness about 0.0003 mm), about 6 by 20 mm in area were mounted over slits cut in cardboard which was 0.45 mm thick. Both sides of these strips were painted with a thin coat of lampblack and covered with soot from a sperm candle. The thermopile receiver was 1.8 by 16 mm. Slits of bright aluminum 0.85 mm thick were placed in front of the thermopile or in front of the blackened platinum strips when they were in front of the thermopile. The distance between the thermopile receiver and the platinum strip (and between the two platinum strips) was 0.45 mm. When the thermopile was exposed directly to a standard of radiation the deflection was 12.15 cm; when one platinum strip intervened the deflection was 5.88 cm; and when the two platinum strips were in front of the thermopile the deflection was 3.57 cm.

The Multiple-Bolometer Receiver.—Since there is but little difference between the radiation sensitivity of a bismuth-silver thermopile and a bolometer, the above tests show that the radiation sensitivity of a bolometer can be increased by 50 per cent by having the receiver (the arm of the bridge) consist of two strips, one back of the other,

⁶⁵ This Bulletin, 9, p. 283; 1913.

⁶⁶ This Bulletin, 11, p. 157; 1914.

⁶⁷ Paschen (Ber. Berliner Akad., p. 409; 1899) appears to be the first to use a hemispherical mirror in front of a bolometer in order to "blacken" it. The device has been used extensively by the writer (this Bulletin, 4, p. 392; 1908). In spectral-radiation work care must be exercised to avoid reflection of radiation from the adjacent parts of the spectrum upon the bolometer strip. In investigations where it is unimportant whether some of the incident beam of radiation falls upon a reflecting surface at the rear of the receiver before it falls upon the receiver—that is, in cases where it is unimportant whether the complete beam of incident radiation is completely intercepted by the receiver—it is possible to place the receiver at the center of an accurately made hollow sphere as just mentioned. Pfund (Phys. Rev., 34, p. 288; 1912) claims a very large increase in sensitivity as the result of using a thermojunction at the focus of a spherical mirror.

the front strip being exposed to radiation; and instead of a galvanometer deflection of 12.15 cm, as just noted, the total deflection would be $(12.15 + 5.88 =) 18$ cm. By using three strips placed one back of another the galvanometer deflection would be increased by about 80 per cent (21.6 cm as noted above) and by using four strips (joined in series, or a single strip folded three times) the deflection will be double that produced by the front strip. The sensitivity of the whole combination would be further increased by placing this multiple receiver at the focus of a hemispherical mirror.

The Bolo-Pile.—This is a combination of a single bolometer strip, close to the back of which is placed the receiver of a thermopile. The latter is so constructed that the pairs of receivers are in two rows corresponding to the two bolometer strips. In this manner the heating produced by the current passing through the bolometer strips will produce no deflection in the thermopile circuit. The manner of connecting the bolometer and the thermopile circuit to the galvanometer will depend upon the relative change in voltage of the two circuits when the receiver is exposed to radiation. If the error due to shunting is too great when the two circuits are joined to the same binding posts, the bolometer current can be passed through one galvanometer coil and the thermopile current through another coil.

The simplest and probably the most useful arrangement is a bolometer consisting of two branches of thin narrow strips of platinum close back of which is placed a thermocouple. In measuring the heat from stars a gain in sensitivity as low as 50 per cent in sensitivity is worth considering.

The Multiple-Thermocouple Receiver.—The use of two thermocouples, joined in series, with the receivers one back of the other, has not yet proved to be so efficient, because of the greater heat capacity of the thermocouple receiver used as compared with a bolometer. The comparison of this combination with the two preceding instruments, and with a single thermocouple (or bolometer) in the focus of a spherical inclosure, in which all the parts are reduced to the dimensions which would be used in measuring stellar radiations, is in progress.

In conclusion it may be added that as a result of the writer's previous measurements of stellar radiation ⁶⁶ the conclusion was reached that, in order to do much successful work in stellar radiometry, it will be necessary to have a hundred-fold greater sensitivity than that previously employed. This gain in sensitivity was to be attained by increasing the light-gathering power of the telescope five times, the sensitivity of the galvanometer ten times, and the radiometer sensitivity two times. In a paper ⁶⁷ just published data are given showing an increase of more than ten times in the galvanometer sensitivity, while the present paper indicates the way to double the radiometer sensitivity. Apparently, then, it remains to find a suitable mirror and funds to operate it.

With the thermocouple and 36-inch reflector used by the writer in 1914, it was possible to measure the radiation from a seventh-magnitude star. The sensitivity was such that a 1-mm deflection would have resulted from sighting a 36-inch telescope upon a candle placed at a distance of 53 miles.

As just mentioned, the sensitivity of the radiometric outfit, alone, can now be increased more than twenty times, so that a 1-mm deflection would be produced by a candle at 240 miles. Or reading to 0.2 mm this means that one can detect the total radiation from a candle at 500 miles, using a 3-foot reflecting telescope; or, using a 6-foot reflecting telescope, thus gaining four times in light-gathering power, this means that one can detect a candle at a distance of 1000 miles. This is on the basis of a galvanometer sensitivity of $i = 1 \times 10^{-11}$ amperes, which is easily attained. The main difficulty will arise in obtaining steady conditions. The modern observatory, with its complicated electrical machinery and power plant, is not the ideal place one might imagine it to be for making delicate radiometric observations.

⁶⁶ This Bulletin, 11, p. 613; 1914.

⁶⁷ This Bulletin, 13, p. 423; 1916.

Appendix 2.—AMPLIFICATION OF THE BOLOMETER CURRENT

The electric current obtained from radiometrically heating a bolometer receiver is exceedingly small. In the foregoing appendix methods are described for increasing this current.

Another method for increasing the response of a bolometer is by amplifying the electric current which would ordinarily pass through the galvanometer by passing it through an audion amplifier. In the present experiments the receiver consisted simply of a thin blackened strip of platinum or gold leaf (and a storage battery), which were suitably connected into the grid circuit of a three-stage audion amplifier. A telephone receiver was connected in the usual manner to the amplifier.

The source of radiation was an acetylene flame. The receiver was exposed to this flame through a rotating sector disk having 15 openings. This combination formed a radiophone.

When a sensitive platinum bolometer receiver was used as a radiophone, the sound produced in the telephone was not very audible. This is no doubt attributable to the great heat capacity of the material, which prevented the rapid changes in resistance and hence in electric current from being of sufficient magnitude to affect the telephone receiver.

Using a lightly smoked strip (6 by 2.5 mm) of gold leaf, the ends of which were clamped between thin (0.02 mm) strips of tin, the sound produced in the telephone receiver was as loud as was observed in a photophone made of selenium.⁷⁰ This device was mounted in a glass bulb, which could be evacuated. As was to be expected, there was no marked difference in the intensity of the sound produced when operated in air and in a vacuum.

In the gold-leaf radiophone, as used, the limit of audibility was attained for a light (radiant power) intensity of 4.8×10^{-8} watts. Using a larger receiver and amplifier, and a larger current (which was 0.2 ampere in the present tests) through the receiver, the sensitivity could be greatly increased.

Appendix 3.—ELIMINATION OF SCATTERED RADIATIONS IN SPECTRAL-ENERGY MEASUREMENTS

For completeness of discussion of the subject of methods of radiometry, as well as for the reason that, among experimenters, especially beginners in spectroradiometric work, the importance of eliminating scattered radiations in emission, reflection, and transmission measurements is not fully realized, it is relevant to discuss this topic.

One method of reducing the intensity of the scattered radiations of various wave lengths from the energy measurements at a given point in the spectrum is to operate two spectroscopes in series. This is most useful in spectral energy measurements, e. g., in the spectrum of a black body. This method has been used by Langley in his measurements of the spectral radiaton from the moon, and also by Paschen in some of his determinations of the constant of spectral radiation.

In the measurement of the spectral reflection from, or transmission through, various substances, a simpler procedure is to use screens which are opaque to most of the radiations excepting the spectral region in which measurements are to be made. In view of the fact that in transmission or reflection measurements the ratio of two intensities is desired and not the absolute intensity of the source it is unnecessary to know the exact transmission of the screen at the point in the spectrum at which measurements are being made.

In visual work (spectrophotometry) it is necessary to eliminate simply the scattered rays which affect the eye by placing gelatin films stained with dyes,⁷¹ or colored

⁷⁰ Jour. Wash. Acad. Sci., 7, p. 525, 1917; this Bulletin, 14, p. 591, 1918.

⁷¹ Uhler and Wood, Atlas of Absorption Spectra, Publication No. 71, Carnegie Institution of Washington, 1907.

glasses in the path of the rays. When using a selective radiometer (e. g., a photoelectric cell) which does not respond to infra-red rays it is not necessary to eliminate these rays scattered over the visible and ultra-violet part of the spectrum.

When spectral energy measurements are made in the visible and ultra-violet by means of a thermopile or bolometer, the infra-red rays can be eliminated by placing before the spectrometer slit a weak solution of cupric chloride⁷² or a glass which is opaque to infra-red rays.⁷³

A further device is a shutter which is opaque to the radiations in that part of the spectrum which is under investigation, but which is transparent to the scattered rays. This procedure is particularly valuable in spectral energy measurements. In his infra-red investigations Rubens used shutters of quartz, fluorite, and rock salt which are as opaque as a metal shutter for wave lengths which are greater, respectively, than 8μ , 12μ , and 30μ , but which are transparent for radiations which are less than 4μ , 7μ , and 12μ , respectively. The scattered radiations which are transmitted by these shutters impinge upon the radiometer practically with the same intensity, whether or not the shutter is in place before the spectrometer slit and hence do not affect the energy measurements.

For spectroradiometric measurements in the yellow to the violet, a shutter of Corning red glass and in the ultra-violet a shutter of amber (or Crooke's neutral tint) glass should be used. Clear glass and mica are opaque to rays less than 0.3μ . These glasses are very transparent to part of the visible and the infra-red rays and they are opaque for radiations of wave lengths less than 0.58μ and 0.4μ , respectively.⁷⁴

The use of rough surfaces which scatter the short wave-length radiations and specularly reflect the long wave-length radiations is a further method for investigating the extreme infra-red, which is suggested by the investigations of Gorton.⁷⁵

Appendix 4.—USE OF A ROTATING SECTORED DISK IN RADIOMETRY

In photometric and radiometric measurements involving high intensities a common practice is to reduce the incident radiations by means of a rotating sectored disk.

In photometry it is well established⁷⁶ that the response of the eye is of such a nature that, physiologically as well as physically, a rotating sectored disk transmits light proportional to the angular openings in the disk.

Recently Kunz⁷⁷ has shown that the photoelectric cell functions so as to indicate that the light transmitted by a rotating sectored disk is proportional to the mechanically measured apertures; that is, Talbot's law holds for the photoelectric cell.

Some years ago the applicability of the rotating sector disk for reducing the intensity of the energy incident upon a bolometer was tested by the writer, who found⁷⁸ that the energy transmitted is appreciably greater than the theoretical value; that is, the value indicated by the mechanical measurement of the openings in the disk. Extensive experimental data were obtained showing that this increased radiation through the openings was a function of the speed of the disk and to some extent a function of the distance intervening between the disk and a screen which faced the bolometer. The explanation offered was that diffraction of the rays of great wave length produced an apparent aperture which was larger than the mechanically measured opening in the disk. This explanation does not, however, fully explain the observations when the distance was varied between the disk and the screen diaphragm.

⁷² This Bulletin, 14, p. 229; 1917. For investigating the extreme infra-red solar spectrum, Fowle, Smithsonian Miscell. Coll., vol. 68, No. 8, 1917, used an absorption screen of solid iodine.

⁷³ See Technologic Paper No. 93, issued by this Bureau.

⁷⁴ Coblentz and Emerson, Technologic Paper No. 93, this Bureau; Luckiesh, Trans. Illum. Eng. Soc., p. 472, 1914; Bell, Proc. Amer. Acad. Arts and Sci., 46, p. 669, 1911.

⁷⁵ Gorton, Phys. Rev. (2), 7, p. 66; 1916.

⁷⁶ Hyde, this Bulletin, 2, p. 1; 1906.

⁷⁷ Kunz, Astrophys. Jour., 46, p. 69; 1917.

⁷⁸ This Bulletin, 4, p. 455; 1907.

These observations were so novel at the time that part of the manuscript was not published. The published data brought forth communications from observers who had not found this discrepancy in their work. This difference in observations has remained unsettled until recently, when Fowle⁷⁹ published a verification of the writer's observations, showing that the energy transmitted by a rotating sector is always greater than the theoretical value. For instance, his 0.333 sector had an aperture of 0.344 as determined radiometrically.

From these experiments it is evident that the rotating sector disk can not be used for reducing intensities in radiometric work unless the disk is mounted close to a stationary diaphragm and operated⁸⁰ at a slow speed as described in the writer's original communication.

In a recent investigation Mendenhall⁸¹ employed a rotating sector disk and thermopile in determining the constant of spectral radiation. Aside from the temperature scale used (m. p. of Pd. 1549 instead of 1555° C), recently adopted by experimenters, it would be of interest to know whether the rotating sector disk contributed to the production of a high value ($C_2=14\,400$) of this constant.

⁷⁹ Fowle, *Smithsonian Miscell. Coll.*, 68, No. 8, p. 14; 1917.

⁸⁰ This Bulletin, 7, p. 249; 1911.

⁸¹ Mendenhall, *Phys. Rev. (2)*, 10, p. 515; 1917.

ADDITIONS TO THE FORMULAS FOR THE CALCULATION OF MUTUAL AND SELF INDUCTANCE

By Frederick W. Grover

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I. INTRODUCTION

Since the appearance of the Bureau of Standards' collection of formulas for the calculation of mutual and self inductance¹ a number of papers have been published upon this subject, some of which have given formulas for cases which had not been previously

¹This Bulletin, 8, p. 1; Scientific Paper No. 169.

treated, while others have given additional formulas for cases which had already received attention.

The present paper has for its purpose the presentation of the more generally useful of the new formulas with the purpose of supplementing the previous collection. The recent rate of production of formulas has, however, been such as to render it impossible to keep the subject up to date for any length of time. The selection of the material has been necessarily restricted to those cases in which the formulas lend themselves readily to numerical calculations or to those more complicated formulas for which auxiliary tables are available. In certain important cases formulas have been here omitted which may be of great service when suitable tables have been prepared for simplifying the numerical work.

It is to be emphasized, also, that although in the case of certain forms of circuit very complete formulas are available, yet in certain other cases which are important in practical alternating-current work and in wireless telegraphy formulas are still lacking or only imperfectly developed.

In general, the classification of the material here is uniform with that of Scientific Paper No. 169. New formulas are designated by the letter *A*, while formulas in Scientific Paper No. 169, to which reference is made, bear the numbers by which they are designated in that publication. The nomenclature adopted in the formulas is as far as possible the same in both papers.

II. MUTUAL INDUCTANCE OF PARALLEL COAXIAL CIRCLES

This case is of great practical importance, since it is the fundamental form from which, by integration, formulas for the inductance of solenoids and of circular coils of finite cross section have been derived.

In what follows the radii of the two circles are denoted by *A* and *a*, *A* being greater than *a*, *d* is the distance between their planes, while the auxiliary moduli *k* and *k'* are given in the equations

$$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + d^2}}$$

$$k' = \sqrt{1 - k^2} = \frac{\sqrt{(A-a)^2 + d^2}}{\sqrt{(A+a)^2 + d^2}}$$

BUTTERWORTH'S FORMULAS

Butterworth² has recently published five series formulas for the calculation of the mutual inductance of two parallel coaxial circles, and has shown³ that these furnish all the possible essentially different hypergeometric developments of Maxwell's elliptic integral formula (1).

His formula (A) is the same as (5).³

His formula (B) is a much abbreviated form of Havelock's formula (17) and has the further advantage over the latter that the general term is known

$$M = \frac{\pi^2 \mu^3}{4} \sqrt{Aa} \left[1 - \frac{3^2}{2 \cdot 6} \mu^2 + \frac{3^2 \cdot 5^2}{2 \cdot 4 \cdot 6 \cdot 8} \mu^4 - \frac{3^2 \cdot 5^2 \cdot 7^2}{2 \cdot 4 \cdot 6^2 \cdot 8 \cdot 10} \mu^6 + \dots \right] \quad (1A)$$

where

$$\mu^2 = \frac{k^2}{1 - k^2} = \frac{4Aa}{(A - a)^2 + d^2} \quad (2A)$$

The general expression for the coefficient of μ^{2n} is the same as that for the coefficient of k^{2n} in the preceding formula, except that the terms are here alternately positive and negative. This formula converges only for $k^2 < 1/2$, that is, for circles far apart.

The formula (C) of Butterworth's paper is best suited for circles near together, although it converges for circles at all distances. It is written in the form

$$\frac{M}{4\pi\sqrt{Aa}} = k^2 \left[\phi_0 + \frac{3^2}{2^2} k'^2 \phi_1 + \frac{3^2 5^2}{2^2 4^2} k'^4 \phi_2 + \dots \right] \quad (3A)$$

in which

$$\phi_0 = \log_e \frac{4}{k'}, \quad \phi_1 - \phi_0 = \frac{1}{1} - \frac{2}{3}, \quad \phi_2 - \phi_1 = \frac{1}{2} - \frac{2}{5}$$

$$\phi_n - \phi_{n-1} = \frac{1}{n} - \frac{2}{2n+1} = \frac{1}{n(2n+1)}$$

and the general term is

$$\left[\frac{3 \cdot 5 \cdot 7 \dots (2n+1)}{2 \cdot 4 \cdot 6 \dots 2n} \right]^2 k'^{2n} \phi_n$$

² Phil. Mag., 21, p. 276; 1916.

³ An error appears in the expression for the general term of this equation as it appears in the earlier issues of Scientific Paper No. 169. The general term of (5) for the series in the parentheses should read

$$\left[\frac{3 \cdot 5 \cdot 7 \dots (2n+1)}{4 \cdot 6 \cdot 8 \dots 2n} \right]^2 \frac{k^{2n}}{(n+2)(2n+2)}$$

If we put for the difference of the radii $c = A - a$, and for k and k' their values in terms of A , a , and d , and expand in powers of c/a and d/a , the resulting expression is the Maxwell's series formula (10) or (14). The latter formula has been much used and has been extended at different times by Nagaoka, Rosa, and Cohen, and later still by Coffin (p. 541). The expression (3A) is in a much more favorable form for numerical calculation, the advantage being very marked for the higher order terms.

Butterworth writes his formula (D) in the form

$$\frac{M}{4\pi\sqrt{Aa}} = \frac{1}{k} \left[\phi_0' + \frac{1^2}{2^2} k'^2 \phi_1' + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} k'^4 \phi_2' + \frac{1^2 1^2 3^2}{2^2 4^2 6^2} k'^6 \phi_3' + \dots \right] \quad (4A)$$

where

$$\begin{aligned} \phi_0' &= \log_e \frac{4}{k}, -2, \quad \phi_1' - \phi_0' = \frac{1}{1} + \frac{2}{1}, \quad \phi_2' - \phi_1' = \frac{1}{2} - \frac{2}{1} \\ \phi_n' - \phi_{n-1}' &= \frac{1}{n} - \frac{2}{2n-3} = -\frac{3}{n(2n-3)}. \end{aligned}$$

The general term is

$$\left[\frac{1 \cdot 1 \cdot 3 \cdot \dots \cdot (2n-3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \right]^2 k'^{2n} \phi_n'$$

Like (3A), this formula converges for circles at all distances, but especially well for circles near together. It is not difficult to prove that (4A) is the same as a new formula previously obtained by Bromwich,⁴ but which was not known to Butterworth. This expression, (3') in Bromwich's paper, was shown by him to be essentially the same as that of Weinstein ((7) in Scientific Paper No. 169). Bromwich's equation is slightly more convergent than that of Weinstein and may be obtained from the latter simply by multiplying by the identity

$$\frac{r_1}{2\sqrt{Aa}} = \frac{r_1}{\sqrt{r_1^2 - r_2^2}},$$

in which r_1 and r_2 have the values given under (2) of Scientific Paper No. 169. Bromwich's expression for the general term of (4A) is equivalent to that found by Butterworth, although expressed in a somewhat different form.

⁴ Quart. Jour. Pure and Applied Math., No. 176, p. 381; 1913.

Butterworth's final expression, (E), is

$$\frac{M}{4\pi\sqrt{Aa}} = \varphi_0'' + \frac{1 \cdot 3}{2^2} \frac{\varphi_1''}{\mu^2} - \frac{1^2 \cdot 3 \cdot 5}{2^2 \cdot 4^2} \frac{\varphi_2''}{\mu^4} + \frac{1^2 \cdot 3^2 \cdot 5 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2} \frac{\varphi_3''}{\mu^6} - \dots \quad (5A)$$

in which

$$\varphi_0'' = \log_e 4\mu - 2, \quad \varphi_1'' - \varphi_0'' = \frac{1}{1} - \frac{1}{3} + \frac{1}{1}$$

$$\varphi_2'' - \varphi_1'' = \frac{1}{2} - \frac{1}{5} - \frac{1}{1}$$

$$\varphi_n'' - \varphi_{n-1}'' = \frac{1}{n} - \frac{1}{2n+1} - \frac{1}{2n-3} \quad \varphi_3'' - \varphi_2'' = \frac{1}{3} - \frac{1}{7} - \frac{1}{3}$$

and μ^2 has the same value as in (2A).

The general term is

$$\left[\frac{1 \cdot 3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \cdot 6 \dots 2n} \right]^2 \frac{\varphi_n''}{(2n-1)(2n+1)\mu^{2n}}.$$

If, in this formula, we substitute the value of μ , it is easy to see that this series is the same as that given as (16), of which the first terms were found by Havelock.⁵ Note that $\frac{1}{\mu^2} = \frac{r^2}{4Aa} = \frac{\alpha}{4}$ in (16). The general term of (16) was first found by Bromwich, in the paper already referred to,⁴ in terms of the variable $p = \frac{\alpha}{4}$ in his nomenclature. It is advantageous to write the formula in terms of this variable rather than α , since the coefficients are thereby simplified.

Evidently the general term in Coffin's equation (13), which is the form taken by (16), when the circles have equal radii, will be the same as that given above for (5A). For this case, $a = A$, $r = d$, and $\alpha = d^2/a^2$. Like (16), this converges only when $\mu > 1$, or, for equal circles, when $d < 2a$.

COFFIN'S EXTENSION OF MAXWELL'S FORMULA (14)

Coffin⁶ has shown how to obtain further terms in (14). He bases his method on (16). Putting $A = a + c$ and expanding (16) in terms of the small quantities c/a and d/a he has obtained three new terms in each part of (14). These new terms are

⁵ Phil. Mag., 16, p. 332; 1908.

⁶ Phys. Rev. (2), 2, p. 65; 1913.

$$\begin{aligned}
& 4\pi a \left[\log \frac{8a}{r} \left\{ \frac{89c^6 + 45c^4d^2 - 345c^2d^4 + 35d^6}{128^2a^6} \right. \right. \\
& \quad - \frac{109c^7 - 63c^5d^2 - 525c^3d^4 + 175cd^6}{2 \cdot 128^2a^7} \\
& \quad + \frac{8921c^8 - 13692c^6d^2 - 43050c^4d^4 + 32900c^2d^6 - 1575d^8}{2 \cdot 128^3a^8} \Big\} \\
& \quad - \left(\frac{2629c^6 + 12045c^4d^2 - 17445c^2d^4 + 1235d^6}{30 \cdot 128^2a^6} \right. \\
& \quad - \frac{27833c^7 + 71169c^5d^2 - 225225c^3d^4 + 50575cd^6}{420 \cdot 128^2a^7} \\
& \quad \left. \left. + \frac{5204309c^8 + 5304852c^6d^2 - 45499650c^4d^4}{840 \cdot 128^3a^8} \right) \right] \quad (6A)
\end{aligned}$$

Now that the general term of (16) is known Coffin's method may be extended to obtain further terms of any desired order. The algebraic work becomes, however, very tedious for the higher order terms, and, as has already been noted, the equivalent formula (3A) is much easier to use for numerical calculations.

SERIES EXPANSION OF ELLIPTIC INTEGRAL FORMULA (4)

Another method of procedure, when k' is too large for (4A) to be conveniently used, is to employ (4), which involves the modulus k'_0 , which is smaller than k' . We may write (4) in the form

$$M = \frac{2\pi\sqrt{Aa}}{(1-k'_0)} \left[(1+k'_0)^2 F_0 - 4E_0 \right] \quad (7A)$$

and expressing F_0 and E_0 in terms of k'_0 , by means of the general series formulas (3), we find

$$\begin{aligned}
M = \frac{2\pi\sqrt{Aa}}{(1-k'_0)} & \left[(1+2k'_0) \log_0 \frac{4}{k'_0} - 4 - \frac{3}{4} k'^2_0 \left(\log \frac{4}{k'_0} - 1 \right) \right. \\
& + \frac{k'^3_0}{2} \left(\log \frac{4}{k'_0} - 1 \right) - \frac{23}{64} k'^4_0 \left(\log \frac{4}{k'_0} - \frac{51}{46} \right) \\
& + \frac{9}{32} k'^5_0 \left(\log \frac{4}{k'_0} - \frac{7}{6} \right) - \frac{59}{256} k'^6_0 \left(\log \frac{4}{k'_0} - \frac{427}{354} \right) \\
& \left. + \frac{25}{128} k'^7_0 \left(\log \frac{4}{k'_0} - \frac{37}{30} \right) - \frac{2775}{16384} k'^8_0 \left(\log \frac{4}{k'_0} - \frac{2783}{2220} \right) + \dots \right] \quad (8A)
\end{aligned}$$

where

$$k'_0 = \frac{1-k}{1+k} = \frac{k'^2}{(1+k)^2}$$

The first two terms of this equation will have to be calculated with a good deal of care, and as the series involves odd powers of the modulus, as well as even, it will be about as easy to use (4A) or (6) and include terms of higher order than are included in (8A).

RANGE OF APPLICATION OF THE SERIES FORMULAS

Since the general term is available in all the series formulas, it is possible to calculate the mutual inductance of any two coaxial circles whatever without having recourse to the elliptical integral formulas. In some cases (see pp. 8 and 9 of Scientific Paper No. 169) the values calculated by the series formulas are more accurate than those obtained by the exact formula, and in any case the series are valuable for obtaining a check, not to mention their adaptability to the carrying out of integrations.

Those cases which it is most difficult to treat by means of a series formula are those where the moduli k and k' are nearly equal. For such cases the elliptical integral formulas will be easy to use, or we may obtain the result with accuracy by means of Nagaoka's series formula (8). Of the hypergeometric formulas (5) and (4A) require only 20 or 30 terms to be calculated (even in the most unfavorable case) to obtain a result correct to the seventh decimal place, and if we previously write down the successive factors which must be multiplied into each term to obtain the next, the calculation is much simplified. For the higher order terms the ratio of successive terms is nearly constant, so that these terms may be obtained with very little labor.

Instead of using (5) or (4A) we may also use (6), which is an expansion of the elliptic integral formula (2), expressed in terms of the modulus $k_1 = \frac{1-k'}{1+k'} = \frac{k^2}{(1+k')^2}$, which is smaller than k . This formula covers very satisfactorily those cases for which k and k' are nearly equal.

As already explained, (8A) gives the mutual inductance in terms of the modulus k_0' , which is smaller than k' . This formula may be used when k' is so large as to render the use of (3A) and (4A) difficult.

The following examples are appended to give an idea of the relative advantages of the different formulas in several rather extreme cases:

EXAMPLES

Example 1.— $k = k' = \frac{1}{\sqrt{2}}$. Assuming the radii $A = 25$, $a = 20$,

the distance between their planes corresponding to the above condition is $d = \sqrt{1975} = 44.441$. For calculating the mutual inductance in this case the series formulas (4A), (6), and (8A) are applicable. Also for this special case the formula (19) gives a very searching numerical test.

The following table gives the value of $M/4\pi\sqrt{Aa}$ calculated by each of these formulas, together with a record of the number of terms which have to be calculated to obtain a result accurate to seven places. Since, however, six-place logarithms were used in the computation, the values obtained differ in some cases by a little in the seventh place. These discrepancies are of the order of magnitude to be expected in work with six-place logarithms:

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(8A)	8	0.1128884
(4A)	11	0.1128885
(6)	4	0.1128887
(19)		0.112888542...

Example 2.— $k' = 0.6$, $k = 0.8$. To these corresponds the modulus $k_0' = \frac{1}{9}$. Assuming the same radii as before, namely, $A = 25$, $a = 20$, we find $d = \sqrt{1100} = 33.1663$. The results found for this case are

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(8A)	7	0.20113986
(4A)	10	0.20113987
(6)	5	0.20113983

Example 3.—Assuming $k^2 = \frac{1}{3}$ and $a/A = \frac{1}{2}$, the distance corresponding is $d = a\sqrt{15}$. One pair of circles satisfying these conditions is $A = 25$, $a = 12.5$, $d = 48.4123$.

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(2A)	14	0.911653
(1A)	13	0.911653
(6)	3	0.911652

Example 4 (examples 8 and 14, Scientific Paper No. 169).— $A = 25$, $a = 20$, $d = 10$, $\frac{1}{\mu^2} = \frac{1}{16}$

Formula	Number of terms	$\frac{M}{4\pi\sqrt{Aa}}$
(5A)	6	0.885387
(4A)	4	0.885387
(3A)	7	0.885387

III. MUTUAL INDUCTANCE OF PARALLEL ECCENTRIC CIRCLES

A knowledge of the mutual inductance of two circles whose planes are parallel, but whose axes are not coincident, is required in the calculation of the inductance of certain standards whose inductance may be given a continuous variation by moving one coil so that its plane remains constantly parallel to that of another coil, the distance between the planes of the two coils being kept constant. This is the construction employed in the Campbell variable inductance.

On account of the relatively large and uncertain correction necessary to apply to take into account the finite cross section of the coils, the value of the inductance in such a case will best be obtained by measurement rather than by calculation. Nevertheless, formulas allowing of the calculation of the inductance, at least approximately, are valuable for purposes of design. In the case of actual coils the current may be regarded as concentrated in a circular filament at the center* of the cross section.

The only formulas yet available for the mutual inductance of eccentric circles are those recently published by Butterworth.⁷ He rests his method on the theorem that any formula applicable to this case must be a solution of Laplace's equation, and for the special case that the circles are coaxial must reduce to one of the formulas for coaxial circles given in the previous section.

There follow not only Butterworth's general formulas but also the simpler formulas which hold for the cases of circles of equal radii and coplanar circles.

In these expressions we put

$$\begin{aligned} A &= \text{radius of the larger circle.} \\ a &= \text{radius of the smaller circle.} \\ d &= \text{the distance between their planes} \\ \rho &= \text{the distance between their axes.} \\ \mu &= \cos \theta = d/r \quad c = A - a \\ r^2 &= d^2 + \rho^2 \end{aligned}$$

$P_n(\mu)$ = the zonal harmonic of the n th order with argument μ .

⁷ Phil. Mag., 81, p. 443; 1916.

For calculating the zonal harmonics we have the well-known expressions

$$\begin{aligned}
 P_0(\mu) &= 1 \\
 P_1(\mu) &= \mu \\
 P_2(\mu) &= \frac{1}{2}(3\mu^2 - 1) \\
 P_3(\mu) &= \frac{\mu}{2}(5\mu^2 - 3) \\
 P_4(\mu) &= \frac{1}{8}(35\mu^4 - 30\mu^2 + 3) \\
 P_5(\mu) &= \frac{\mu}{8}(63\mu^4 - 70\mu^2 + 15) \\
 P_6(\mu) &= \frac{1}{16}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5) \\
 P_7(\mu) &= \frac{\mu}{16}(429\mu^6 - 693\mu^4 + 315\mu^2 - 35) \\
 P_8(\mu) &= \frac{1}{128}(6435\mu^8 - 12012\mu^6 + 6930\mu^4 - 1260\mu^2 + 35)
 \end{aligned} \tag{9A}$$

BUTTERWORTH'S FORMULAS FOR UNEQUAL CIRCLES

For unequal circles far apart

$$M = 2\pi^2 a \left(\frac{a}{r} \right) \left(\frac{A}{r} \right)^2 \left[P_2(\mu) - \frac{3}{2} K_1 P_4(\mu) \frac{A^2}{r^2} + \frac{3.5}{2.4} K_2 P_6(\mu) \frac{A^4}{r^4} - \frac{3.5 \cdot 7}{2.4 \cdot 6} K_3 P_8(\mu) \frac{A^6}{r^6} + \dots \right]$$

where

$$\begin{aligned}
 K_1 &= 1 + \frac{a^2}{A^2}, \quad K_2 = 1 + 3 \frac{a^2}{A^2} + \frac{a^4}{A^4} \\
 K_3 &= 1 + 6 \frac{a^2}{A^2} + 6 \frac{a^4}{A^4} + \frac{a^6}{A^6}, \quad K_n = F\left(-n-1, -n, 2, \frac{a^2}{A^2}\right)
 \end{aligned} \tag{10A}$$

F is the hypergeometric series. (See p. 17, Scientific Paper No. 169.)

This formula converges for values of r greater than $(A+a)$. For the limiting case that the circles are coaxial it goes over into (17), Havelock's formula for circles far apart. The general term of (10A) is known, but the higher-order terms become tedious to calculate. Table A, taken from Byerleys "Fourier's Series and Spherical Harmonics," will be found helpful in making calculations with (10A) and the succeeding formulas.

For unequal circles near together

$$M = 4\pi\sqrt{Aa} \left[(\lambda - 2) + \frac{3}{16} \frac{c^2}{Aa} \left\{ \left(\lambda - \frac{1}{3} \right) \varphi_2 - \chi_2 \right\} - \frac{15}{1024} \frac{c^4}{A^2 a^2} \left\{ \left(\lambda - \frac{31}{30} \right) \varphi_4 - \chi_4 \right\} + \dots \right] \quad (11A)$$

in which

$$\lambda = \log_e \frac{16\sqrt{Aa}}{c(1+\mu)\sqrt{1+\nu^2}}$$

and μ^2 and $-\nu^2$ are the roots of the quadratic

$$t^2 - t \left(1 - \frac{r^2}{c^2} \right) - \frac{d^2}{c^2} = 0$$

and

$$\varphi_2 = \frac{2}{3} - \frac{2}{3} P_2(\mu) P_2(i\nu) = \frac{1}{2} \{ (1 + \mu^2) - \nu^2 (1 - 3\mu^2) \}$$

$$\varphi_4 = \frac{8}{15} - \frac{16}{21} P_2(\mu) P_2(i\nu) + \frac{8}{35} P_4(\mu) P_4(i\nu)$$

$$= \frac{1}{8} \{ (3 + 2\mu^2 + 3\mu^4) - 2\nu^2 (1 + 6\mu^2 - 15\mu^4) + \nu^4 (3 - 30\mu^2 + 35\mu^4) \}$$

$$\chi_2 = \frac{1}{4} (1 - \mu) \{ (1 - \mu) - \nu^2 (1 + 7\mu) \}$$

$$\chi_4 = \frac{1}{96} (1 - \mu) \{ 3(1 - \mu)(7 + 2\mu + 7\mu^2) - 6\nu^2 (5 + \mu - \mu^2 + 59\mu^3) + \nu^4 (21 + 241\mu - 113\mu^2 - 533\mu^3) \}$$

The general solution of the quadratic is

$$t = \frac{1}{2} \left(1 - \frac{r^2}{c^2} \right) \pm \frac{1}{2} \sqrt{\left(1 - \frac{r^2}{c^2} \right)^2 + 4 \frac{d^2}{c^2}}$$

the positive root is taken as μ^2 , while the negative root is $-\nu^2$.

Butterworth gives the method for obtaining the general term of (11A), but the calculation of the quantities φ and χ for the higher-order terms becomes very tedious. An idea of the importance of the terms omitted in (11A) may be obtained from an examination of the convergence of the formula (16) with the same values of A , a , and d . The convergence for the eccentric circles will be at least as good as that of (16), which holds for the limiting case of coaxial circles.

Similar considerations hold for (10A) and (17) if in the latter d be taken as of the same value as r in the former.

BUTTERWORTH'S FORMULAS FOR EQUAL ECCENTRIC CIRCLES

If the circles have the same radius a and their distance apart is large compared with a ,

$$M = 2\pi^2 a \left(\frac{a}{r}\right)^3 \left[P_2(\mu) - 3 P_4(\mu) \frac{a^2}{r^2} + \frac{75}{8} P_6(\mu) \frac{a^4}{r^4} - \frac{245}{8} P_8(\mu) \frac{a^6}{r^6} + \dots \right] \quad (12A)$$

For equal circles near together

$$M = 4\pi a \left[(\lambda_1 - 2) + \frac{3}{16} \frac{r^2}{a^2} \left\{ \left(\lambda_1 - \frac{1}{3} \right) P_2(\mu) - \psi_2 \right\} - \frac{15}{1024} \frac{r^4}{a^4} \left\{ \left(\lambda_1 - \frac{31}{30} \right) P_4(\mu) - \psi_4 \right\} + \frac{35}{(128)^2} \frac{r^6}{a^6} \left\{ \left(\lambda_1 - \frac{247}{210} \right) P_6(\mu) - \psi_6 \right\} - \dots \right] \quad (13A)$$

• in which

$$\lambda_1 = \log_e \frac{16a}{r(1+\mu)}, \quad \mu = \frac{d}{r}$$

$$\psi_2 = -\frac{1}{4} (1-\mu) (1+7\mu)$$

$$\psi_4 = \frac{1}{96} (1-\mu) (21+241\mu-113\mu^2-533\mu^3)$$

$$\psi_6 = -\frac{1}{960} (1-\mu) (185-2957\mu+3728\mu^2+18008\mu^3-3247\mu^4-18107\mu^5)$$

BUTTERWORTH'S FORMULAS FOR COPLANAR CIRCLES

(a) Unequal coplanar circles far apart.

$$M_0 = -\pi^2 a \left(\frac{a}{A}\right) \left(\frac{A}{r}\right)^3 \left[1 + \frac{3^2}{2.4} K_1 \frac{A^2}{r^2} + \frac{3^2 \cdot 5^2}{2.4^2 \cdot 6} K_2 \frac{A^4}{r^4} + \frac{3^2 \cdot 5^2 \cdot 7^2}{2.4^3 \cdot 6^2 \cdot 8} K_3 \frac{A^6}{r^6} + \dots \right] \quad (14A)$$

the values of the K 's being the same as in (10A).

(b) Unequal coplanar circles near together, but $r > (A-a)$; i. e., the distance between their centers greater than the difference of their radii.

$$M_0 = 4\pi \sqrt{Aa} \left[(\lambda_0 - 2) - \frac{3}{32} \frac{r^2}{Aa} \left(\lambda_0 - \frac{5}{6} \right) \left(1 - 2 \frac{c^2}{r^2} \right) - \frac{45}{8192} \frac{r^4}{A^2 a^2} \left\{ \lambda_0 \left(1 - \frac{8}{3} \frac{c^2}{r^2} + \frac{8}{3} \frac{c^4}{r^4} \right) - \left(\frac{97}{60} - \frac{214}{45} \frac{c^2}{r^2} + \frac{214}{45} \frac{c^4}{r^4} \right) \right\} - \dots \right] \quad (15A)$$

with

$$\lambda_0 = \log_e \frac{16\sqrt{Aa}}{r}, \quad c = A - a$$

This formula holds even when the circles intersect. A negative sign in the formulas for the mutual inductance in this section indicates that the electromotive force induced in either coil, due to a change of current in the other, is of opposite sign to that which would be induced, under the same circumstances, if the coils were in the coaxial position.

(c) unequal coplanar circles for which $r < (A - a)$ —that is, where one circle is entirely within the other.

$$M_0 = 4\pi \sqrt{Aa} \left[(\lambda_2 - 2) + \frac{3}{32} \frac{c^2}{Aa} \left\{ \left(\lambda_2 - \frac{1}{3} \right) (1 + \mu^2) - \frac{1}{2} (1 - \mu)^2 \right\} \right. \\ \left. - \frac{15}{8192} \frac{c^4}{A^2 a^2} \left\{ \left(\lambda_2 - \frac{31}{30} \right) (3 + 2\mu^2 + 3\mu^4) - \frac{1}{4} (1 - \mu)^2 (7 + 2\mu + 7\mu^2) \right\} + \dots \right] \quad (16A)$$

$$\lambda_2 = \log_0 \frac{16\sqrt{Aa}}{c(1 + \mu)}, \quad \mu^2 = 1 - \frac{r^2}{c^2}$$

(d) unequal coplanar circles, one touching the other internally; that is, $r = A - a = c$.

$$M_0 = 4\pi \sqrt{Aa} \left[(\lambda_3 - 2) + \frac{3}{32} \frac{c^2}{Aa} \left(\lambda_3 - \frac{5}{6} \right) - \frac{45}{8192} \frac{c^4}{A^2 a^2} \left(\lambda_3 - \frac{97}{60} \right) + \dots \right] \quad (17A)$$

$$\lambda_3 = \log_0 \frac{16\sqrt{Aa}}{c}$$

(e) equal coplanar circles far apart.

$$M_0 = -\pi^2 a \left(\frac{a}{r} \right)^3 \left[1 + \frac{9}{4} \frac{a^2}{r^2} + \frac{375}{64} \frac{a^4}{r^4} + \frac{8575}{512} \frac{a^6}{r^6} + \dots \right] \quad (18A)$$

(f) equal coplanar circles near together.

$$M_0 = 4\pi a \left[(\lambda_0 - 2) - \frac{3}{32} \frac{r^2}{a^2} \left(\lambda_0 - \frac{5}{6} \right) - \frac{45}{8192} \frac{r^4}{a^4} \left(\lambda_0 - \frac{97}{60} \right) - \frac{175}{(512)^2} \frac{r^6}{a^6} \left(\lambda_0 - \frac{251}{140} \right) - \dots \right] \quad (19A)$$

$$\lambda_0 = \log_0 \frac{16a}{r}$$

EXAMPLES

Example 5 (Formula (10A)).— $A = 25$, $a = 20$, $d = 40$, $r = 100$.

Thence $c = 5$, $\mu = 0.4$, $A/r = \frac{1}{4}$, $\frac{a}{A} = 0.8$

$$\begin{array}{ll} P_2(\mu) = -0.260000 & K_1 = 1.64 \\ P_4(\mu) = -0.113000 & K_2 = 3.3296 \\ P_6(\mu) = 0.2926 & K_3 = 7.5597 \\ P_8(\mu) = -0.2670 & \end{array}$$

Thence

$$\begin{aligned} M &= 2\pi^2 \times 0.8 \times 20 \times \frac{1}{64} [-0.260000 + 0.01737 + .00484 + .00108] \\ &= \frac{\pi^2}{2} [-.23671] \end{aligned}$$

The corresponding coaxial case $A = 25$, $a = 20$, $d = 100$ gives, using (17)

$$\begin{aligned} M &= 2\pi^2 \times 0.8 \times 20 \times \frac{1}{64} [1 - 0.15375 + 0.02439 - .00404] \\ &= \frac{\pi^2}{2} (0.86660) \end{aligned}$$

The convergence of (10A) is about as good as (17), and an estimate of the neglected terms may be gained from an examination of the same terms in (17).

Example 6 (Formula (11A)).— $A = 25$, $a = 12.5$, and $\rho = d = \frac{12.5}{\sqrt{2}}$
 $= 8.8375$.

Thence $c = 12.5$, $\frac{d^2}{c^2} = \frac{1}{2}$, $\frac{c^2}{Aa} = \frac{1}{2}$

$$\begin{array}{ll} \varphi_2 = 1.25 & \chi_2 = -0.18735 \\ \varphi_4 = 1.0935 & \chi_4 = -0.41549 \\ \mu^2 = \nu^2 = \frac{1}{\sqrt{2}} & \lambda = \log_e 9.40754 = 2.24151 \end{array}$$

Thence by (11A)

$$\begin{aligned} \frac{M}{4\pi\sqrt{Aa}} &= 0.24151 + 0.24118 - 0.00636 + \dots \\ &= .47633 \end{aligned}$$

For the coaxial case $A = 25$, $a = 12.5$, $\frac{d}{a} = \frac{1}{\sqrt{2}}$ we find in (16),

$$a = \frac{3}{4}, \log_e \frac{8\sqrt{Aa}}{\sqrt{c^2 + d^2}} = 2.22453 \text{ and}$$

$$\begin{aligned} \frac{M}{4\pi\sqrt{Aa}} &= 0.22453 + 0.26595 - 0.00982 + 0.00095 \\ &= 0.48161 \end{aligned}$$

Here, again, the importance of higher-order terms neglected in the eccentric case may be estimated by calculating these terms for the coaxial case. Thus, the next term in (11A) should amount in this problem to about +0.0006.

Example 7 (Formula (13A)).—Equal circles near together $A = a = 25$, $r = 16$, and $\frac{d}{r} = 0.7$

Thence $\frac{r}{a} = \frac{16}{25}$, $\mu = 0.7$, $d = 11.2$

$$\begin{array}{ll} P_2(\mu) = 0.23500 & \psi_2 = -0.4425 \\ P_4(\mu) = -0.41206 & \psi_4 = -0.15153 \\ P_6(\mu) = -0.1253 & \psi_6 = -0.71742 \\ P_8(\mu) = 0.3067 & \lambda = \log_0 \frac{16 \times 25}{16 \times 1.7} = 2.68825 \end{array}$$

Accordingly

$$\frac{M}{4\pi a} = 0.68825 + 0.07648 + 0.00130 + 0.00008 = 0.76611$$

Formula (13) gives for the coaxial case $A = a = 25$, $d = 16$, $\frac{M}{4\pi a} = 0.52573 + 0.16838 - 0.00367 + 0.00020 = 0.69064$

IV. SELF-INDUCTANCE OF A SINGLE-LAYER COIL OR SOLENOID

The inductance of a single-layer coil or solenoid is most easily obtained by basing the solution of the problem on that for a cylindrical current sheet. The latter may be regarded as being equivalent to a single-layer helical winding of flat tape of negligible thickness, the adjacent turns being separated by insulation of infinitesimal thickness.

Rosa⁸ has shown how the difference between the inductance of such an ideal winding and that of an actual winding of round wire may be computed and has prepared tables for facilitating the calculation of this correction.⁹

The following formulas of this section apply only to cylindrical current sheets, and the results obtained by their use require correction by Rosa's method to reduce to the case of an actual winding.

⁸ This Bulletin, 2, pp. 161-187; 1906.

⁹ Scientific Paper No. 169, pp. 122, 197, and 199.

BUTTERWORTH'S FORMULAS FOR INDUCTANCE OF A CYLINDRICAL CURRENT SHEET

In a recent article Butterworth¹⁰ has obtained the differential equation of which Lorenz's absolute formula (72) for the inductance of a solenoid is a solution, and has developed from this the eight possible hypergeometrical series expansions. Of these, his formulas G, I, J, and M resemble closely those denoted by the letters F, H, K, and L, respectively, but the latter formulas are the more convergent, and these only will be considered here. Formulas F and K are new, and H and L are expressed by Butterworth in a form especially convenient for calculation.

There is an advantage in expressing the inductance of a cylindrical current sheet in such a manner as to make clear the connection with the well-known formula for the inductance when the length of the current sheet is infinite; that is, $L_1 = 4\pi^2 a^2 n^2 / b$, and each of the following formulas is so written as to indicate this relation. The quantities k and k' have already been defined on page 120, Scientific Paper No. 169.

Butterworth's formula (F) gives

$$L = L_1 \frac{\sqrt{4a^2 + b^2}}{b} \left[1 - \frac{4}{3\pi} k - \frac{3}{8} k^2 - \frac{5}{64} k^4 - \frac{35}{1024} k^6 - \dots \right] \quad (20A)$$

the general term in the parentheses being

$$-\frac{[1 \cdot 3 \cdot 5 \dots (2n+1)][1 \cdot 1 \cdot 3 \cdot 5 \dots (2n-3)]}{2^{2n} n! (n+1)!} k^{2n}$$

This converges for coils of all lengths, but most rapidly for long coils (k small).

Formula (H) of Butterworth's paper is

$$L = L_1 \left[1 - \frac{4}{3\pi} \frac{2a}{b} + \frac{1}{8} \left(\frac{2a}{b} \right)^2 - \frac{1}{64} \left(\frac{2a}{b} \right)^4 + \frac{5}{1024} \left(\frac{2a}{b} \right)^6 - \dots \right] \quad (21A)$$

in which the general term is

$$(-1)^{n+1} \frac{[1 \cdot 3 \cdot 5 \dots (2n-1)][1 \cdot 1 \cdot 3 \cdot 5 \dots (2n-3)]}{2^{2n} n! (n+1)!} \left(\frac{2a}{b} \right)^{2n}$$

an expression which agrees with that given by Webster and Havlock, formula (79), but which appears here in a somewhat simpler form than that given by them.

¹⁰ Phil. Mag., 81, p. 276; 1926.

The formula (21A) converges only for coils whose length b is greater than the diameter $2a$.

Butterworth's formula (K) may be written

$$L = L_1 \cdot \frac{2}{\pi k'} \left(\frac{2a}{b} \right) \left[\frac{2}{3} (1 - k'^2) + k'^2 \psi_1 - \frac{1 \cdot 3}{2 \cdot 4} k'^4 \psi_2 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6} k'^6 \psi_3 - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 4 \cdot 6 \cdot 8} k'^8 \psi_4 - \dots \right] \quad (22A)$$

in which

$$\begin{aligned} \psi_1 &= \log_e \frac{4}{k'} - \frac{3}{2}, \quad \psi_4 - \psi_3 = \frac{1}{6} + \frac{1}{8} - \frac{1}{3} - \frac{1}{7} \\ \psi_3 - \psi_1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{1} - \frac{1}{3}, \quad \psi_n - \psi_{n-1} = \frac{1}{2n-2} + \frac{1}{2n} - \frac{1}{2n-5} - \frac{1}{2n-1} \\ \psi_3 - \psi_2 &= \frac{1}{4} + \frac{1}{6} - \frac{1}{1} - \frac{1}{5} \end{aligned}$$

and the general term is

$$- \frac{[1 \cdot 1 \cdot 3 \cdot 5 \dots (2n-5)] [3 \cdot 5 \cdot 7 \dots (2n-1)]}{[2 \cdot 4 \cdot 6 \dots (2n-2)] [4 \cdot 6 \cdot 8 \dots (2n)]} k'^{2n} \psi_n$$

This formula converges for all values of k' , but especially well for short solenoids (k' small).

Finally, we have Butterworth's formula (L), which gives for the inductance the value

$$L = L_1 \cdot \frac{2}{\pi} \left(\frac{b}{2a} \right) \left[\psi_1' + \frac{1 \cdot 1}{2 \cdot 4} \psi_2' \left(\frac{b}{2a} \right)^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4 \cdot 6} \left(\frac{b}{2a} \right)^4 \psi_3' + \frac{1 \cdot 1 \cdot 3 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 4 \cdot 6 \cdot 8} \left(\frac{b}{2a} \right)^6 \psi_4' - \dots \right] \quad (23A)$$

in which

$$\begin{aligned} \psi_1' &= \log_e \frac{8a}{b} - \frac{1}{2}, \quad \psi_4' - \psi_3' = \frac{1}{6} + \frac{1}{8} - \frac{1}{3} - \frac{1}{5} \\ \psi_2' - \psi_1' &= \frac{1}{2} + \frac{1}{4} + \frac{1}{1} - \frac{1}{1}, \quad \psi_n' - \psi_{n-1}' = \frac{1}{2n-2} + \frac{1}{2n} - \frac{1}{2n-5} - \frac{1}{2n-3} \\ \psi_3' - \psi_2' &= \frac{1}{4} + \frac{1}{6} - \frac{1}{1} - \frac{1}{3}, \quad = - \frac{3}{n(2n-5)} \frac{[(2n-3)(2n-2)-1]}{(2n-3)(2n-2)} \end{aligned}$$

and the general term is

$$\frac{[1 \cdot 1 \cdot 3 \dots (2n-5)] [1 \cdot 3 \cdot 5 \dots (2n-3)]}{[2 \cdot 4 \cdot 6 \dots (2n-2)] [4 \cdot 6 \cdot 8 \dots 2n]} \psi_n' \left(\frac{b}{2a} \right)^{2n-2}$$

This is Rayleigh and Niven's formula (69), which was extended by Coffin, formula (71), but the general term has not previously

been given. The formula converges only for coils whose length is less than the diameter.

The preceding four formulas cover between them the whole range of possible solenoids, although in the case of coils in which the length and diameter are nearly equal the number of terms necessary to be calculated is not inconsiderable, as may be noted from the following examples. In such cases the calculation of the higher-order terms is materially simplified, if one obtains the expression for the factor necessary to apply to each term in order to obtain the term next following.

EXAMPLE

Example 8.—For the case in which the length equals the diameter, we have $k = k' = \frac{1}{\sqrt{2}}$.

Using formula (20A), 14 terms gives $L = 0.6884229 L_1$

Using formula (22A), 13 terms gives $L = 0.6884230 L_1$

where $L_1 = 4\pi^2 a^2 n^2 / b$

For the coil of example 60 of Scientific Paper No. 169, $a = 27.0862$, $b = 30.551$, we find from eight terms of (23A) the value $L = 0.5546957 L_1$, and seven terms in (22A) give $L = 0.554697 L_1$. The values found by Nagaoka's formulas (77) and (76) differ from the first of these values by only one and three units, respectively, in the last place.

DISK COILS

An extension to the Rayleigh and Niven formula (70) for a disk coil of radial dimension c and mean radius a has been found by Lyle. The additional terms are

$$4\pi n^2 a \left[\left(\frac{11}{46080} \frac{c^4}{a^4} + \frac{103}{105.256^2} \frac{c^6}{a^6} \right) \log \frac{8a}{c} + \frac{1}{2400} \frac{c^4}{a^4} + \frac{98579}{(131072)(44100)} \frac{c^6}{a^6} \right] \quad (70A)$$

Formula (70) with the additional terms of (70A) suffices for values of c as great as the mean radius a .

For those coils where c is large compared with a —that is, for disk coils in which the inner radius is small compared with the outer radius—a formula has been developed by Spielrein¹¹ who puts

$$L = n^2 A f(\alpha) \quad (24A)$$

¹¹ Arch. für Elektrotechnik, 8, p. 187; 1915.

in which

A = The outer radius of the disk

α = the ratio of inner radius to outer radius

$$G = \left(\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots \right) = 0.9159656 \dots$$

$$u_n = \frac{\pi}{2} \left[\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \right]^2$$

and

$$\begin{aligned} f(\alpha) = & \frac{8\pi}{3} \frac{1}{(1-\alpha)^3} \left[(2G-1) - \sum_1^{\infty} \frac{2u_n}{(2n-1)(2n+1)} \right. \\ & - \alpha^3 \left\{ \pi \log 2 - (2G-1) - \frac{\pi}{4} + \frac{\pi}{2} \log \frac{1}{\alpha} \right. \\ & \left. \left. - \sum_1^{\infty} \frac{2u_n(2n+1)}{2n(2n+2)} \alpha^{2n} \right\} \right] \\ = & \frac{1}{(1-\alpha)^3} \left[6.96957 - \alpha^3 (30.3008 \log \frac{1}{\alpha} + 9.08008) \right. \\ & + 1.48044 \alpha^5 + 0.33045 \alpha^7 \\ & \left. + 0.12494 \alpha^9 + 0.06038 \alpha^{11} + 0.0337 \alpha^{13} + \dots \right] \end{aligned}$$

Spielrein gives a second formula for values of α between 0.5 and 1, but if we remember that $\frac{c}{2a} = \frac{1-\alpha}{1+\alpha}$, it is easy to show that this is equivalent to (70) and (70A), lacking the term in c^6/a^6 .

In Table E is reproduced a collection of values of $f(\alpha)$ calculated by Spielrein for a number of values of α .

V. SELF-INDUCTANCE OF A CIRCULAR COIL OF RECTANGULAR CROSS SECTION

For the calculation of the inductance of a circular coil of rectangular cross section, whose dimensions b and c are relatively small compared with the mean radius a , the most accurate formula previously available has been that of Weinstein,¹² which appears as (88) in Scientific Paper No. 169. In the next volume of Wiedemann's *Annalen*, after that containing Weinstein's article, Stefan¹³ published what is the same formula, but so arranged as to facilitate numerical calculations. (See (90) in Scientific Paper No. 169.) There is nothing in Stefan's article to show that he was acquainted with Weinstein's work.

¹² Wied. Ann., 21, p. 329; 1884.

¹³ Wied. Ann., 22, p. 113; 1884.

The formula given by Stefan is reproduced here.

$$L = 4\pi an^2 \left\{ \left(1 + \frac{3b^2 + c^2}{96a^2} \right) \log \frac{8a}{\sqrt{b^2 + c^2}} - \gamma_1 + \frac{b^2}{16a^2} \gamma_2 \right\} \quad (90)$$

Values of the two quantities γ_1 and γ_2 are given by Stefan in tables which he prepared to make his formula more generally useful. Changing his nomenclature to agree with that of this paper, the two equations given by him for the calculation of γ_1 and γ_2 are

$$\begin{aligned} \gamma_1 &= \frac{\pi x}{3} - \frac{1}{12x^2} \log(1+x^2) - \frac{x^2}{12} \log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3}\left(x - \frac{1}{x}\right) \tan^{-1}x - \frac{1}{12} \\ \gamma_2 &= \frac{1}{6} \left[\frac{69}{20} + \frac{221}{60} \cdot \frac{1}{x^2} - \frac{1}{10x^4} \log(1+x^2) + \frac{x^2}{2} \log\left(1 + \frac{1}{x^2}\right) \right. \\ &\quad \left. - \frac{8\pi x}{5} + \frac{16x}{5} \tan^{-1}x \right] \end{aligned} \quad (25A)$$

in which $x = b/c$, the ratio of the axial dimension of the cross section to the radial.

Stefan's formula and tables have been reproduced in a number of handbooks, including the Bureau of Standards' collection, with the statement that both γ_1 and γ_2 are unchanged, when b and c are interchanged; that is, that γ_1 and γ_2 are the same functions of c/b that they are of b/c .

This statement is true of γ_1 , but does not hold for γ_2 , as may be seen from the defining equations (25A).

MODIFICATION OF STEFAN'S FORMULA FOR THE CASE ($c > b$)

The formula (25A) shows that γ_2 grows rapidly larger as c is increased relatively to b , and approaches infinity as its limit when the ratio b/c approaches zero. In such cases interpolation of the values of γ_2 becomes difficult. This difficulty may, however, be avoided if, for the case $c > b$, we write Stefan's formula in the form¹⁴

$$L = 4\pi an^2 \left\{ \left(1 + \frac{3b^2 + c^2}{96a^2} \right) \log \frac{8a}{\sqrt{b^2 + c^2}} - \gamma_1 + \frac{c^2}{16a^2} \gamma_3 \right\} \quad (26A)$$

In this equation the quantity γ_3 is related to γ_2 by the equation $\gamma_3 = b^2/c^2 \cdot \gamma_2$, and sufficient values of γ_3 are included in Appendix B to allow of accurate interpolation. The defining equation for γ_3 is

$$\begin{aligned} \gamma_3 &= \frac{1}{6} \left[\frac{69x^2}{20} + \frac{221}{60} - \frac{8\pi x^3}{5} + \frac{16x^3}{5} \tan^{-1}x \right. \\ &\quad \left. - \frac{1}{10x^2} \log(1+x^2) + \frac{x^4}{2} \log\left(1 + \frac{1}{x^2}\right) \right] \end{aligned} \quad (27A)$$

¹⁴ This form of the equation, with tables for computation, was first given in the 1916 revision of Scientific Paper No. 169.

LYLE'S FORMULA

The Weinstein-Stefan formula was obtained by integrating the series expression (10), for the mutual inductance of two coaxial circles, over the rectangular cross section of the coil in question. In this integration are included terms of second order only in c/a and b/a . If the dimensions of the cross section are small, relatively to the mean radius of the coil, this approximation will suffice. In a good many cases, however, the further terms are not negligible, and in any case it is desirable to be able to prove that they are negligible.

To carry out the integration of (14) so as to include higher-order terms is a difficult matter on account of the large number of terms which must be treated. In a recent paper Lyle¹⁵ has shown how to simplify the work so that further terms may be obtained in Weinstein's formula, and has published the expressions for the terms of fourth and sixth order, together with tables for calculating the fourth-order term.

The author of the present paper has called Prof. Lyle's attention to an error in one of the coefficients of the sixth-order terms of the extension of (14) upon which the integration was performed, and he has very kindly repeated his work and supplied the correction of the single term affected.

In addition Prof. Lyle has been so good as to communicate to us additional tables; not heretofore published, with his permission for their incorporation with this paper. The following is quoted from Prof. Lyle's letter:

In a former paper I have extended Maxwell's and Weinstein's formula for the self-inductance of a circular coil of rectangular section to the sixth, and, following Stefan, have given tables by means of which the result, up to the fourth order, may easily be applied to the calculation of inductances. I have lately recalculated the figures given in one of these sets of tables and extended the latter to the sixth order.

Thus, if uniform current density over the whole section of the coil be assumed, its self-inductance may be written in the form

$$L = 4\pi an^2 \left[\left(1 + m_1 \frac{d^2}{a^2} + m_2 \frac{d^4}{a^4} + m_3 \frac{d^6}{a^6} \right) \log \frac{8a}{d} - l_0 + l_1 \frac{d^2}{a^2} + l_2 \frac{d^4}{a^4} + l_3 \frac{d^6}{a^6} \right] \quad (28A)$$

in which

a is the mean radius of the coil
 n the number of turns
 $d^2 = b^2 + c^2$, where
 b = the axial width of the coil
 c = the radial depth of the coil

¹⁵ Phil. Trans., 218A, pp. 421-435; 1914.

In Appendix C Lyle's values of $m_1, m_2, m_3, l_0, l_1, l_2$, and l_3 are given for different values of c/b for thick coils—that is, those in which b is greater than c —and in Appendix D are given their values for different values of b/c for thin coils—that is, those in which b is less than c .

The following relations exist between Lyle's constants and the quantities γ_1, γ_2 , and γ_3 of formulas (90) and (26A):

$$\begin{aligned} l_0 &= \gamma_1 \\ \gamma_2 &= 16 l_1 (1 + c^2/b^2) \\ \gamma_3 &= 16 l_1 (1 + b^2/c^2) \end{aligned}$$

A second form was given by Lyle to his formula in his original paper. Formula (28A) has, however, the advantage that it differs from Stefan's formula only in that the fourth and sixth order terms are added. Therefore, in any given case, a rough preliminary calculation will suffice to show whether the higher-order terms are of importance. In a great many cases it may thus be shown that Stefan's formula is sufficient, and only in extreme cases (coils of relatively very large cross section) will the sixth and higher order terms be important. In the latter case no other formula is yet available for obtaining such an accurate value with so little labor. Lyle's formula, however, fails for the case of coils whose length b is considerably greater than the mean radius a .

BUTTERWORTH'S FORMULA FOR SELF-INDUCTANCE OF A LONG MULTIPLE-LAYER COIL

Butterworth¹⁶ has developed a series formula for the case of a long coil whose winding depth is rather large. Its region of convergence, for coils whose length is greater than four times the outer radius and whose winding depth is greater than about one-fifth the mean radius, covers the case of coils whose cross section is so great that Stefan's and Lyle's formulas are not sufficiently convergent.

Changing Butterworth's nomenclature to agree with that previously employed in this paper, his method may be summarized as follows:

Writing L_1 = the inductance of an infinite cylindrical current sheet of the same mean radius as the coil, the inductance of a finite solenoid having the same length as the given coil is $L_s = KL_1$,¹⁷

¹⁶ Proc. Phys. Soc., London, 27, p. 371; 1915.

¹⁷ Scientific Paper No. 169, p. 119, formula (75).

and we may write for the inductance of the actual coil, $L = L_2 + \Delta L$, in which

$$\frac{\Delta L}{L_1} = -\frac{1}{3} \frac{c}{a} \left[1 - \frac{c}{4a} - \frac{1}{2\pi} \frac{c}{b} \left(\log \frac{8a}{c} - \frac{23}{12} \right) + \frac{1}{160\pi} \frac{c^3}{a^3} \frac{a}{b} \left(\log \frac{8a}{c} - \frac{1}{20} \right) - \frac{1}{4} \frac{c}{a} \frac{a^2}{b^2} \left(1 - \frac{7}{4} \frac{a^2}{b^2} + \frac{17}{4} \frac{a^4}{b^4} - \dots \right) - \frac{1}{96} \frac{c^3}{a^3} \frac{a^2}{b^2} \left(1 - \frac{39}{10} \frac{a^2}{b^2} + \dots \right) - \dots \right] \quad (29A)$$

EXAMPLES

Example 9.—As an example, we may consider one of the coils treated by Butterworth, viz, $b/a = 4$, $c/a = 0.2$, the value of the mean radius and the number of turns being so chosen, for simplicity, that $\frac{\pi^2 n^2 a^2}{b^2} = 1\,000\,000$. This gives $L_1 = 16$ millihenrys.

The value of K for $2a/b = 0.5$ is 0.818136 ,¹⁸ and, therefore, $L_2 = 13.09017$.

$$\begin{aligned} \frac{\Delta L}{L_1} &= -\frac{1}{15} \left[1 - .05 - \frac{1}{40\pi} \left(\log_e 40 - \frac{23}{12} \right) + \frac{1}{80000\pi} \left(\log_e 40 - \frac{1}{20} \right) - \frac{1}{320} \left(1 - \frac{7}{64} + \frac{17}{1024} \right) - \frac{1}{192000} \left(1 - \frac{39}{160} \right) \right] \\ &= -\frac{1}{15} \left(1 - .05 - .014103 + .000014 - .002845 - .000005 \right) = -.062204 \\ \text{or } \Delta L &= -.09526 \text{ mh.} \end{aligned}$$

The only other formula available as a check is Rosa's formula (91), which gives the result $\Delta L = -1.0207$ mh, or a difference from Butterworth's formula of 0.0255 mh, or about two parts in a thousand of the total inductance. Lyle's formula can not be used in this case.

Butterworth explains the above difference as being due to the neglect of the curvature in the geometric mean-distance formulas used in obtaining B_s in Rosa's formula (91).

For the coil $b = 10$, $c = 1$, $a = 10$, $n = 1000$ we find

$$\begin{aligned} \frac{L}{4\pi} &= 15.53984 \text{ Lyle's formula to fourth order} \\ &= 15.54071 \text{ Lyle's formula to sixth order} \\ &= 15.5361 \text{ Rosa's formula (91) (see example 66 of Scientific Paper No. 169).} \end{aligned}$$

Butterworth's formula is not applicable to a coil as short as this.

¹⁸ Scientific Paper No. 169, Table XXI.

Here, again, Rosa's formula gives a result somewhat too small, although the difference in this case is only 3 in 10 000.

- These checks on Rosa's formula are valuable, since it is the only formula yet available in the region where neither Lyle's formula (28A) nor Butterworth's formula (29A) converges well.

The error due to the neglect of the curvature in applying the geometric mean-distance formulas in obtaining a result by Rosa's formula will not usually be regarded as important. It would not, however, be difficult to obtain a correction for this effect, although the formula thus obtained would not be so simple to use as (91).

NOTE ON COHEN'S APPROXIMATE FORMULA (92)

Cohen's formula (92) is applicable to a coil of several layers. The formula presupposes that the rectangular cross section is divided into a number of equal axial rectangles equal to the number of layers, and the formula for the inductance involves the radii of the layers.

Butterworth has shown that assuming a coil of given cross-sectional dimensions the inductance as calculated by (92) comes out quite different according to the number of layers assumed in the cross section. He goes on to show that this may be explained by the fact that in the derivation of (92) the approximations made at certain points of the demonstration are not sufficient to give the accuracy claimed by Cohen. However, for a certain choice of the number of sections, different in each case, and not a priori determinate, the result may lie quite close to the true result.

As an example of these points Butterworth has calculated, by means of Cohen's formula, the inductance of the coil in the example next preceding but one for different assumptions with regard to the quantity m in (92).

$$m = \quad 1 \quad \quad 2 \quad \quad 3 \quad \quad 4 \quad \quad 5 \quad \quad 10 \text{ infinite.}$$

$$L = 12.70 \quad 12.11 \quad 12.07 \quad 12.06 \quad 12.09 \quad 12.14 \quad 12.19 \text{ millihenrys.}$$

The correct value of the inductance for this case is, to four significant figures, 12.09 millihenrys.

VI. SELF AND MUTUAL INDUCTANCE OF LINEAR CONDUCTORS

Formulas are given in section 8 of Scientific Paper No. 169 for the calculation of the self-inductance of straight wires of different cross section and for the mutual inductance of two such conductors when placed parallel to one another.

Such cases are easily treated by the method of the geometric mean distance. For the calculation of the self-inductance of a straight conductor of any desired cross section we have only to calculate the mutual inductance of two parallel straight filaments placed at a distance apart equal to the geometric mean distance of the cross section from itself.

Similarly the mutual inductance of two parallel straight conductors is equal to the mutual inductance of two parallel straight filaments whose distance apart is taken equal to the geometric mean distance of the area of cross section of one conductor from the cross section of the other.

The calculation of the self-inductance of any straight conductor or any pair of parallel straight conductors may, therefore, be accomplished by substituting the proper geometric mean distance for R in the formula

$$M = 2l \left[\log \frac{2l}{R} - 1 + \frac{R}{l} \right] \quad (30A)$$

which is the expression (99) of Scientific Paper No. 169 for the mutual inductance of two filaments of length l , at a distance R apart, which is small compared with their length. In most practical cases the last term of (30A) may safely be neglected.

To aid in making calculations by this method, the formulas for geometric mean distance, in a number of important cases, are presented in section 9 of Scientific Paper No. 169.

The inductance of a circuit composed of a number of linear conductors may, in general, be found by taking the sum of the self-inductances of the individual conductors and the mutual inductances of each wire on all the others. In the case of a return circuit—that is, a circuit consisting of two parallel wires in which the direction of the current in one is opposed to the direction of the current in the other—the inductance of the remainder of the circuit being negligible in comparison, $L = L_1 + L_2 - 2M$, in which L_1 and L_2 are the self-inductances of the two wires and M is their mutual inductance.

This equation, taken in connection with (30A), if the last term in the latter be neglected, gives as a general formula for a return circuit

$$L = 2l [2 \log R_{12} - \log R_1 - \log R_2] \quad (31A)$$

in which R_1 and R_2 are, respectively, the geometric mean distances of the cross sections of the two wires on themselves, and R_{12} is the geometric mean distance of the cross sections of the two wires.

If the cross sections of the two wires are the same, this formula becomes

$$L = 4l \log \frac{R_{12}}{R_1} \quad (32A)$$

These formulas have been employed, in conjunction with those of section 8 of Scientific Paper No. 169, to obtain the inductance of a considerable number of the special circuits treated in that section.

INDUCTANCE OF SHUNTS

In recent years the use of shunts of large carrying capacity for measuring the current in alternating-current circuits has lent a very practical importance to a knowledge of the inductance in such cases.

As such shunts are constructed, it is true, the inductance is very small (of the order of a few abhenrys), but since the resistance is often less than a thousandth of an ohm, the phase angle between electromotive force and current may, even with such a small inductance, depart widely from zero, so that the assumption that such apparatus is noninductive may cause very serious error in the measurement of current and power.

We will consider here shunts of two main types—(a) shunts of flat metal strip, bent so as to form a return circuit whose parallel elements are very close together, and (b) tubular shunts.

(a) *Shunts of flat strip*.—If we neglect the thickness of the strip, in comparison with its width and the distance apart of the two parallel conductors, we may calculate $\log R_1$ and $\log R_2$ from (123) and $\log R_{12}$ by (132). The expression resulting from the substitution of these quantities in (31A) may, however, be put in a more serviceable form, if we expand the logarithmic and inverse trigonometric functions. Putting w for the width of the strip and g for the distance between the strips, then, if g/w is small,

$$L = 4l \left[\frac{\pi g}{w} + \frac{g^2}{w^2} \log \frac{g}{w} - \frac{3}{2} \frac{g^3}{w^2} - \frac{1}{12} \frac{g^4}{w^4} - \dots \right] \quad (33A)$$

Since g/w is not always small, we should, in the case of strips at some distance apart, use the exact expressions for $\log R_1$ and $\log R_{12}$.

More often, however, we will be unable to neglect the thickness of the strip. Silsbee¹⁹ has recently treated this case by calculating $\log R_1$ by (124), the formula for the geometric mean distance of a

¹⁹ Scientific Paper No. 281, pp. 375-421; 1916.

rectangle from itself, and $\log R_{12}$ by Gray's formula for two parallel rectangles.²⁰ Expanding these quantities in series involving b/w , and g/w , the thickness of the strip being denoted by b and the thickness of the insulating space between the two strips by g , he finds finally

$$L = 4l \left[\frac{\pi}{3}(3\beta - \delta) - \frac{25}{12}\beta^2 - \frac{1}{12}\beta^4 - \frac{1}{12}\beta^2\delta^2 + \frac{1}{12\delta^2}(\alpha^4 \log \alpha - 2\beta^4 \log \beta + \gamma^4 \log \gamma - 2\delta^4 \log \delta) \right] \quad (34A)$$

in which

$$\alpha = \frac{2b+g}{w}, \beta = \frac{b+g}{w}, \gamma = \frac{g}{w}, \delta = \frac{b}{w}.$$

This expression reduces to the preceding expression (33A) if we let δ approach zero.

To show that, in practice, the difference between the two formulas (33A) and (34A) may be large, we may consider an example given by Silsbee:

$$l = 35.62, b = 0.1064, w = 4.986, g = 0.0336.$$

Here, although the metal used is only about 1 mm thick, the thickness of the insulation between the two legs of the shunt is only about one-third of this, so that b/w is about three times as great as g/w , instead of being negligible, as (33A) supposes.

Making the calculation by Silsbee's formula, the terms taken in order are as follows:

$$\begin{aligned} &4l (0.065867 - 0.001642 - 7 \times 10^{-8} - 3 \times 10^{-8} - 0.003282 \\ &\quad + 0.001288 - 0.000002 + 0.000292) \\ &= 4l (0.062521) = 8.91 \times 10^{-9} \text{ henry} \end{aligned}$$

The value found by (33A) is

$$4l (0.021171 - 0.000227 - 0.000068) = 4l (0.020876)$$

which is less than one-third of the correct value. This example, then, illustrates the fact that formula (33A) should be used only in those cases where the quantity b/w can be shown to be negligible in comparison with g/w . Such cases are likely to be rare, except when the distance of the strips apart is comparable with the width of the strip, a condition not conducive to good design. Formula (33A) is, therefore, of limited usefulness.

Silsbee has also treated the case where the return circuit is made up of two strips of different thickness. Suppose two strips

²⁰ This Bulletin, 8, p. 6, 1907, formula (8).

of the same width w , but of different thicknesses b and c , the thickness of the insulating space between them being g , then if we denote by L_1 the inductance of a length l of the conductor of thickness b , the other conductor serving as a return, and by L_2 the inductance of a length l of the conductor of thickness c , with its return through the other conductor, Silsbee shows (page 378 of his article) that $L_1 = 2l (\log R_{12} - \log R_1)$, where R_1 is the geometric mean distance of strip b from itself and R_{12} is the geometric mean distance of the cross sections of the two strips.

Calculating these geometric mean distances by the formulas used in the previous case, and expanding the resulting expressions, he finds

$$L_1 = 2l \left[\pi \left(\gamma + \frac{\delta}{6} + \frac{\eta}{2} \right) - 25 \left(\frac{\eta\delta}{24} + \frac{\gamma\alpha}{12} + \frac{\eta^2}{36} + \frac{\delta^2}{72} \right) + \frac{1}{12\delta\eta} \left(\alpha^4 \log \alpha - \kappa^4 \log \kappa - \lambda^4 \log \lambda + \gamma^4 \log \gamma - 2\delta^2\eta \log \delta \right) \right] \quad (35A)$$

where

$$\alpha = \frac{b+c+g}{w}, \quad \gamma = \frac{g}{w}, \quad \delta = \frac{b}{w}, \quad \eta = \frac{c}{w}, \quad \kappa = \frac{b+g}{w}, \quad \lambda = \frac{c+g}{w}$$

The inductance L_2 is found by interchanging the letters δ and η , and the inductance of the complete circuit is $L = L_1 + L_2$.

Equation (34A) can be derived from (35A) by letting $b = c$ in the complete expression for $(L_1 + L_2)$.

(b) *Tubular shunts*.—These are generally constructed of two concentric tubes of resistance metal, one of which forms a return for the other, the two potential leads being attached at points which differ in different designs. In case one or both of the potential leads are so disposed that an electromotive force is induced in the lead, this will change the effective reactance of the shunt, which may be defined as the ratio of the quadrature component of the voltage between those ends of the potential leads which are attached to the measuring apparatus, to the current in the shunt. For a full treatment of this question the reader is referred to page 378 of Silsbee's article.

Silsbee has attacked the problem by calculating directly the linkages of the magnetic flux with the different elements of the shunt, and has given in series form the inductance for four practical designs of tubular shunt. In accordance with his suggestion that these formulas more clearly illustrate the procedure which may be adopted in deriving formulas for similar cases not included in these examples, if their relation to the geometric

mean distances involved be made clear, this method of treatment will here be outlined.

The geometric mean distance of an annulus of inner radius a_2 and outer radius a_1 is given by formula (129)

$$\log R_1 = \log a_1 - \frac{a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} + \frac{1}{4} \frac{(3a_2^2 - a_1^2)}{(a_1^2 - a_2^2)}$$

which Silsbee develops in the series form

$$\log R_1 = \log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^5}{40} + \dots \quad (36A)$$

useful in the case when the ratio $t = (a_1 - a_2)/a_1$ of the thickness to the outer radius is small.

The geometric mean distance of an annulus from any area entirely inside of it is by (135), if for the outer radius we put a_3 and for the inner radius a_4 ,

$$\log R_{12} = \frac{a_3^2 \log a_3 - a_4^2 \log a_4}{a_3^2 - a_4^2} - \frac{1}{2}$$

or

$$\log R_{12} = \log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60} + \dots \quad (37A)$$

in terms of the quantity $s = \frac{a_3 - a_4}{a_3}$. This formula gives the geo-

metric mean distance of the cross sections of two concentric rings.

For the design *a* of tubular shunt, treated by Silsbee (p. 400), in which one tube forms a return circuit for the other, and the potential leads are brought out in the same plane at a distance l from the junction of the tubes,

$$\begin{aligned} L &= L_1 + L_2 - 2M \\ &= 2l[2 \log R_{12} - \log R_1 - \log R_2] \\ &= 2l \left[2 \left(\log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60} + \dots \right) \right. \\ &\quad \left. - \left(\log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^5}{40} + \dots \right) \right. \\ &\quad \left. - \left(\log a_3 - \frac{s}{3} + \frac{s^3}{30} + \frac{s^5}{40} + \dots \right) \right] \end{aligned}$$

This may be reduced by the relations

$$\begin{aligned}\log a_3 - \log a_1 &= \log \frac{a_3}{a_4} + \log \frac{a_4}{a_1} \\ \log \frac{a_4}{a_3} &= \log \frac{[a_3 - (a_3 - a_4)]}{a_3} = \log (1 - s) \\ \log \frac{a_1}{a_4} &= \log \frac{[a_4 - (a_4 - a_1)]}{a_4} = \log (1 - u)\end{aligned}$$

and expanding $\log (1 - s)$ and $\log (1 - u)$, we obtain Silsbee's equation

$$L = l \left[\frac{2}{3}t + 2u + \frac{2}{3}s + u^2 + \frac{2}{3}s^2 + \frac{3}{5}s^3 + \frac{2}{3}u^3 - \frac{t^3}{15} + \dots \right] \quad (38A)$$

For his case *b*, in which the potential leads are attached to the outer tube at points a distance l apart and are carried away at right angles to the axis of the tube

$$\begin{aligned}L &= L_2 - M = 2l (\log R_{12} - \log R_2) \\ &= 2l \left[\left(\log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60} + \dots \right) - \left(\log a_3 - \frac{s}{3} + \frac{s^3}{30} + \frac{s^4}{40} \right) \right] \\ &= l \left(-\frac{s}{3} - \frac{s^2}{6} - \frac{s^3}{15} - \frac{s^4}{60} + \dots \right)\end{aligned} \quad (39A)$$

Silsbee's case *c* is like the preceding, except that the potential leads are attached to the inner tube and are brought perpendicularly out through holes in the outer tube. For this arrangement

$$\begin{aligned}L &= L_1 - M = 2l [\log R_{12} - \log R_1] \\ &= 2l \left[\left(\log a_3 - \frac{s}{2} - \frac{s^2}{12} + \frac{s^4}{60} \right) - \left(\log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^4}{40} \right) \right]\end{aligned}$$

which, remembering that

$$\log \frac{a_1}{a_3} = \log \frac{a_1}{a_4} + \log \frac{a_4}{a_3} = \log (1 - u) + \log (1 - s)$$

gives on expansion of the logarithms

$$L = l \left[2u + s + \frac{2}{3}t + u^2 + \frac{5}{6}s^2 + \dots \right] \quad (40A)$$

Silsbee's final case *d* uses potential leads attached to the inner tube at a distance l apart, one of them being carried away inside the inner tube parallel to its axis. It is necessary, therefore, in

this case to take into account the electromotive force induced in this lead, which depends upon the geometric mean distance of the inner tube on an area inside of it.

The inductive effect of the outer tube on the inner tube is equal and opposite to the effect of the outer tube on the potential lead, so that we find for the inductance simply

$$\begin{aligned}
 L &= L_1 - M_o = 2l [\log R_o - \log R_1] \\
 &= 2l \left[\left(\log a_1 - \frac{t}{2} - \frac{t^2}{12} + \frac{t^4}{60} + \dots \right) - \left(\log a_1 - \frac{t}{3} + \frac{t^3}{30} + \frac{t^4}{40} + \dots \right) \right]_{(41A)} \\
 &= l \left[-\frac{t}{3} - \frac{t^2}{6} - \frac{1}{15}t^3 - \frac{t^4}{60} + \dots \right]
 \end{aligned}$$

It is to be noted that in cases *b* and *d* the inductance comes out negative; that is, that the potential between the terminals lags behind the current in phase.

The method used in deriving formulas (38A) to (41A), inclusive, may be used to derive the inductance in other cases not here treated.

WASHINGTON, July 27, 1917.

TABLE B.—Values of y_2 and y_3 in Formulas (90) and (26A)

[Radial Depth of Cross Section Greater than the Axial Breadth]

$\frac{b}{c}$	y_2	y_3	Δ_1	Δ_2
0	∞	0.59722	136	237
0.05	239.43	.59858	373	191
.10	60.231	.60231	564	153
.15	27.020	.60795	717	124
.20	15.378	.61512	841	100
.25	9.9765	.62353	941	80
.30	7.0327	.63294	1021	67
.35	5.2502	.64315	1088	54
.40	4.0876	.65403	1142	47
.45	3.2861	.66545	1189	41
.50	2.7093	.67734	1230	37
.55	2.2798	.68964	1267	35
.60	1.9509	.70231	1302	35
.65	1.6931	.71533	1337	33
.70	1.4871	.72870	1370	34
.75	1.3198	.74240	1404	36
.80	1.1819	.75644	1440	34
.85	1.0669	.77084	1474	39
.90	.9698	.78558	1513	38
.95	.8872	.80071	1551	
1.00	.8162	.81622		

TABLE C.—Constants in Lyle's Formula (28A), Thick Coils, $b > c$

Communicated by Prof. Lyle.

c/b	100 m_1	10 ⁴ m_2	10 ⁴ m_3	1 ₀	100 l_1	10 ⁴ l_2	10 ⁴ l_3	c/b
0.00	3.125000	-9.7656	76.29	0.5000000	0.781250	6.5104	-69.30	0.00
.025	3.123699	-9.7463	76.01	.5252663	.783689	6.4896	-68.94	.025
.05	3.119805	-9.6886	75.14	.5489951	.790984	6.4274	-67.88	.05
.10	3.104373	-9.4613	71.79	.5924342	.819830	6.1838	-63.77	.10
.15	3.079157	-9.0942	66.50	.6310248	.866769	5.7944	-57.35	.15
.20	3.044872	-8.6040	59.67	.6652018	.930230	5.2827	-49.20	.20
.25	3.002451	-8.0115	51.78	.6953236	1.008207	4.6774	-39.99	.25
.30	2.952982	-7.3402	43.36	.7217163	1.098406	4.0102	-30.44	.30
.35	2.897643	-6.6144	34.88	.7446891	1.198386	3.3128	-21.17	.35
.40	2.837644	-5.8573	26.77	.7645392	1.305696	2.6145	-12.70	.40
.45	2.774168	-5.0903	19.34	.7815523	1.417987	1.9404	-5.39	.45
.50	2.708333	-4.3316	12.82	.7960019	1.533097	1.3107	+ .55	.50
.55	2.641155	-3.5961	7.32	.8081473	1.649113	.7400	+ 5.05	.55
.60	2.573529	-2.8951	+ 2.89	.8182324	1.764399	+ .2378	8.17	.60
.65	2.506224	-2.2366	- .51	.8264842	1.877606	- .1912	10.03	.65
.70	2.439877	-1.6260	- 2.94	.8331124	1.987664	- .5460	10.82	.70
.75	2.375000	-1.0656	- 4.52	.8383088	2.093763	- .8287	10.73	.75
.80	2.311992	- .5563	- 5.37	.8422476	2.195318	-1.0437	9.97	.80
.85	2.251149	- .0970	- 5.61	.8450864	2.291944	-1.1966	8.73	.85
.90	2.192680	+ .3141	- 5.37	.8469663	2.383421	-1.2939	7.16	.90
.95	2.136717	+ .6800	- 4.75	.8480134	2.469663	-1.3425	5.42	.95
1.00	2.083333	+1.0037	- 3.85	.8483397	2.550686	-1.3490	3.62	1.00
1.05	2.032551	+1.2888	- 2.75	.8480444	2.626593	-1.3199	1.83	1.05
1.10	1.984351	+1.5387	- 1.53	.8472152	2.697542	-1.2613	+ .13	1.10

TABLE D.—Constants in Lyle's Formula (28A), Thin Coils, $c > b$

Communicated by Prof. Lyle.

b/c	100 m ₁	10 ⁴ m ₂	10 ⁶ m ₃	l ₀	100 l ₁	10 ⁴ l ₂	10 ⁶ l ₃	b/c
0.00	1.041667	2.3872	14.97	0.5000000	3.732639	4.1667	17.05	0.00
.025	1.042978	2.3913	15.01	.5252663	3.732506	4.1614	17.00	.025
.05	1.046862	2.4035	15.13	.5489951	3.731810	4.1434	16.81	.05
.10	1.062294	2.4508	15.59	.5924342	3.727159	4.0584	15.88	.10
.15	1.087510	2.5237	16.25	.6310248	3.716052	3.8971	14.13	.15
.20	1.121795	2.6140	16.96	.6652018	3.696644	3.6550	11.60	.20
.25	1.164216	2.7115	17.56	.6953236	3.667845	3.3359	8.45	.25
.30	1.213685	2.8057	17.91	.7217163	3.629250	2.9510	+4.98	.30
.35	1.269024	2.8859	17.89	.7446891	3.581036	2.5161	+1.49	.35
.40	1.329023	2.9430	17.42	.7645392	3.523847	2.0499	-1.73	.40
.45	1.392498	2.9694	16.49	.7815523	3.458662	1.5715	-4.42	.45
.50	1.458333	2.9601	15.11	.7960019	3.386676	1.0990	-6.44	.50
.55	1.525512	2.9119	13.36	.8081473	3.309190	+ .6479	-7.68	.55
.60	1.593137	2.8239	11.33	.8182324	3.227522	+ .2308	-8.16	.60
.65	1.660442	2.6971	9.11	.8264842	3.142942	- .1431	-7.93	.65
.70	1.726790	2.5337	6.83	.8331124	3.056619	- .4677	-7.09	.70
.75	1.791667	2.3372	4.58	.8383088	2.969599	- .7400	-5.77	.75
.80	1.854675	2.1114	+2.44	.8422476	2.882783	- .9592	-4.09	.80
.85	1.915518	1.8608	+ .48	.8450864	2.796929	-1.1265	-2.21	.85
.90	1.973987	1.5898	-1.24	.8469663	2.712655	-1.2448	- .23	.90
.95	2.029950	1.3028	-2.69	.8480134	2.630449	-1.3175	+1.74	.95
1.00	2.083333	1.0037	-3.85	.8483397	2.550686	-1.3490	+3.62	1.00
1.05	2.134116	.6963	-4.71	.8480444	2.473638	-1.3437	+5.34	1.05
1.10	2.182315	.3839	-5.28	.8472152	2.399492	-1.3062	+6.86	1.10

TABLE E.—Values of $f(\alpha)$ in Formula (24A)

α	$f(\alpha)$	α	$f(\alpha)$	α	$f(\alpha)$
0	6.969573	0.35	14.19157	0.70	28.08799
0.05	7.715806	.40	15.64577	.75	31.20646
.10	8.555811	.45	17.2339	.80	34.91552
.15	9.487594	.50	18.9740	.85	39.52880
.20	10.51246	.55	20.8897	.90	45.74241
.25	11.63398	.60	23.01363	1.00	∞
.30	12.85776	.65	25.39097		

THERMAL EXPANSION OF ALPHA AND OF BETA BRASS BETWEEN 0 AND 600° C, IN RELATION TO THE MECHANICAL PROPERTIES OF HETEROGENEOUS BRASSES OF THE MUNTZ METAL TYPE

By P. D. Merica and L. W. Schad

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I. INTRODUCTION

In the course of recent investigation of the cracking or fracturing of brass articles of the type composition, 60 per cent copper and 40 per cent zinc, it has been possible to give adequate explanation for failure in most of the individual cases. These failures have been ascribed to excessive initial or service stress in conjunction with corrosion, or to the improper execution of the forging operation by which the article was formed. A number of instances of cracking in brass of this type have, however, come to the attention of the authors, for which these explanations can not be applicable. Such an instance is the following:

A $\frac{3}{8}$ -inch diameter naval brass hook bolt, which had been heated to "cherry red" and quenched in warm water, was used in the support of a strainer plate in a water-filter plant, under a tensional stress of from 10 000 to 15 000 pounds per square inch. After about 60 days it broke off, with practically no elongation. There was little initial stress in this bolt, and the load stress was not above the proportional limit; the subsequent cracking of the bolt appears, therefore, rather mysterious.

Several naval brass rivets have been examined which had fractured at the shoulder, under a tensional stress no greater than that caused by the restrained thermal expansion of the rivet after heating.

These cases have been described more fully.¹

In all of these instances, in which the usual mode of explanation of cracking has failed, it was discovered that the brass article

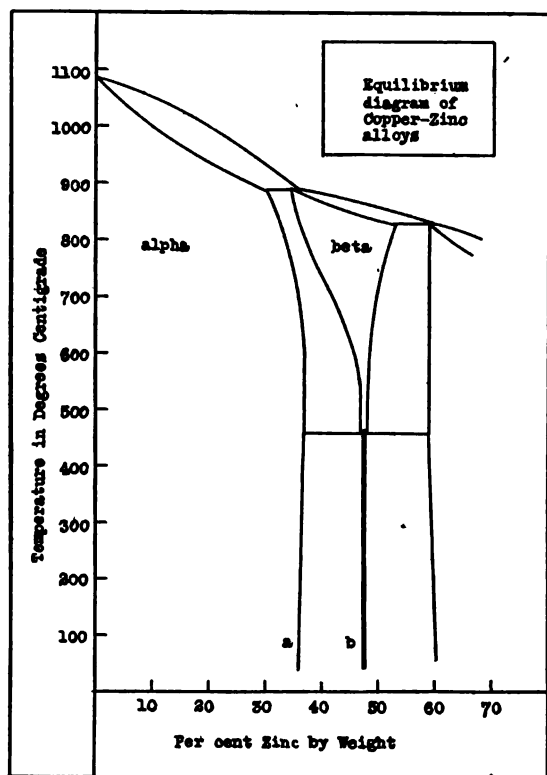


FIG. 1.—Portion of equilibrium diagram of copper-zinc alloys

had been subjected, at some time, to a very rapid cooling or quenching. Since the 60 : 40 (60 per cent copper and 40 per cent zinc) alloy is heterogeneous in structure, consisting of the two constituents, alpha and beta, in approximately equal proportions, the thought occurred to the authors that local stresses between the alpha and the beta constituents, caused by their unequal thermal expansion, might be developed during its rapid cooling. It was

¹ Merica and Woodward, Bureau of Standards, Technologic Paper No. 82, 1916; *Trans. Am. Inst. Metals*, 9, 298, 1915.

with the idea of obtaining some information on this point that the work described below was undertaken, with the purpose of determining the difference between the unit thermal expansion values of the two constituents, alpha and beta, of 60 : 40 brass.

Apparently there has been, hitherto, no investigation along this line, either with brass or with other heterogeneous alloys. Some determinations² have been made of the thermal expansion of brass of different compositions, particularly at ordinary temperatures.

It was desired in this work to compare the thermal expansion of alpha and beta brass³ of compositions which are normally in equilibrium with each other at ordinary temperatures. In the case of the pure copper-zinc alloy, compositions such as those at *a* and *b* in Fig. 1 were chosen; in a slowly cooled alloy of any intermediate composition the concentrations of copper or zinc in the alpha and in the beta phases will be those given by these points. The preparation of homogeneous alloys of these compositions is not possible in general without appropriate heat treatment. The cast alpha brass must be annealed for some time at approximately 500° C and slowly cooled; the beta alloy may require to be quenched from above the transformation range and drawn to relieve stresses.

TABLE 1.—Composition of Brass Samples

Number of alloy	Chemical analysis					Phase
	Copper	Zinc	Tin	Lead	Iron	
	Per cent	Per cent	Per cent	Per cent	Per cent	
215 ^a	65.6	32.9	1.3	0.2	0.1	Alpha
216 ^a	54.5	43.9	1.3	.1	.2	Beta
250 ^a	53.5	45.6	.9	Trace	Trace	Alpha
252 ^a	54.7	44.5	.7	.1	.1	Do
256 ^a	65.3	34.5	.23	.2	Trace	Beta
257 ^a	67.2	32.0	.7	.1	.1	Do
259 ^b	64.6	35.4	.06	.04	< .04	Alpha
261 ^b	55.5	44.5	.07	.05	< .04	Beta

^a These samples were cast from some remelted copper and Horsehead spelter.

^b These samples were cast from electrolytic copper and Horsehead spelter; they contained, therefore only slight amounts of impurities.

² Dittenberger, Zeit. Ver. deutsch. Ing., 46, p. 1532, 1902; Benoit, Journ. de Phys., 8, p. 471, 1889; Henning, Ann. d. phys., 22, p. 631, 1907; Price, Trans. Am. Int. Metals, X, p. 233, 1916.

³ Hereafter the terms "alpha" and "beta" will be used to denote the alpha and beta phases, respectively, of brass.

TABLE 2.—Heat Treatment of Alpha and Beta Alloys After Casting

Specimen	Time of heating	Temperature	Cooling	Time of heating	Temperature	Cooling	Time of heating	Temperature	Cooling
	Minutes	°C		Minutes	°C		Minutes	°C	
215A	10	700	f-c	60	600	f-c	180	600	f-c
216C	15	750	q-w	60	100-150	f-c	10	300-400	f-c
215C	10	700	f-c	60	600	f-c	180	600	f-c
250A	45	350-450	f-c						
250B	45	350-450	f-c						
252A	45	350-450	f-c						
252B	45	350-450	f-c						
256A	240	600	f-c						
256B	240	600	f-c						
257A	240	600	f-c						
257B	240	600	f-c						
259A	15	750	f-c						
261F	15	750	q-o	120	325	f-c			

f-c indicates furnace cooled.

q-w indicates quenching in water.

q-o indicates quenching in oil.

II. THERMAL EXPANSION OF ALPHA AND OF BETA BRASS

1. PREPARATION OF ALLOYS

The samples for the measurements were cast in chill molds, heat treated in order to produce a homogeneous, stress-free alloy, then machined and tested.

The specimens 259 and 261 were made of pure electrolytic copper and Horsehead zinc; the others were made of some remelted copper, together with Horsehead zinc and Straits tin. These samples, unfortunately, contained some lead and iron. The chemical analyses of the samples are given in Table 1.

After casting, the alpha brasses were annealed and the beta brasses treated as indicated in Table 2 in order to homogenize the alloy and relieve it of stress. The specimens were heat treated until they were homogeneous as determined microscopically; this required several periods of heating for some specimens. An idea of the homogeneity of the alloys may be obtained from the micrographs, Figs. 6 to 11.

2. MEASUREMENT OF EXPANSION

The thermal expansivity measurements were made in special apparatus designed and built at the Bureau of Standards for the purpose of obtaining good temperature uniformity and high accuracy in measuring length changes.

The specimens tested to 500° C or over were heated in air in an electric furnace in which the temperature as determined by differential thermoelements was uniform to 0.1° C throughout the entire chamber containing the specimen. The absolute tempera-

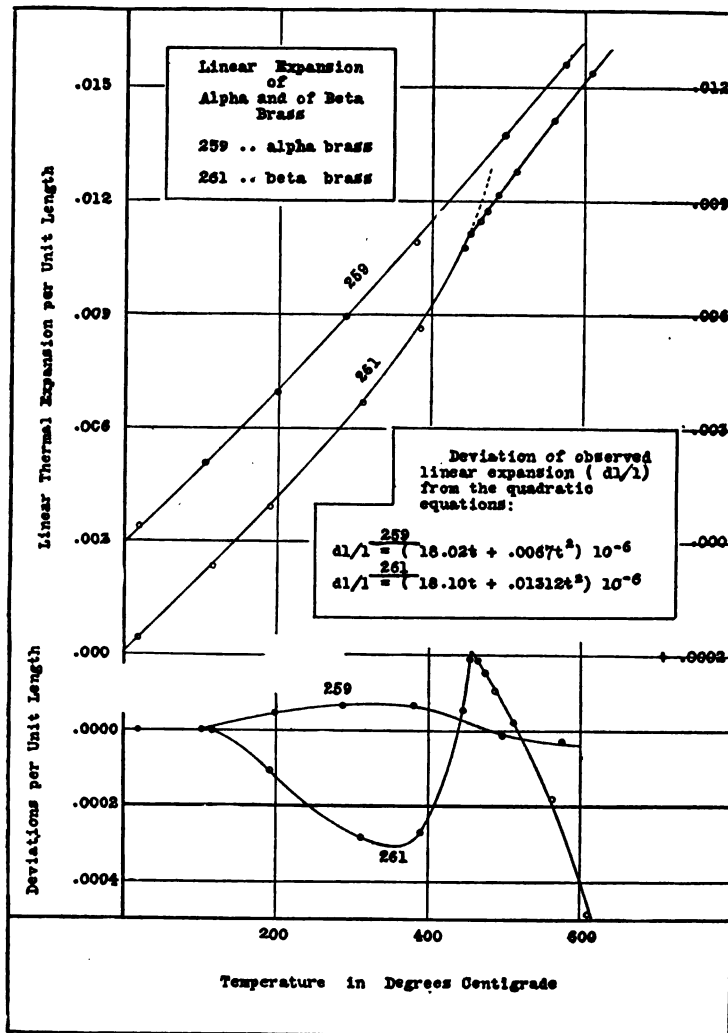


FIG. 2.—Expansion curves of brass

tures were determined by a carefully calibrated platinum platinum-rhodium thermoelement.

The tests to 300° or less were made in an oil bath in which the temperature variation was perhaps less than 0.1° C over the entire specimen. In this case the absolute temperatures were determined

with a copper-constantan thermoelement immersed in the oil at the side of the specimen.

The length changes were determined with a special comparator, consisting of two microscopes rigidly clamped on an invar bar at a

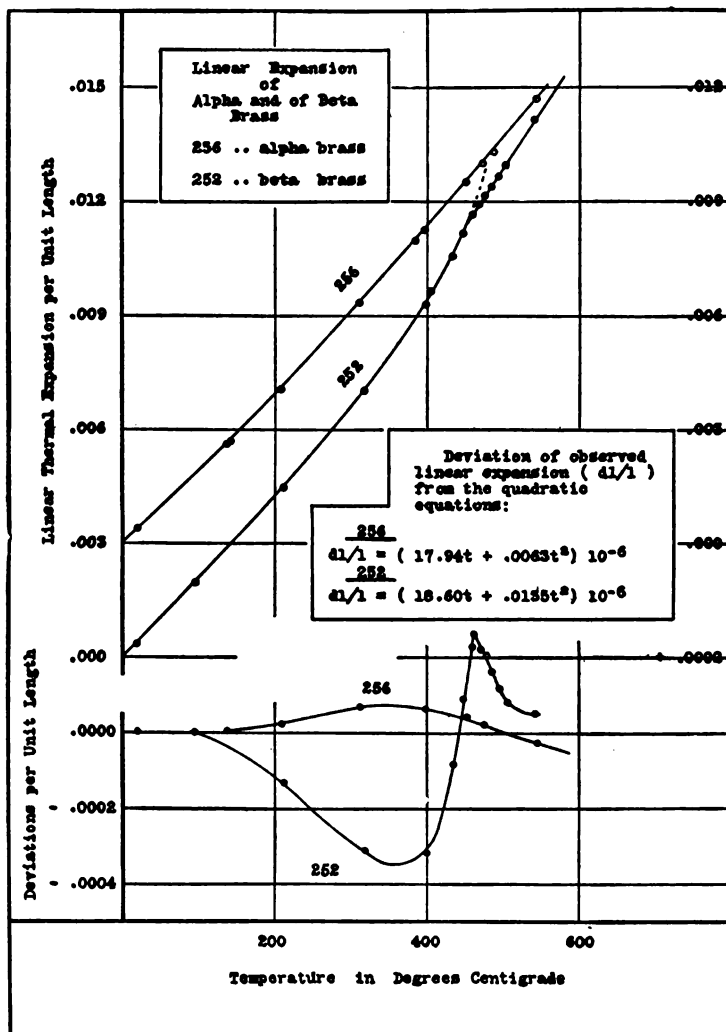


FIG. 3.—Expansion curves of brass

distance from each other equal to the length of the specimen and so arranged that they could first be sighted on a standard-length bar kept at a constant temperature and then on the 1 mil wires dropped over the ends of the specimen under test.⁴

⁴ Journal of the Washington Academy of Sciences, 11, p. 248; 1912.

The length changes measured in this way are accurate to about ± 0.001 mm, which on a specimen of 300 mm long (the standard length of the specimen) would be ± 0.0003 per cent.

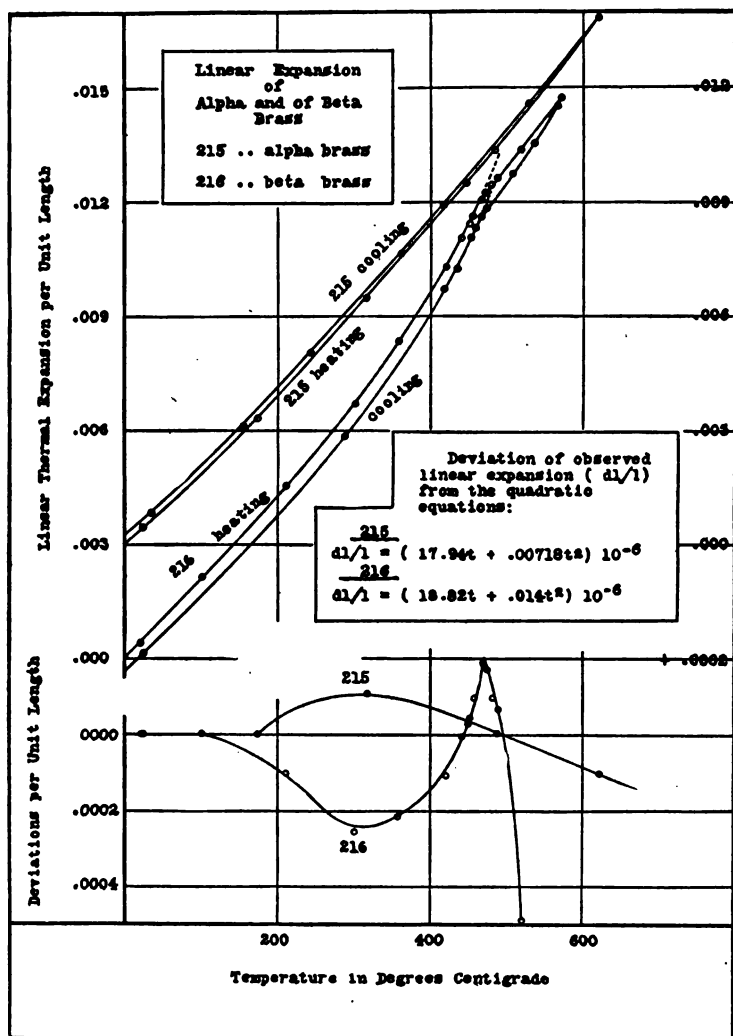


FIG. 4.—Expansion curves of brass: ○=points taken on heating; ●=points taken on cooling

For specimens heated in oil to 300° or less each test was completed in about five hours; the test to 600° C in the air furnace was completed in from three to five days. Experience seemed to show that the rapidity with which the test was made had little or no effect upon the behavior of the specimen.

3. RESULTS AND DISCUSSION

The data are given in Figs. 2 to 5, in which is shown in each case the actual unit linear thermal expansion (linear expansion per unit length) and also the deviation of this observed expansion from that computed from a quadratic equation, which in each case best fits the series of observations.

The curves for alpha are continuous in each case up to 600° C, whereas the transformation of beta into beta prime is readily

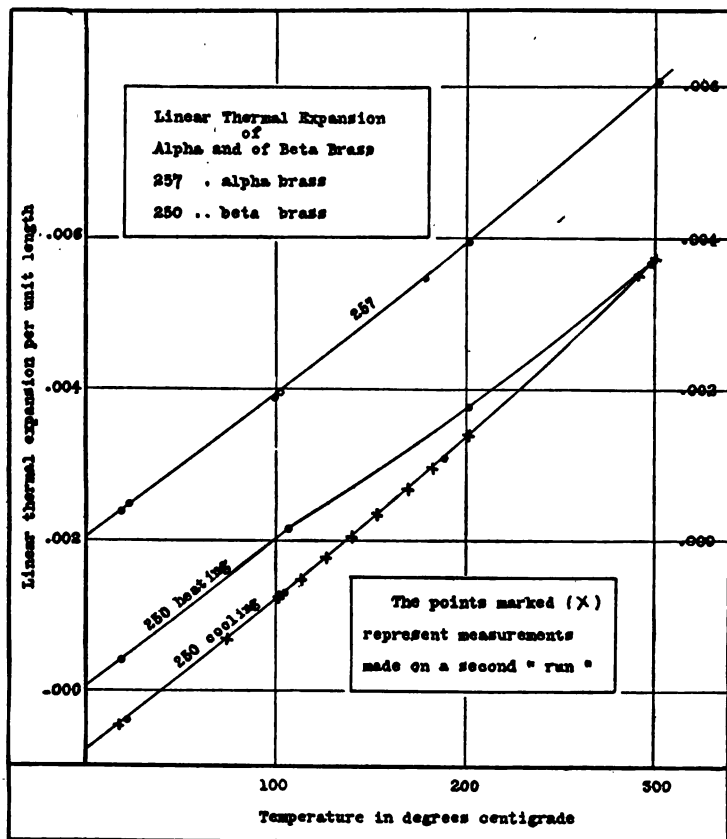


FIG. 5.—Expansion curves of brass

seen in the change in the slope of the beta curve at from 450 to 460° C. The first terms of the quadratic equation representing the thermal expansion are very nearly equal, although with a high tin content the value for beta becomes higher. The second term of the equation for alpha is, however, consistently lower than that of the beta; indeed, it is only approximately one-half of the second term of the beta equation. Thus, although from

0° to about 300° C the expansions of alpha and beta brass are almost equal, beyond this temperature and up to 460° C the expansion of the beta becomes almost twice that of the alpha. This is shown in Table 3. Above the transformation temperature the curve of the expansion of the beta is almost linear and runs nearly parallel to that of the alpha.

The quadratic equations best fitting the observed expansions are given on the figures. It is to be noted:

1. That the portion of the beta curve up to 460° C would be better fitted with a cubic curve.

2. That above this transformation point the curve is approximately linear (to 600° C) and might be represented as follows:

TABLE 3.—Expansion of Brass Over Different Temperature Intervals

Specimen	Unit linear expansion per degree centigrade between—					
	20 and 100°	100 and 200°	200 and 300°	300 and 400°	400 and 450°	500 and 600°
215A; alpha.....	19.2×10 ⁻⁶	20.0×10 ⁻⁶	22.0×10 ⁻⁶	22.5×10 ⁻⁶	23.5×10 ⁻⁶	24.5×10 ⁻⁶
216C; beta.....	21.6×10 ⁻⁶	21.8×10 ⁻⁶	22.8×10 ⁻⁶	29.6×10 ⁻⁶	35.0×10 ⁻⁶	30.5×10 ⁻⁶
250A; alpha.....	20.1×10 ⁻⁶					
252A; beta.....	22.8×10 ⁻⁶	19.4×10 ⁻⁶	23.5×10 ⁻⁶	27.5×10 ⁻⁶	39.2×10 ⁻⁶	26.9×10 ⁻⁶
256A; alpha.....	18.7×10 ⁻⁶	20.0×10 ⁻⁶	22.0×10 ⁻⁶	22.5×10 ⁻⁶	23.0×10 ⁻⁶	23.7×10 ⁻⁶
259A; alpha.....	22.8×10 ⁻⁶	22.2×10 ⁻⁶	21.9×10 ⁻⁶	22.2×10 ⁻⁶	23.4×10 ⁻⁶	23.6×10 ⁻⁶
261F; beta.....	20.0×10 ⁻⁶	21.0×10 ⁻⁶	23.6×10 ⁻⁶	28.0×10 ⁻⁶	35.0×10 ⁻⁶	27.0×10 ⁻⁶

$$\text{For brass No. 261 } \frac{dl}{l} = 26.5 \times 10^{-6} t$$

$$\text{For brass No. 252 } \frac{dl}{l} = 27.9 \times 10^{-6} t$$

$$\text{For brass No. 216 } \frac{dl}{l} = 25.2 \times 10^{-6} t$$

The second term of a quadratic fitting these curves would be small (and negative in value for No. 261).

Very interesting are the deviations observed by subtracting the values computed from the quadratic equations noted in the curves (Figs. 2 to 5) from the observed expansions; these are plotted in the same figure on a larger scale. It is seen that the deviation of the alpha brass is at first zero, then at about 100 to 150° it becomes positive, attaining a maximum at from 300 to 350°, falling off thereafter to 0 at about 500° and becoming negative. The beta brass pursues almost the reverse course; the deviations attain a negative maximum at about 350°, rise to a sharp positive maximum at about 400°, and drop again rapidly. These deviations are quite regular and occur in all of the samples.



FIG. 6.—No. 215, alpha brass, cast and annealed. $\times 100$



FIG. 7.—No. 216, beta brass, cast and annealed. $\times 100$

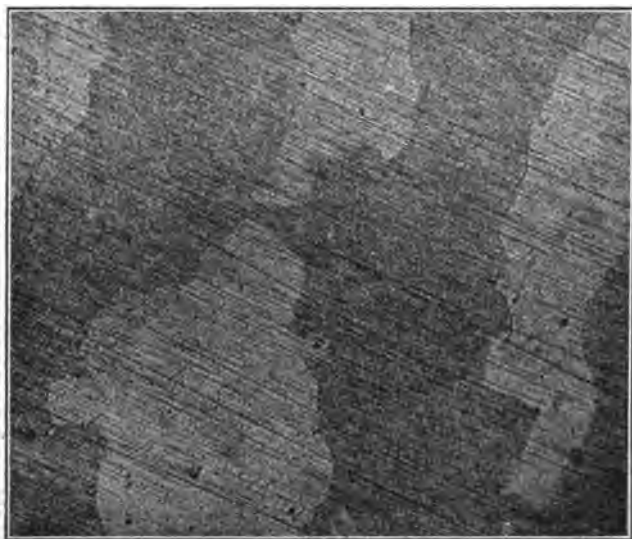


FIG. 8.—No. 256, alpha brass, cast and annealed. $\times 100$



FIG. 9.—No. 252, beta brass, cast and annealed. $\times 100$
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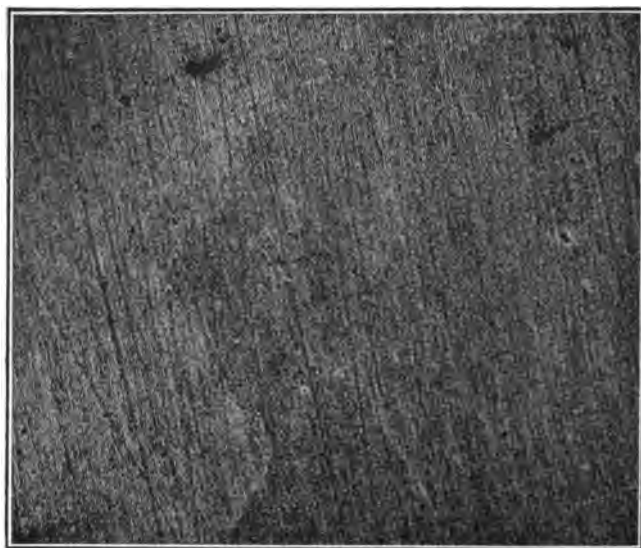


FIG. 10.—No. 259, alpha brass, cast and annealed. $\times 100$



FIG. 11.—No. 261, beta brass, cast and annealed. $\times 100$

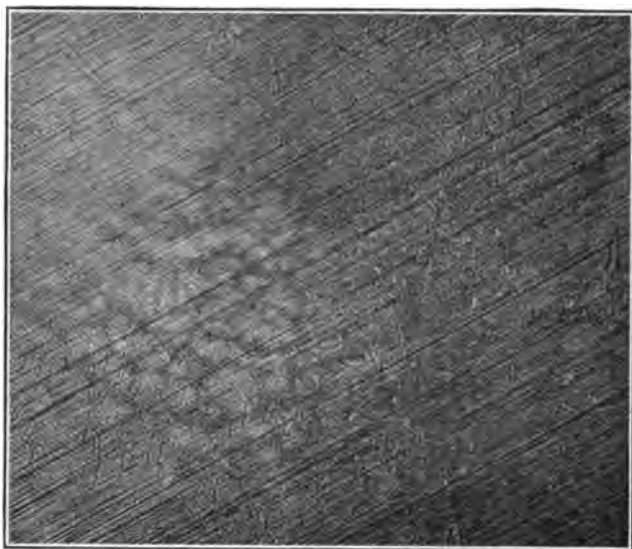


FIG. 12.—No. 261, beta brass, after heating to 600° C. $\times 100$

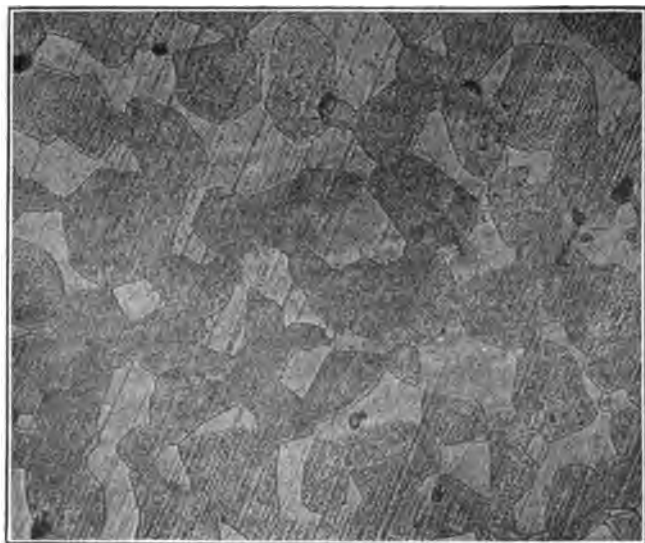


FIG. 13.—No. 169 Q, naval brass, quenched. $\times 500$



FIG. 14.—No. 169 R, naval brass, quenched and drawn. $\times 500$



FIG. 15.—No. 169 V, naval brass, slowly cooled. $\times 500$

They are related closely to the transformation point in the case of the beta brass.

In the case of the alpha brass a slight permanent elongation was noticed in some cases after heating to 600° C and cooling, whereas in the beta brass a shrinkage always took place. This shrinkage is most marked in specimen 250, Fig. 5, heated to 300° C; the corresponding alpha brass showed no change in length whatever. The second determination on the sample 250, indicated in Fig. 5, gave an expansion coinciding exactly with the "down" curve of the first.

It may be noted that some precipitation of alpha took place in the sample 261 upon heating between 400 and 600° C; this is shown in the micrograph, Fig. 12, showing the structure of the thermal expansion specimen after a 600° C heating.

This precipitation of alpha (which has a slightly different density than beta) would alter the observed coefficient through the temperature range of its precipitation.

Thus, if

V = specific volume of the alloy

V_α = specific volume of the alpha constituent

V_β = specific volume of the beta constituent

m = fraction of alpha by weight,

then $V = mV_\alpha + (1-m) V_\beta$.

$$\frac{dv}{dt} = m \frac{dV_\alpha}{dt} + V_\alpha \frac{dm}{dt} + (1-m) \frac{dV_\beta}{dt} - V_\beta \frac{dm}{dt}$$

$$= \frac{dm}{dt} (V_\alpha - V_\beta) + m \left(\frac{dV_\alpha}{dt} - \frac{dV_\beta}{dt} \right) + \frac{dV_\beta}{dt}$$

Measurements showed the densities at ordinary temperatures of 259 (alpha) and 261 (beta) to be as follows:

$$\begin{array}{l} 259 \dots\dots\dots 8.294 \\ 261 \dots\dots\dots 8.226 \end{array} \left(V = \frac{1}{\text{density}} \right)$$

If it is assumed that 5 per cent of alpha has been formed between 460 and 480° C,

$$\frac{\Delta V}{\Delta T} = \frac{0.05}{20} \left(\frac{1}{8.294} - \frac{1}{8.226} \right) + \frac{dV_\beta}{dT} = -2.45 \times 10^{-6} + \frac{dV_\beta}{dT}$$

$$\therefore \frac{1}{L} \frac{dL}{dT} = \frac{1}{3} \frac{dV}{dT} = (-0.82 + 26.5) \times 10^{-6}$$

The precipitation of alpha has therefore a small but appreciable effect on the thermal expansion coefficient.

4. LOCAL "GRAIN STRESSES" DUE TO DIFFERENCES IN THERMAL EXPANSION

It is much beyond the scope of this paper to attempt a discussion of the magnitude and distribution of stresses arising during the cooling of a heterogeneous aggregate of particles of different coefficients of thermal expansion. A particle imbedded in a homogeneous mass having a greater unit thermal expansion than it has, will be, after rapid cooling, essentially in hydrostatic or isotropic compression. The ground mass surrounding it will, at the surface of division, be subject to a system of tangential tensional stresses parallel to it. The magnitude and distribution of these stresses will depend on the size, shape of the imbedded particles and their distances apart.

One may gain a very rough idea of the magnitude of such stresses by assuming that the two constituents are in the form of bars rigidly clamped parallel to each other. When the temperature of such a duplex bar is lowered there will be in each constituent bar (if of equal cross section) a stress equal to

$$\frac{\Delta dL}{L} \frac{E}{2}$$

where $\frac{\Delta dL}{L}$ = difference in thermal expansion per unit length between the constituents.

E = modulus of elasticity.

Such stresses are calculated on the basis of E equal to 15×10^6 pounds per square inch and are given in Table 4. The values may be considered as representing the tensional stress in the beta and the compressional stress in the alpha bar which remain after cooling rapidly from the temperatures indicated. The reverse stresses would be temporarily caused by rapid heating. As the amount of beta in a brass is increased, the average stress from this cause in the beta would decrease; one would expect, therefore, other things being equal, to find a greater effect of such stresses in an alloy of beta than in one such as manganese bronze; a quenched alloy with alpha grains surrounded by beta envelopes should show most noticeably any effect of tensional stress in the beta constituent.

It is noted that the development of stress, due to unequal thermal expansion, during the cooling of a 60:40 brass, is greatest within the temperature range from 300 to 500° C., at which experience has indicated, annealing and relief of stresses take place fairly readily. One may assume, therefore, that during slow cooling of such an alloy from 500 to 600°, the alloy constituents yield locally under the stresses produced through the range 600 to 300°. Below 300° the contraction of the two constituents is almost equal. During rapid cooling, however, the time may not be sufficient to allow of the local yielding to any extent of the constituents; they arrive at ordinary temperature, therefore, in a state of stress described above.

TABLE 4.—Calculated Stresses Due to Difference in Thermal Expansion Between Alpha and Beta Brass

	For 215 and 216B	For 256 and 252B	For 259 and 261B
Difference in total unit linear expansion between 0° and 300°.....	0.00050	0.0005	0.00026
Corresponding stress pounds per square inch <i>c</i>	3750	3750	1950
Difference in total unit linear expansion between 0° to 400°.....	0.0012	0.0010	0.00075
Corresponding stress pounds per square inch <i>c</i>	9000	7500	5600
Difference in total unit linear expansion between 0° to 500° ..	0.0022	0.0021	0.00162
Corresponding stress pounds per square inch <i>c</i>	16 500	15 800	12 200
Difference in total unit linear expansion between 0° to 600°.....	0.0022		0.00190
Corresponding stress pounds per square inch <i>c</i>	16 500		14 200

c Assuming two bars, one of A and one of B, rigidly clamped and heated from 0° C to the temperature noted, or cooled through this range; E is assumed to be 15×10^6 pounds per square inch.

III. EFFECTS OF CERTAIN HEAT TREATMENTS ON MECHANICAL PROPERTIES OF 60:40 BRASS

Some experiments were undertaken with a view to ascertaining directly whether the "grain" stresses described above exerted an appreciable effect upon the properties of the brass.

Six-inch lengths of drawn brass 1-inch diameter rods 164, 169, 171, 173, and 175⁵ of the compositions and properties given in Table 5, were given the heat treatments described in Table 6, which consisted largely in the heating of the specimen to various temperatures between 400 and 600° C, and the quenching of them in water or oil. The samples were then submitted to the mercurous nitrate test. This test indicates the presence of initial stress of significant extent caused by drawing or working brass. None of the samples cracked during the quenching or in the test.

⁵ Described more fully in Tables 2, 3, and 5 of Bureau of Standards Technologic Paper No. 82.

A few 1 inch diameter samples of naval brass and of Muntz metal were quenched or slowly cooled from 500° or from 400° C. These were then machined to 0.505-inch diameter and tested in tension. The results of these tests are shown in the Table 7. Without exception the quenched samples have a slightly lower proportional limit and a higher tensile strength than the samples which were slowly cooled from the same temperature or were quenched and drawn back to 400° C in order to relieve the local stresses.

It must be observed that the full effect of differential grain stresses on the mechanical properties of a brass might not be developed except when the brass was at the same time under additional tensional stress. The latter would increase the local tensional stresses and diminish the local compressional stresses such that an incipient local surface crack might develop throughout the brass. Information on this point could be obtained from the results of corrosion tests under stress such as are being conducted at present at the Bureau.

In the case of the naval brass the samples heated to 400° or thereabouts suffered in ductility, a fact which was readily explained by a microscopic examination of such samples. At that temperature the hard delta constituent forms at the edge of the beta grains. This can be seen in the micrographs Figs. 13 to 15. At higher temperatures the quenched specimens did now show any delta.

TABLE 5.—Chemical Composition of Brasses Which Were Heat Treated and Tested in Tension

Num- ber	Material	Chemical analysis				Physical properties	
		Copper	Zinc	Tin	Iron	Ultimate strength	Elonga- tion in 2 inches
		Per ct.	Per ct.	Per ct.	Per ct.	lbs./in. ²	Per ct.
164	Muntz metal	61.1	38.5			72 000	^a 16
169	Naval brass.....	61.2	37.5	1.2		83 000	13
171	Muntz metal.....	59.4	40.2			80 000	21
173	Manganese bronze.....	56.8	39.9	1.6	1.3	100 500	
175	Manganese bronze.....	57.5	40.9	1.0	.6	77 000	27

^a In 3 inches.

TABLE 6.—Heat Treatment of Brass Samples Submitted to Mercurous Nitrate Test

Specimen	Heat treatment
164A.....	1 hour at 430°, water quenched.
164B.....	1½ hours at 430°, oil quenched.
164C.....	1½ hours at 480°, water quenched.
164D.....	1½ hours at 480°, oil quenched.
169C.....	1½ hours at 480°, water quenched.
169D.....	1½ hours at 480°, oil quenched.
169K.....	20 hours at 625°, water quenched.
169L.....	Do.
169M.....	Do.
169N.....	30 minutes at 620–600°, water quenched, { tested. annealed 1 hour at 330°.
169O.....	
169P.....	20 minutes at 625 to 500°, water quenched.
169T.....	20 minutes at 400°, quenched.
169W.....	40 minutes at 400°, furnace cooled.
171F.....	20 minutes at 500°, water quenched.
171G.....	20 minutes at 500°, 20 minutes at 440°, water quenched.
171K.....	20 minutes at 500°, 20 minutes at 440°, furnace cooled.
175A.....	30 minutes at 445°, water quenched.
175B.....	30 minutes at 440°, oil quenched.
171C.....	1½ hours at 480°, water quenched.

TABLE 7.—Mechanical Properties of Heat-Treated Naval Brass and Muntz Metal

Specimen	Heat treatment	Physical properties				
		Ultimate strength	Proportional limit	Elastic modulus	Elongation in 2 seconds	Reduction of area
		Lbs./in. ²	Lbs./in. ²	Lbs./in. ²	Per cent	Per cent
169Q	30 minutes at 600–620° tested.....	59.7x10 ³	14.0x10 ³	14.4x10 ⁶	34.5	33.0
	Followed by 20 minutes at 500°, water quench, 1 hour at 330°.....	56.6	18.0	16.0	21.0	21.6
169S	15 minutes at 428°, water quench.....	63.2	16.5	14.3	26.5	25.0
169V	20 minutes at 428°, furnace cooled.....	61.6	20.0	14.8	30.0	25.1
171D	20 minutes at 500°, { 2 seconds at water quench. { 400°.	59.3	16.2	13.9	53.0	57.1
171E	{ tested.....	60.4	14.0	14.3	53.0	57.6
171H	20 minutes at 500° { water quench.....	58.8	12.5	13.7	54.0	58.6
171J	Followed by 20 minutes at 440° { furnace cooled.....	57.7	16.2	13.9	54.5	59.1

IV. CONCLUSION

The difference in the thermal expansion of alpha and of beta brass of compositions which normally are in equilibrium in such alloys as Muntz metal, naval brass, etc., has clearly been shown by the measurements made. Fundamental variations in behavior as regards thermal expansion at temperatures up to 600°C

were noted, due to the occurrence of a transformation in the beta constituent.

The effect of the local or, as they might be called, "grain" stresses, on the physical properties and service behavior has been only incompletely indicated. Tests showed that stresses of this sort produced by quenching commercial drawn 60:40 brass 1-inch diameter rod did not cause cracking in mercurous nitrate. On the other hand, a lowering of the proportional limit of the alloy amounting to about 2000 pounds per square inch resulted from this treatment.

Manufacturers of brass never quench 60:40 brasses; in fact, among them there seems to exist a disposition to regard this as dangerous practice. However, no definite data other than that recorded above are known to the authors, which would show clearly the ill effects of such sudden cooling.

It is found that naval brass or manganese bronze when quenched in such a manner as to leave the beta grains surrounded by alpha envelopes, is generally both weak and brittle, and the fracture intercrystalline, a condition which has been ascribed to the alpha envelope. Now it is known that alpha brass is weaker than beta, but not more ductile, such that the authors suggest that as an explanation for this brittleness may more readily be assigned to existence of tangential tensional stresses in the beta grains immediately adjacent to the alpha envelope.

It would appear to the authors that further investigation into this general question of the expansion behavior of different constituents of other alloys might reveal causes of mysterious failures and weakness now considered quite obscure. Such materials as hypereutectoid steels, cast iron, type metal, and bearing metals contain two constituents. In many cases one of these constituents is brittle, a fact which would accentuate the effect of local contraction stresses. The authors hope to be able to present some data later along these lines, indicating also more definitely the physical effect of such stresses.

WASHINGTON, August 15, 1917.

PHOTOELECTRIC SENSITIVITY OF BISMUTHINITE AND VARIOUS OTHER SUBSTANCES

By W. W. Coblentz, Associate Physicist

The present paper summarizes the results of an examination¹ of various substances, to determine their electrical sensitivity to light, and describes the results of a more detailed examination of the photoelectric sensitivity of bismuthinite Bi_2S_3 and molybdenite, MoS_2 .

All of the herein-described substances were examined for change in electrical conductivity caused by the action of light upon them, and two of them were examined for photoelectrical activity when they were charged to a negative potential in an evacuated bulb, and exposed to light.

When the substances were examined for an increase in electrical conductivity, a potential of 2 to 6 volts was connected, through a resistance of zero to 100 000 ohms, into a circuit containing a d'Arsonval galvanometer and the substance under investigation as illustrated in Fig. 1. In most cases the substances were slightly conducting when not exposed to light, so that the "dark current" caused an appreciable galvanometer deflection, which had to be annulled by passing an electric current in the opposite direction through the galvanometer. This counter current was obtained by shunting across a resistance of 100 ohms, which was

¹ Some of these substances were examined with the assistance of W. B. Emerson and the results published under joint authorship in the Jour. Wash. Acad. Sci., 7, p. 525; 1917.

in series with a cell of 2 volts and a variable resistance of zero to 70 000 ohms.

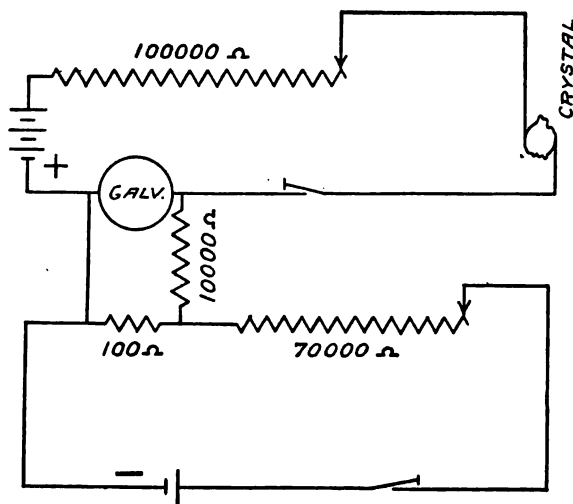


FIG. 1.—Arrangement of electrical connections

The source of light, when not otherwise specified, was a 16 cp carbon incandescent lamp, placed at a distance of 10 cm from the substance under investigation. One disappointing feature of this investigation is that no substance was found which is comparable with the potassium photoelectric cell and the selenium cell in sensitivity.

Part I.—PHOTOELECTRIC SENSITIVITY TESTS OF VARIOUS SUBSTANCES

Gallium.—The material examined was the highly purified metal prepared and supplied by Dr. H. S. Uhler. This metal was solid, thus differing from the impure material, which is a liquid. It was melted and solidified over a platinum wire sealed into a glass bulb, thus forming the negative electrode of a photoelectric cell from which the air was exhausted. The anode was a loop of platinum wire, situated at a distance of about 12 mm above the gallium electrode.

A potential of 340 volts was applied to the cell which was connected to a sensitive ($i = 5 \times 10^{-10}$ amp) ironclad Thomson galvanometer.

The results obtained proved quite disappointing, this metal being quite insensitive to light. When the cell was exposed to daylight the photoelectric current produced a galvanometer de-

flection of only 4 to 5 mm, whereas, similarly exposing a potassium photoelectric cell, the photoelectric current was sufficient to give a deflection beyond the range of the scale.

Silver Sulphide.—The sample examined was a thin flexible strip, 6 by 10 mm in area, prepared by G. W. Vinal.² In one test the silver sulphide formed the negative electrode of a photoelectric cell (evacuated glass bulb about 5 cm. diameter; with a ring of platinum wire for the anode) similar to the gallium cell just described. It was connected through an ironclad Thomson galvanometer to a battery of 340 volts. When exposed to daylight a deflection of perhaps 1 to 2 mm was observed, but no deflection resulted from exposure to the standard carbon lamp.

In the second test, copper wires were melted to the ends of a strip (3 by 5 by 0.3 mm) of silver sulphide which was connected in series with a high resistance, a storage cell of 2 volts, and a d'Arsonval galvanometer. When exposed to the standard lamp, the galvanometer deflection was 10 mm. In another sample about 2 cm long, the ends joining the copper wires were covered to prevent thermoelectric currents. The exposed area was 14 by 4 mm. The radiation from the standard lamp produced a deflection of 13 to 17 mm. Both samples were quick acting, and after exposure to light there was no lag such as obtains in selenium in recovering its dark resistance.

Selenium.—A crystal of selenium, prepared by Dr. F. C. Brown,³ having a receiving surface of less than 1 mm² area, when exposed to the standard lamp, gave a deflection of more than 50 cm, which shows its great sensitivity as compared with other substances.

The mounting of the selenium crystal consisted of metal electrodes between which the crystal was held by compression. When operated as a photophone, by connecting the selenium crystal with an audion amplifier, as described subsequently, a loud musical note was obtained.

Pyrites.—FeS₂. The sample examined was a fragment of a crystal, 1.2 by 1.5 by 4 mm, soldered to copper wires.

This substance gave no photoelectric response when exposed to the standard lamp; neither did it produce an audible sound when operated as a photophone in connection with an audion amplifier.

Tellurium.—This metal is said to change in resistance when exposed to light. The present tests were made upon a mirror of tellurium deposited upon a glass plate by cathode disintegration.

² Vinal, this Bulletin, 14, p. 331; 1917.

³ Brown, Phys. Rev. (2), 4, p. 85; 1914.

Suitable terminals were attached to a sample about 4 by 50 mm. No change in electrical conductivity was observed when exposed to light.

Silicon.—From the Carborundum Co., Niagara Falls, N. Y. The crystal, 1 by 1.5 by 3 mm, was attached to copper wires by means of solder.

No photoelectric effect was observed when exposed to the standard lamp; and no audible sound was produced when operated as a photophone. Thermoelectric effects were obtained by heating the connections with the copper-lead wires.

Boleite.—The sample of boleite ($3\text{PbCl}(\text{OH})\text{CuCl}(\text{OH}) + \text{AgCl}$; from Boleo, Mexico) examined was a single rectangular crystal, 3 by 3 by 1.5 mm, held by compression between copper electrodes. No change in conductivity was observed when the crystal was exposed to daylight or to the standard incandescent lamp.

Cylindrite.—This is a lead-antimony-tin sulphide, which occurs in cylindrical form, mixed with inhomogeneous material.

The sample examined was about 1.5 mm in diameter and 6 mm long.

The standard lamp produced a deflection of 10 to 11 cm. A similar decrease in resistance was produced by the heat radiated from the hand, and by warming the substance by touching it with a hot wire.

Stibnite.—Samples of this same specimen of stibnite, Sb_2S_3 , were supplied to Elliot⁴ for investigation. The purpose of the present investigation was to obtain a comparison of its sensitivity with other substances containing stibnite.

The size of one sample examined was 4 by 7 by 0.5 mm. Terminals were attached to it by heating a copper wire to incandescence in a gas flame and bringing it in contact with the plate of stibnite.

The standard carbon lamp caused a deflection of 5 cm. A deflection equally as large as this was obtained by exposing the crystal to the heat radiated by the hand. Stibnite may be considered as sensitive as boulangerite, but the deflection drifted, due to a decrease in resistance with time (heating), as noticed by other observers.

Boulangerite.—The specimen of boulangerite ($3\text{PbS}, \text{Sb}_2\text{S}_3$, Irkutsk, Siberia) investigated was obtained from the Smithsonian Collection, No. 78395. Several samples were examined. In one

⁴ Elliot, *Phys. Rev.* (2), 5, p. 55; 1915.

sample, 4 by 7 by 0.8 mm, the electrodes consisted of copper wires melted into the material, as just described. The radiation from the standard lamp gave a deflection of 10 to 20 cm.

Another sample, 1 by 1.2 by 2 mm, held by compression between two heavy electrodes of copper, when exposed to the standard incandescent lamp, produced a deflection of 2 to 3 cm, which is comparable with the preceding when one considers the size of the exposed surfaces.

Although this substance seems fairly sensitive, it did not appear to be sufficiently so to investigate its spectral sensitivity, with a view of using this mineral as a selective radiometer.

Jamesonite.— $2\text{PbS}, \text{Sb}_2\text{S}_3$; Smithsonian Collection, No. 12500; from Cornwall, England. The sample examined (size 2 by 7 by 1 mm) had the copper wire terminals attached by fusing the incandescent wire into the material. The standard lamp gave a deflection of only 1 to 2 cm, which seems to indicate that this material is not so light sensitive as is *boulangerite*.

Mixtures of galena, PbS , and Sb_2S_3 of various proportions were melted in a crucible and poured upon a plate of metal. Several samples, 5 by 10 by 0.5, were examined but none of them gave any indication of light sensitiveness (resistance change) when exposed to daylight or to the standard incandescent lamp.

Bismuthinite.— Bi_2S_3 was obtained from the Smithsonian Collection, No. 85 071, from Jefferson County, Mont. This is the most interesting substance examined, in view of the diverse results obtained and the explanation offered therefor.

The sample of bismuthinite examined consisted of a nonhomogeneous mass of acicular crystals, which was easily crushed into numerous fine needlelike crystals. The first sample examined was a small mass of crystals (size 1 by 1 by 0.7 mm) held by compression between two heavy electrodes of copper. When the crystal was exposed to the standard carbon lamp no change in conductivity could be detected with certainty.

A second sample, 3 by 6 by 1 mm, had the copper wire terminals attached by fusion, as already described. The emf's applied were the same as for the preceding sample. When exposed to the standard lamp no change in conductivity was observed. These results being contradictory to those published by Case⁶ who used a three-stage audion amplifier to detect the change in conductivity of the crystals, the foregoing experiments were

⁶ Case, *Phys. Rev.*, 9, p. 305: 1917.

repeated in the manner described by him. For this purpose the light from an acetylene flame, shining through a slit 2 by 10 mm, was focused upon the crystal by means of a triple achromatic lens, 6 cm in diameter and 18 cm focal length. The light was interrupted by means of a sectored disk having 15 openings, operated by means of an electric motor, the speed of which could be varied. The usual speed used gave 240 interruptions per second. The crystal was connected to a three-stage audion amplifier and telephone receiver. A crystal of selenium or a selenium cell produced a loud note; but the samples of boulangerite and jamesonite, which by previous tests were light sensitive, did not give a musical sound in the telephone.

The sample of bismuthinite, with electrodes sealed on, produced no audible note when exposed to light.

At least a dozen samples of bismuthinite, held by compression between heavy copper electrodes, were examined in connection with the amplifier. Of this number only two samples appeared to be light sensitive. One sample produced only a faint sound in the telephone receiver. The second sample produced a loud note in the telephone. The sound was the loudest when the crystal was exposed along the line of contact with the copper electrode. Covering the crystal with red glass did not reduce the loudness of the note very much, indicating that the effect is due to heating of the material. Unfortunately, this crystal was crushed while under investigation. Prolonged tests on other samples gave negative results as regards the production of sound. (See further tests, described in Part II of this paper; using samples of smaller heat capacity and electrodes which prevented loss of heat by conduction, etc.)

In connection with these tests on bismuthinite, the following experiments on thin strips of metals are of interest.

Platinum and Gold.—It is of interest to record the results obtained when using thin, blackened strips of platinum and of gold leaf as radiophones, by connecting them through a battery to an amplifier.

These blackened strips were warmed intermittently by exposing them, through a rotating sectored disk, to the acetylene flame, as already described.

When a sensitive platinum bolometer receiver was used as a radiophone, the sound produced in the telephone was not very audible. This no doubt was owing to the great heat capacity of the material, which prevented the rapid alternations in resistance,

and hence in electric current, from being of sufficient magnitude to affect the telephone receiver.

Using a lightly smoked strip (6 by 2.5 mm) of gold leaf, the ends of which were clamped between thin (0.02 mm) strips of tin, the sound produced in the telephone receiver was as loud as was observed in the photophone made of selenium, just described.

This device was mounted in a glass bulb, which could be evacuated. As was to be expected, there was no marked difference in the intensity of the sound produced when operated in air and in a vacuum.

In the gold-leaf radiophone as used the limit of audibility was attained for a light (radiant power) intensity of 4.8×10^{-5} watts. Using a larger receiver and amplifier and a larger current (which was 0.2 amp in the present tests) through the receiver, the sensitivity could be greatly increased.

Part II.—ADDITIONAL PHOTOELECTRIC SENSITIVITY TESTS OF BISMUTHINITE AND MOLYBDENITE

In view of the fact that some of the results obtained with bismuthinite seemed somewhat contradictory, a more detailed examination of its photoelectric sensitivity was undertaken. The method of supporting the crystals by compression between copper electrodes was unsatisfactory, because of the fragility of the crystals, which were easily crushed. Moreover, the noise produced in the telephone (when using the amplifier) as the result of poor contact at the electrodes also militated against the use of such a holder. It was found that while ordinary solder usually does not adhere to bismuthinite, it can be melted in a large globule, around the end of a crystal and onto a copper wire. On solidifying, the solder holds the crystal very securely. In this manner it was possible to mount thin, fragile crystals (0.5 mm in diameter and 2 to 5 mm in length) which would have been crushed in the compression mounting just mentioned.

The thin lamina of molybdenite are easily mounted, because the molten solder readily adheres to the surface of this substance.

In the preceding tests the crystals (which have a massive, fibrous structure) were placed with the needle crystals parallel to the copper electrodes; and the question arose as to whether the photoelectric sensitivity depended upon the direction of the current through the crystal. In the present tests the electrodes being attached to the ends of the bundles of needle-like crystals,

the current passed along the needles instead of across them. However, as will be shown presently, it was then found that the high photoelectric sensitivity observed with an audion amplifier occurs in small spots in the crystal, usually at the boundary or intergrowth of two masses of acicular crystals.

It is important to distinguish between the purely thermal and the so-called photoelectric effect of radiation upon the electrical characteristics of selenium, and the various sulphides just described, which have a negative temperature coefficient of resistance.

A selenium cell having a resistance of 1 000 000 ohms at 20°C, has a resistance of about 3 000 000 ohms at 0°C. When such a cell is exposed to diffuse daylight, its resistance decreases one-tenth to one-fiftieth of its value when unexposed to light, in spite of the fact that its temperature has not changed by an appreciable amount. Furthermore, this photoelectric change (decrease) in resistance is greatest for radiations of wave lengths which are the least absorbed; and it is not produced by an appreciable amount for rays of wave lengths greater than 1μ . If appreciable radiation were absorbed, the temperature would rise and the resistance would decrease. It is therefore an easy matter to distinguish the purely thermal effect from that produced by exposure to radiant energy.

On the other hand, the herein-described tests show that the electrical resistance of bismuthinite and other sulphides mentioned decreases with rise in temperature. The effect of radiant energy is also a decrease in resistance, and in view of the fact that it is not very marked (as compared with selenium) it is a difficult task to distinguish between the photoelectric and the purely thermal change of resistance. This is especially true since the infra-red rays of wave length greater than 1μ have an appreciable effect (as shown by the insertion of absorption screens in the path of the rays), whereas selenium is not sensitive to infra-red rays beyond 1.5μ .

Herewith follows a description of the behavior of various crystals of bismuthinite and molybdenite, soldered to copper terminals and tested for photoelectrical activity (1) by continuous exposure to the radiations from the 16 cp standard lamp, and (2) by interrupted exposure to an acetylene flame, in connection with an audion amplifier, as previously described.

For convenience in examination the samples were soldered, two at a time, to copper wires (No. 26), which were securely attached to light wooden mountings designated as *A*, *B*, etc., in presenting these data.

The image of the slit which was in front of the acetylene flame or of a Nernst glower was about 1.5 mm wide. This enabled the observer to project the light upon different parts of the various crystals (3 to 5 mm in length) and thus locate the most sensitive spots.

BISMUTHINITE (Bi_2S_3)

For convenience in discussion, the various samples of bismuthinite examined may be classified into four types—(1) straight rods of material of fibrous structure, or needles, 3 to 5 mm long and 0.3 to 1.5 mm in thickness; (2) spindle-shaped (sometimes flattened) masses of fibrous material consisting of two bundles of acicular crystals joined end to end, 2 to 5 mm long and 1 to 2 mm wide at the center; (3) flat bars consisting of two straight masses of acicular crystals joined together, end to end (usually at the center of the rod), at an angle of about 120° ; and (4) straight rods (1 to 2 mm in thickness) of material, composed apparently of short crystals, which appeared fibrous in structure, but which were easily broken into lengths of 1 mm to 2 mm.

Samples A_1 , A_2 , and B_2 , type 1, produced no sound when operated as a radiophone, in connection with the audion amplifier and telephone receiver.

Sample B_1 , length 3.5 mm between the solder electrodes, type 4, produced a loud sound in the radiophone. On one side of the crystal the sound was loudest at the center of the crystal; on the opposite side, the sound was loudest near the end.

This crystal became detached and in resoldering it the distance between the electrodes was reduced 2.5 mm. A loud radiophonic sound was produced as before, especially in the center, at which point, later on, the crystal separated, leaving very ragged edges along the line of contact.

Samples C_1 and D_1 , type 1, size 0.5 by 4 mm, gave no sound in the radiophone.

Sample C_2 , type 3, size 1.5 by 4.5 mm, at first produced no sound. On cutting it thinner, a slight sound was produced when one side was exposed to radiation.

Sample C_3 , size 1.5 by 5 mm, seemed to be a mixture of short crystals (type 4), which was light-sensitive in spots. No sound was produced when the light was projected on the ends of the crystal, showing that the phenomenon is not localized at the electrodes, as might be inferred from the earlier experiments with samples held by compression between copper electrodes.

This crystal had a small region on one side, near the center, which produced a fairly loud sound in the telephone receiver; and a faint sound was observed on the opposite side near the end. When this crystal was (accidentally) broken the fracture was along the line joining these two points. The fracture was very ragged and somewhat conical shaped.

Some interesting experiments were made on the light-sensitive spot in this crystal. Painting the ends of the crystal with lamp-black did not affect the sound production; but on painting the sensitive spot no sound was produced. Removing the lampblack restored the sound, which became louder when the sensitive spot or the whole crystal and the electrodes were covered with alcohol. On immersing this sample in alcohol, the crystal separated into two parts (no force having been applied), and the question is whether, in this case, the sound was the result of a variation in resistance (at the surface of contact) with fluctuation in temperature.

Sample C_4 , type 3, size 1.5 by 3 mm, gave a faint sound when one side was exposed to radiation, and a loud sound on the opposite side. The sensitive line between these two points marked the line of contact and partial intergrowth of two fibrous masses of crystals.

When exposed to the standard incandescent lamp, Fig. 1, a galvanometer deflection of 5 to 10 cm was produced, indicating a decrease in electrical resistance.

Sample C_5 , type 2, flat rod, produced a faint sound when one side was exposed to radiation; but no audible sound was produced when the opposite side was similarly exposed.

Exposing either side to the standard lamp produced a deflection of about 3 cm. In comparison with C_4 , the size of the deflection seemed to be in proportion to the sound produced in the radiophone, though tests on other crystals do not indicate any relation between radiophonic sensitivity and change in conductivity produced by exposure to the standard lamp.

Sample D_2 , club-shaped, size 1 by 1.5 by 2.5 mm, produced a faint sound when the thick end was exposed to radiation; loudest on one side containing a projecting fragment of crystal.

The standard incandescent lamp produced a deflection of 10 cm.

Samples E_1 , E_2 , E_3 , E_4 , and E_5 produced no audible sound. E_4 was a very fine crystal, 0.2 mm by 4 mm, of type 1. E_5 gave a faint sound at one end. It was of the same type as D_2 , and after

removing a protruding edge no sound was audible. E_7 , size 1.5 by 2.5 mm, produced a faint sound; and when broken the fracture occurred along the sensitive spot. E_8 produced no audible sound. E_9 produced a faint sound. This crystal broke very easily; type 4. E_{10} consisted of one end (length 2 mm) of E_9 remounted. It seemed more structureless than the average sample, having very short crystals. It produced a faint audible sound on all sides exposed; but one side, which appeared rough and structureless, produced a quite audible sound. The standard lamp gave a deflection of 6 to 8 cm whichever side was exposed.

Sample F_1 , type 4, 1 by 1.2 by 2 mm, gave a loud sound when one side was exposed to radiation, but no sound was audible when the opposite side was exposed. The cleavage seemed rather structureless. The standard lamp produced the same galvanometer deflection, 2.5 cm, whichever side was exposed. The heat radiated by the hand held near the crystal produced a similar deflection, indicating a decrease in resistance as a result of warming of the crystal. This change in resistance was slow (exposure to light 15 to 30 sec.) and appeared to be the result of warming of the crystal.

F_2 produced no audible sound when exposed to the interrupted light.

Sample G_1 , 1.5 x 3 mm, produced a fairly audible sound. The standard lamp produced a deflection of about 1 cm. In G_2 , size 0.6 by 2.5 mm, no audible sound was produced. The standard lamp gave a deflection of about 3 cm. G_3 , size 1.5 by 2 mm, produced a fairly audible sound when one side was exposed to radiation. The standard lamp caused a deflection of 2.5 cm. Radiation from the hand, or from a warm soldering copper, held near the crystal produced a similar deflection. This is not a thermoelectric effect, as was shown by heating alternately the junctures where the crystal touched the copper wires.

The results of this examination show that of the 26 samples of bismuthinite examined by the radiophonic method, 12 samples produced an audible sound. All of the 8 samples showed a decrease in resistance when exposed to the standard lamp or when warmed by radiations (from the hand) of wave lengths dominating at 8μ . Samples which responded to radiation stimuli from the standard incandescent lamp did not produce an audible sound when operated as radiophones; but this might have been owing to lack of intensity of the light stimulus.

Experiments must be made, using stimuli of equal energy of different wave lengths, and noting the heating produced, etc., in order to determine how much of this decrease in resistance is caused by rise in temperature and how much of—and for what wave lengths—this decrease in resistance is caused by a true photoelectric (radioelectric) change in this substance. Bismuthinite is sometimes seleniferous, and it will be of interest to determine whether the presence of selenium has any effect upon its photo-electrical sensitivity.

MOLYBDENITE (MoS_2)

The samples of molybdenite examined were soldered to copper terminals. A number of samples, which were 0.03 to 0.07 mm in thickness, 2 to 3 mm in width, and 5 to 20 mm in length, were tested. When exposed to the standard lamp a deflection of 2 to 3 cm was noted. The radiations from this same source, which remained after passing through red glass produced half this deflection; and large deflections were obtained for radiations from the hand and from a hot soldering copper.

These same samples, also others held by compression between copper electrodes, produced an audible vibration when operated as radiophones in connection with the audion amplifier. The sound produced was to some extent a function of the thickness of the lamina exposed to radiation. Some samples were sensitive in spots, as observed in bismuthinite. Audible vibrations were produced in spots on (1 mm) thick samples which were exposed to radiation.

No marked increase in loudness of the sound was observed after covering a sample of molybdenite with a thin layer of (soot) lampblack. This is probably to be expected in view of the fact that the reflecting power⁶ of molybdenite is only about 30 per cent, so that the effect of the increased absorption caused by the lampblack might be outweighed by the increased heat capacity, etc.

The behavior of molybdenite when operated as a radiophone is similar to that of platinum and gold leaf, already described; with this difference, that in the former the resistance decreases with rise in temperature. From tests made in this laboratory and from the measurements made by Reichenheim,⁷ the temperature coefficient of resistance of molybdenite is about $\alpha = 0.0026$ to 0.003.

⁶ Coblenz, Publication No. 97, p. 14. Carnegie Institution of Washington, 1908.

⁷ Reichenheim, Inaug. Diss. Freiburg, p. 26; 1906.

In the case of cylindrite, molybdenite, etc., the alleged photoelectric effect and the rise in temperature, which results from the absorption of radiant energy, both tend to decrease the electrical resistance of the substance. From rough tests, made with radiations dominating in intensity at wave lengths 0.5μ , 0.75μ , 1.2μ , 2.5μ , and 4.4μ , respectively, the results show that molybdenite is sensitive to all of these radiations, and has a maximum sensitivity at about 1μ . For radiations dominating in intensity at 8μ to 9μ the tests are inconclusive to establish a resistance change which is greater than would result from absorption of radiant energy and the consequent rise in temperature, which, in comparison with selenium, is a criterion for distinguishing these two phenomena.

Further investigations are therefore in progress to determine (1) to what extent this decrease in electrical resistance is caused by rise in temperature when this substance is exposed to radiation, and (2) to what extent it is caused by radioelectric activity when stimulated by radiation of equal energy value but of different wave lengths.*

Molybdenite has a high specific resistance, but from consideration of some of its physical properties it seems to be promising material for a bolometer, in stellar radiometry where a short receiver of high resistance will be useful in connection with a string galvanometer or with an audion amplifier. It is quite sensitive to fluctuations in light intensity, so that, like selenium, it produces an audible vibration in a telephone receiver without the use of an amplifier.

SUMMARY

This paper summarizes the results of an investigation of various substances (1) for an increase in electrical conductivity caused by the action of light upon them, and (2) for photoelectrical activity when they were charged to a negative potential in an evacuated chamber and exposed to light.

Pure gallium and silver sulphide were found to have but small photoelectric discharging action when charged to a negative potential and exposed to light.

No change was observed in the electrical conductivity of tellurium, boleite, pyrites, silicon, and mixtures of the sulphides of lead and antimony when exposed to light.

* As this paper goes to press further data are being published; Amer. Phys. Soc., Apr. 27, 1918.

An increase in electrical conductivity was observed in crystals of selenium, stibnite, boulangerite, jamesonite, bismuthinite, cylindrite, molybdenite, and silver sulphide when exposed to light.

Radiophonic experiments are described in which some of these substances were joined through a battery to the grid circuit of an audion amplifier and a telephone. A change in current in this circuit affected the telephone. The light stimulus was interrupted by means of a rotating sector disk, as used in Bell's selenium photophone. When using a cell or crystal of selenium the fluctuations in light intensity produced a sufficient change in conductivity to cause a musical note in the telephone. Similarly, in several samples of a crystal of bismuthinite a change in conductivity was produced, which caused an audible vibration in the telephone. In the case of bismuthinite this change in conductivity (production of an audible vibration) is confined to quite narrow regions along the length of the crystal; occurring usually at the intergrowth or line of contact of two bundles of acicular crystals. In the case of molybdenite the change in conductivity is to some extent a function of the thickness (heat capacity) of the lamina. Further experiments are in progress to determine to what extent and for what wave lengths this resistance change is a true photo-electric phenomenon.

Experiments are described in which a thin, blackened strip of platinum or gold leaf is joined through a battery to an audion amplifier. The variation in temperature and hence resistance of, and current through the strip, caused by the fluctuation in intensity of the intermittent light, was sufficient in magnitude to produce an audible sound in the telephone receiver.

WASHINGTON, December 17, 1917.

SOME CHARACTERISTICS OF THE MARVIN PYRHELIOMETER

By Paul D. Foote

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I. INTRODUCTION

Pyrheliometers are usually calibrated by comparison observations with a standard pyrheliometer which has been calibrated by exposing the receiver to a measured quantity of electrical energy, or to radiation of long wave lengths emitted by a radiator electrically heated to a temperature slightly above that of the room, both the instrument under test and the standard being sighted upon the sun. There are no published measurements of a calibration made by exposing the receiver to a known amount of energy of radiation from a radiator at high temperatures. Such a calibration is highly desirable, for always the question may arise as to whether the thermal processes taking place in the pyrheliometer receiver are exactly similar during the electrical calibration and during the radiometric use of the instrument. In the present work both methods were employed for the calibration of a Marvin silver-disk pyrheliometer.

II. DESCRIPTION OF INSTRUMENT

The Marvin pyrheliometer has been the "working" instrument of the United States Weather Bureau for several years. A complete description of the instrument and methods of use is now in

preparation by Prof. Marvin, Chief of the Weather Bureau. This pyrheliometer is dynamic in type in that it is necessary to consider the rate at which the receiver gains heat when exposed to radiation, and the rate at which the receiver loses heat when shaded from radiation.¹

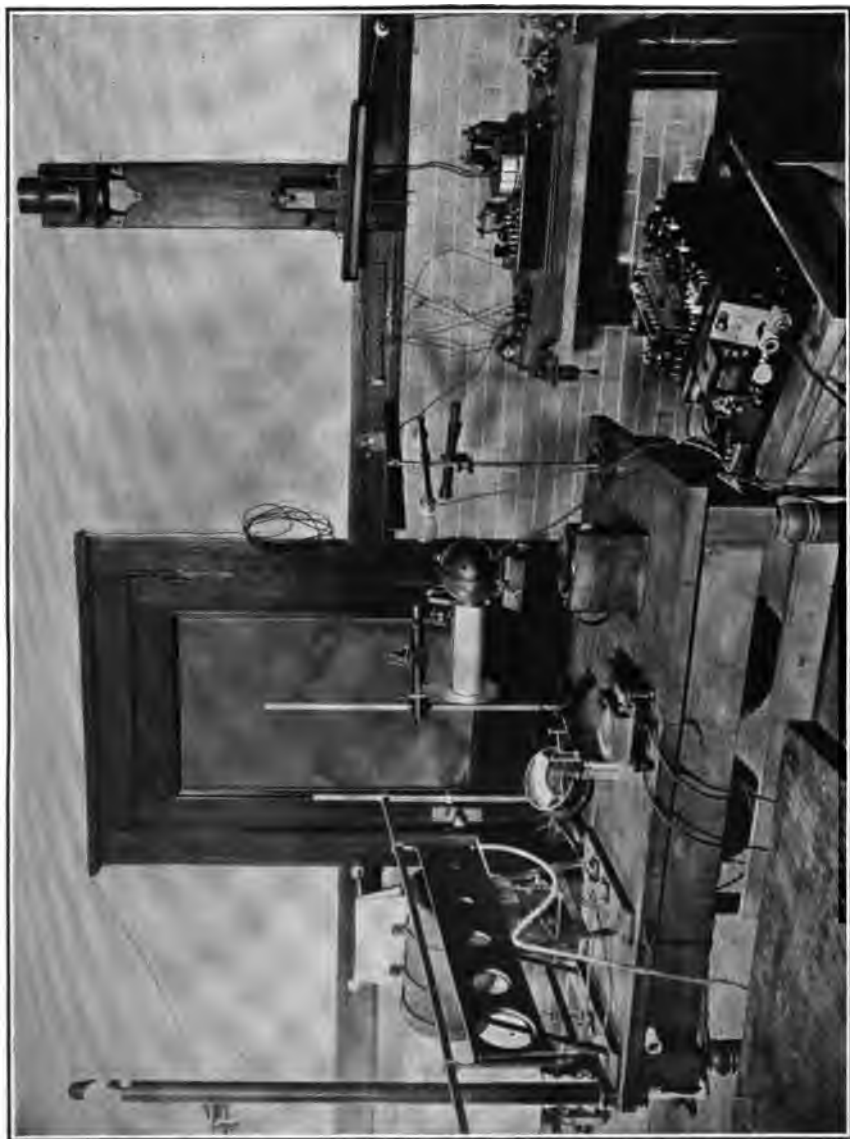
The essential feature of the instrument is the receiver. In the form used in the present work it consisted of a silver disk about 4.5 cm in diameter and 0.3 cm thick, in an annular space inside of which is carefully mounted with the best possible thermal contact a noninductive spirally wound coil of No. 35 silk insulated nickel wire in the form of a 3-lead resistance thermometer, having a total resistance of from 20 to 25 ohms. The coil serves both as the thermometer and as the heater for the purpose of an electrical calibration, the rise in temperature of the thermometer being observed when a known amount of electrical energy is dissipated in the coil. The receiver is mounted within a metal shell, which is incased by a wooden shell in order to reduce local temperature variations to a minimum, and the type of suspension of the receiver is such that thermal losses by conduction are negligible. Before the front face of the receiver a limiting diaphragm is placed and leading from this through a hole in the metal and wooden shells is a diaphragmed and blackened tube which serves the purpose of limiting the cone of rays to a convenient solid angle greater than that subtended by the sun. The end of the tube carries a double-walled aluminum shutter, operated by a magnetic release controlled by a chronograph which may be so set as to open or close the shutter at any desired instant. For solar work the instrument is mounted as an equatorial telescope and is driven by clockwork so that the surface of the receiver is always presented normally to the sun. The instrument is shown in Fig. 1.

III. METHOD OF CALIBRATION, GENERAL

The electrical calibration was made by subjecting the nickel coil of the thermometer to a measured current and observing the change in temperature indicated by the thermometer. The radiometric calibration was made in a similar manner except that the heat was supplied by radiation from an outside source. The source employed was a Lummer-Kurlbaum² black body, or a black body of similar type, electrically heated, with a compensating winding to

¹ Kimball, *Bull. Mount Weather Obs.*, 8, Part 2, p. 73; 1910.

² Waidner and Burgess, *this Bulletin*, 8, p. 163; 1907 (Scientific Paper No. 55).



(806-1)

FIG. 1.—Apparatus used for radiometric calibration of pyrheliometer
From left to right: Furnace, water-cooled diaphragm, pyrheliometer, wheatstone bridge, potentiometer

reduce the temperature gradient and to approximate temperature uniformity. The temperature of the inner inclosure, from which the radiation was taken, was measured by standard platinum, platinum-rhodium thermocouples, accurately calibrated in terms of the melting points of zinc (419.4°), antimony (630.5°), and copper (1083°). A water-cooled diaphragm was mounted directly in front of the opening to the furnace. This diaphragm acts as the effective source of radiation. The equation of rate of energy transfer from the furnace to the pyrheliometer receiver is as follows when R is large compared with $\sqrt{A_1}$ and $\sqrt{A_2}$.

$$J = \frac{\sigma}{\pi} (T^4 - T_o^4) \frac{A_1 A_2}{R^2}$$

where J = energy transferred per unit time from furnace to receiver.

A_1 = area of water cooled diaphragm in front of furnace.

A_2 = area of inmost or effective diaphragm in the pyrheliometer.

T = absolute temperature of furnace.

T_o = absolute temperature of pyrheliometer receiver and surroundings.

σ = the Stefan-Boltzmann coefficient of radiation.

R = distance from A_1 to A_2 .

The quantity T_o^4 is negligible, for the present work, in comparison with T^4 .

The coefficient σ has been determined in a variety of ways by many investigators, with widely discordant results. Within the past two or three years, however, the agreement has been quite satisfactory, values of different observers showing $\sigma = 5.7 \times 10^{-12}$ watts cm^{-2} deg^{-4} . This value is also the mean of all determinations to the present date. The constant is closely connected with the value of the electronic charge, the constant C , of the Planck equation of spectral distribution of the energy of a black body, and with the energy of emission of electrons in photoelectric phenomena. All of these interrelations point to a numerical value of $\sigma = 5.7 \times 10^{-12}$. The most recent work on a direct determination of σ is by Coblentz³ who found as a mean of extended experiments $\sigma = 5.72$. Any error in the assumed value of σ will enter directly into the calibration of the pyrheliometer. It appears unlikely, however, that the value $\sigma = 5.7$ is in error by more than 1 per cent.

³ This Bulletin, 12, p. 553; 1916 (Scientific Paper No. 262).

IV. SIMPLE THEORY OF THE MARVIN PYRHELIOMETER

Pursuing the general treatment given by Kimball⁴ and others for various pyrheliometers, the following describes the action of the Marvin instrument, omitting the consideration of certain corrections which are discussed later. The case considered is for alternate heating and cooling periods of 60 seconds' duration.

Let q = intensity of solar radiation at the earth's surface in cal./cm² min.

S = entire surface of receiver. The front face is assumed to be a perfect absorber.

s = area receiving radiation.

C = water equivalent of receiver.

h = mean coefficient of heat loss from receiver, assuming Newton's law of cooling.

$E = \frac{hS}{C}$ a constant for any particular instrument.

T = temperature of receiver.

T' = temperature of surroundings.

T_1 = temperature at beginning of cooling.

T_2 = temperature at beginning of heating.

t = time in seconds.

The variable thermometric state is expressed by

$$CdT = qsdt - hS(T - T') dt \quad (1)$$

Cooling.—When the shutter is closed $q = 0$, whence

$$\frac{dT}{T - T'} = \frac{d(T - T')}{T - T'} = -\frac{hS}{C} dt \text{ since } T' \text{ is assumed constant.}$$

$$T - T' = Ae^{-\frac{hS}{C}t} = Ae^{-Et}$$

when $t = 0$

$$T = T_1$$

whence $T = T' + (T_1 - T')e^{-Et}$ the equation for cooling (2)

$$\text{Heating.}—\text{From (1) } \frac{dT}{(T - T') - \frac{qs}{EC}} = -E dt$$

and since $T = T_2$ when $t = 0$

$$T = T' + \frac{qs}{EC} + \left(T_2 - T' - \frac{qs}{EC}\right)e^{-Et} \text{ the equation for heating.} \quad (3)$$

The amount of cooling in 60 seconds is found from equation (2) by substituting 0 and 60 for t and subtracting, whence:

$$\text{Cooling in 60 seconds} = \Delta T_0 = T_1 - T' - T_1e^{-60E} + T'e^{-60E} \quad (4)$$

⁴ Bulletin Mount Weather Obs., 8, Part 2; 1910.

By similarly using equation (3)

$$\text{Heating in 60 seconds} = \Delta T_h = T' + \frac{qs}{EC} + \left(T_2 - T' - \frac{qs}{EC} \right) e^{-60E} - T_2 \quad (5)$$

In the use of this instrument the temperature of the receiver is so adjusted that the amount of cooling ΔT_c equals the amount of heating ΔT_h , so that after a two-minute period, consisting of one heating and one cooling, the receiver returns to its original temperature. Whence on equating (4) and (5) one obtains

$$T_1 - T' = -T_2 + \frac{qs}{EC} + T' \quad (6)$$

But $T_1 - T_2 = \Delta T$ = the total change in temperature during a 60-second period. Whence from (6) and (2) it can be shown that

$$\Delta T_{60} = \frac{qs}{EC} \frac{1 - e^{-60E}}{1 + e^{-60E}} = \text{constant} \cdot q = \frac{1}{F} \Delta Q \quad (7)$$

where F is a constant and ΔQ the heat incident upon the receiver in 60 seconds.

Equation (7) states that the change in temperature in 60 seconds equals a constant of the instrument times the intensity of solar radiation and is independent of the room temperature. This conclusion is derived on the assumption of Newton's law of cooling, but since all the temperature differences usually observed are fairly small, there is probably no serious question in regard to the applicability of this law. Questions of greater moment are those regarding the temperature distributions in the receiver under the four different conditions of cooling and heating, electrically and radiometrically, and the lag of the thermometer. These are considered later.

An experimental condition which is of interest, as will appear below, is that in which the readings over the first 10-second period immediately following a shift of the shutter are discarded, and the change in temperature of the receiver is observed for the 50-second interval immediately following. It will be shown that the fall in temperature during such a 50-second interval of cooling is equal to the rise in temperature during the corresponding 50-second interval of heating.

From equation (2), during cooling

$$T_{10} - T_{60} = (T_1 - T') (e^{-10E} - e^{-60E}) \quad (8)$$

From equation (3), during heating

$$T_{60} - T_{10} = - \left(T_2 - T' - \frac{qs}{EC} \right) (e^{10E} - e^{-60E}) \quad (9)$$

But from (2) and (3) evaluated over a 60-second interval we obtain the following:

$$T_1 - T' = \frac{\Delta T_{60}}{2} + \frac{1}{2} \frac{qs}{EC} \quad (10)$$

$$T' - T_2 = \frac{\Delta T_{60}}{2} - \frac{1}{2} \frac{qs}{EC} \quad (11)$$

Substituting (10) and (11) in (8) and (9) it is seen that the cooling in 50 seconds equals the heating in 50 seconds.

If we denote the amount of heating or cooling in the 50-second interval defined above by ΔT_{50} , the relation between ΔT_{50} and ΔT_{60} is given by equations (4) and (8) as follows:

$$\Delta T_{60} = \frac{1 - e^{-60K}}{e^{-10K} - e^{-60K}} \Delta T_{50} \quad (12)$$

Equation (12) is discussed later.

V. METHOD OF CALIBRATION—DETAILED DESCRIPTION

In the radiometric calibration the pyrliometer was alternately exposed to and shaded from the radiation emitted through an opening in a blackened, water-cooled diaphragm placed in front of the black body. The opening through the water-cooled compartment was conical and of greater angle than that required by the pyrliometer. Care was always taken to operate the pyrliometer at such a distance from this diaphragm that the radiation striking the receiver came from the portion of the black body furnace of which the temperature was measured by the thermocouples. The temperature gradient throughout this section of the black body was easily made practically negligible.⁵

The electromotive forces of the two thermocouples were measured by a Leeds and Northrup potentiometer, shown in Fig. 1, and in converting the electromotive force readings to temperature due consideration was given to the correction of the "thermoelectric" scale to the thermodynamic scale.⁶

The resistance of the thermometer of the pyrliometer was measured by the precision thermostated bridge described in detail by Mueller.⁷

This dial bridge reads directly to 0.0001 ohm, and by interpolating the readings of the galvanometer, the sensibility of which

⁵ Waldner and Burgess, this Bulletin, 8, p. 167; 1907 (Scientific Paper No. 55).

⁶ This Bulletin, 9, p. 563; 1913 (Scientific Paper No. 202).

⁷ This Bulletin, 18, p. 547; 1916 (Scientific Paper No. 888).

was determined at all points of the scale used, at least one more figure in the resistance measurement could be relied upon. The scale deflections were always maintained as nearly zero as possible.

The small measuring current passing through the thermometer coil in general had an appreciable heating effect. The current was maintained constant by using rather high voltage and external resistance so that the small variations in resistance of the thermometer were negligible in their effect upon the magnitude of the current. When necessary, the current was measured by the

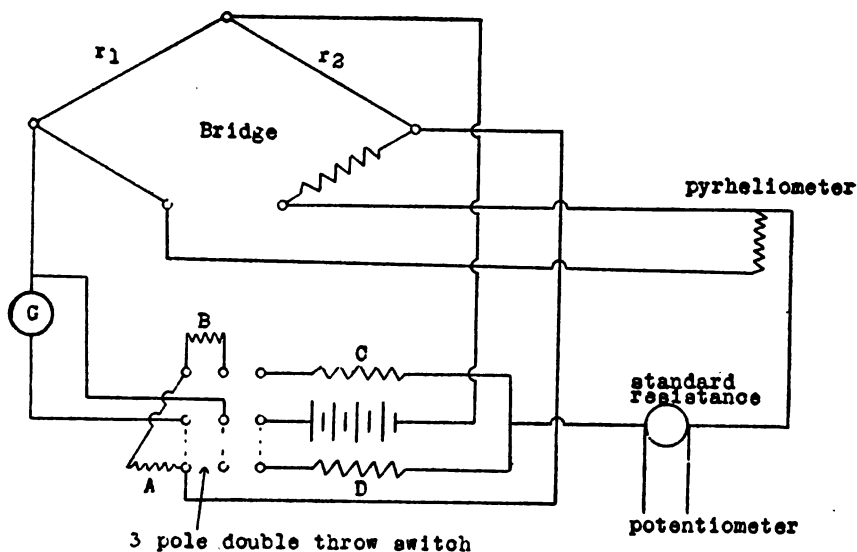


FIG. 2.—Wiring diagram for obtaining observations during both heating and cooling of an electrical calibration

Magnitude of heating and measuring current controlled by rheostats C and D. Galvanometer G critically damped and maintained at the same sensibility during heating and cooling by rheostats A and B.

potentiometer method and controlled by an adjustable low-resistance slide wire. The constant current was allowed to flow during the entire radiometric calibration, during both heating and cooling of the receiver, so that it produced no net effect upon the measurements. Fig. 1 shows the experimental arrangement for a radiometric calibration.

The main difficulty with the electrical calibration was to devise a suitable means for measuring the resistance during the period of heating. The wiring diagram is given in Fig. 2. The maximum current required for heating was small enough to permit the use of the bridge in the manner illustrated. The galvanometer sensibility was maintained approximately the same for the bridge

currents during both heating and cooling, and the galvanometer was critically damped in each case by suitable series and shunt resistances. Both thermometer currents were measured with a potentiometer and were maintained constant by adjusting a slide-wire rheostat. The heating by each current was computed, using the mean resistance of the thermometer coil throughout the periods of heating and of cooling. The net heating effect is equal to the difference between the heat developed during the period of heating and that developed during the period of cooling.

In a large portion of the work, for both the radiometric and the electrical calibrations, readings were taken every 10 seconds in order to determine accurately the forms of the heating and cooling curves. For enabling the observer to obtain readings exactly at the time desired, the chronograph was arranged to tap a bell at the proper instant. One observer operated the bridge and another the potentiometer.

The determination of the relation between the temperature of the thermometer and its resistance requires an independent experiment in which the receiver is removed from the pyrheliometer and mounted in a constant temperature bath, the temperature of which may be varied over the range required. The temperature relation so found may be accurately expressed by a parabolic equation. Thus, for silver block No. II, $R=23.041+0.10033t+0.001093t^2$ and for silver block No. III $R=19.521+0.08394t+0.00010127t^2$ where t is the temperature centigrade. These data were obtained by Prof. H. H. Kimball, of the United States Weather Bureau.

The simple theory derived in Section IV requires certain modifications to fit the actual experimental conditions. The room temperature T' did not remain constant during a run of an hour required for any one set of measurements. The result was that there was a gradual variation back and forth in the equilibrium position of the thermometer temperature so that the thermometer did not return exactly to its initial temperature after a complete cycle of one heating and one cooling. Each observed heating and cooling curve may be readily corrected to the ideal curve by assuming a linear change in temperature over the complete cycle in question. Thus, for example, if, after a cycle of two minutes, the temperature of the thermometer coil had risen 0.001° , the measurements at every 10-seconds interval were corrected by adding $1/12$, $2/12$, $3/12$, etc., of 0.001° to the first, second, third, etc., reading of the cycle. A correction method developed by Prof. Marvin, which

is more convenient and accurate and which will be described in his paper referred to above, was frequently employed. The corrected curves were always used in the computation of the results. In general, variations of the above description tend to eliminate themselves over a long period, the temperature rising for part of the series and falling for the rest of the time. However, runs were obtained for all possible conditions, namely, continually increasing temperature, continually decreasing temperature, and practically constant temperature. No dependence of the results upon such conditions could be observed.

VI. EFFECT OF LAG

A second modification of the simple theory discussed in Section IV is required by the presence of a lag in the temperature or resistance measurements. In order to obtain electrical insulation of the thermometer coil from the silver receiving disk, the wire is silk covered. This silk covering, of course, acts also as a thermal insulator. The result is that the temperature of the thermometer lags behind that of the disk when the heating is radiometric, and leads the temperature of the disk when the heating is electrical. In the case of an entire cooling following a radiometric heating and for a part of the cooling following an electrical heating, the thermometer temperature again lags behind that of the disk. If the lag effect is not corrected for, the experimentally determined value of the constant F (Sec. IV, equation (7)) increases for radiometric exposures and decreases for electrical exposures as the time of exposure is shortened. When the heat is supplied electrically directly in the thermometer coil, the lag effect due to thermal insulation increases the observed ΔT over the value of ΔT for the disk, and thus the measured F is too low. The reverse is true when the heating is radiometric. In order to detect the presence of lag, the receiver was exposed to radiation or to the heating current and then cooled, the periods of heating and cooling being equal and of the following durations: 10, 20, 30, 60, and 120 seconds. Observations were made at the beginning and the end of the coolings and heatings; that is, at the instant of shifting from a cooling to a heating and vice versa. The results of several experiments with Marvin pyrheliometer No. 7, disk No. 3, are given in Fig. 3. Here the ordinates are values of F' obtained by multiplying the value of F , Section IV, equation 7, by $\frac{(5.5)(1.20)}{1.217}$. The constant F' is more convenient than F

in reducing solar measurements to calories/cm². min. and will be used exclusively hereafter. If there were no lag effect whatever, the two curves in Fig. 3 would coincide exactly and would pass through the point $F' = 2.135$ at $t = 60$ seconds. The curve F' versus t then would differ slightly from a horizontal straight line, because of the fact that the cooling of the disk follows Newton's exponential law instead of a linear relation with time. The

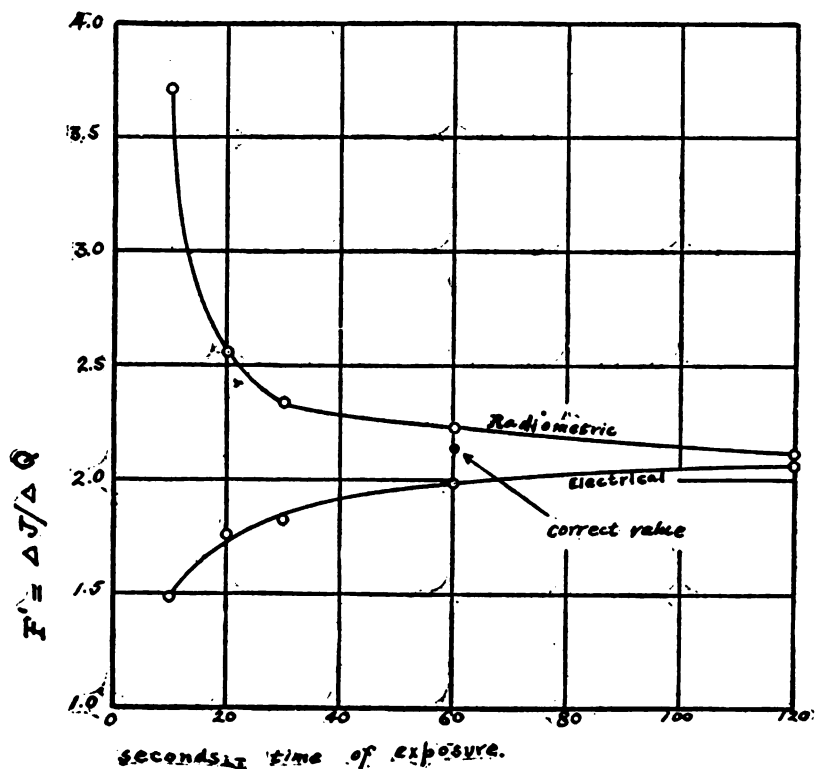


FIG. 3.—Observed F' as a function of time of exposure

If there were no lag, the curves for radiometric and electrical calibrations would be coincident and intermediate to the two curves shown

above experiment demonstrates clearly the presence of a lag effect. The magnitude of this lag may be calculated in the manner shown below.

VII. ANALYSIS OF THE HEATING AND COOLING CURVES

Heatings and coolings were made alternately every minute for a period of one half to one hour, during which time bridge readings of the thermometer temperature were taken every 10 seconds. The heatings and coolings for the entire period were used to form

a typical heating and cooling curve. This curve was found to be an exponential, as would follow from Newton's law of cooling. Thus, in the case of cooling according to Newton's law we have, as in Section IV:

$$\frac{dT}{dt} = -E (T - T') \quad (13)$$

Since resistance is measured directly, instead of temperature, it is more convenient to express equation (13) in terms of resistance R . Over small ranges of temperature, change in temperature is proportional to change in resistance, that is:

$$R = a + bT$$

$$\text{and } \frac{dR}{dt} = b \frac{dT}{dt}$$

Hence

$$\frac{dR}{dt} = -E (R - R') \quad (14)$$

$$R = R' + Ae^{-Et} \quad (15)$$

$$\frac{dR}{dt} = -AEe^{-Et} \quad (16)$$

$$\log_{10} \left(-\frac{\Delta R}{\Delta t} \right) = \log_{10} AE - 0.4343Et \quad (17)$$

If the observations, accordingly, satisfy Newton's law of cooling

the plot of $\log_{10} \left(-\frac{\Delta R}{\Delta t} \right)$ versus t will be a straight line. The slope

of this straight line determines E , the coefficient of cooling. As an illustration of this method the following examples may be cited:

TABLE 1.—Cooling and Heating Curve Data, Electrical Calibration

	Time	($R_t - R_0$) observed		Time	($R_t - R_0$) observed
	Seconds	Ohms		Seconds	Ohms
Heating	0	0.00000	Cooling	10	0.02612
	10	.00690		20	.02053
	20	.01241		30	.01521
	30	.01780		40	.01004
	40	.02296		50	.00495
	50	.02805		60	.00000
	60	.03292			

TABLE 2.—Analysis of Heating and Cooling Curves, Electrical Calibration

Mean time	$-(\Delta R/\Delta t)_{10 \text{ sec.}}$			$\log \left(\frac{-\Delta R}{\Delta t} \right)$		$-\Delta R/\Delta t$, computed
	Heating	Cooling	Mean	Observed	Curve	
Seconds						
5	0.000690	0.000680	0.000685	6.836-10	6.748-10	0.000660
15	531	557	544	.736	.737	546
25	539	534	536	.729	.726	532
35	516	517	516	.713	.714	518
45	509	509	509	.707	.703	505
55	487	495	491	.691	.691	491

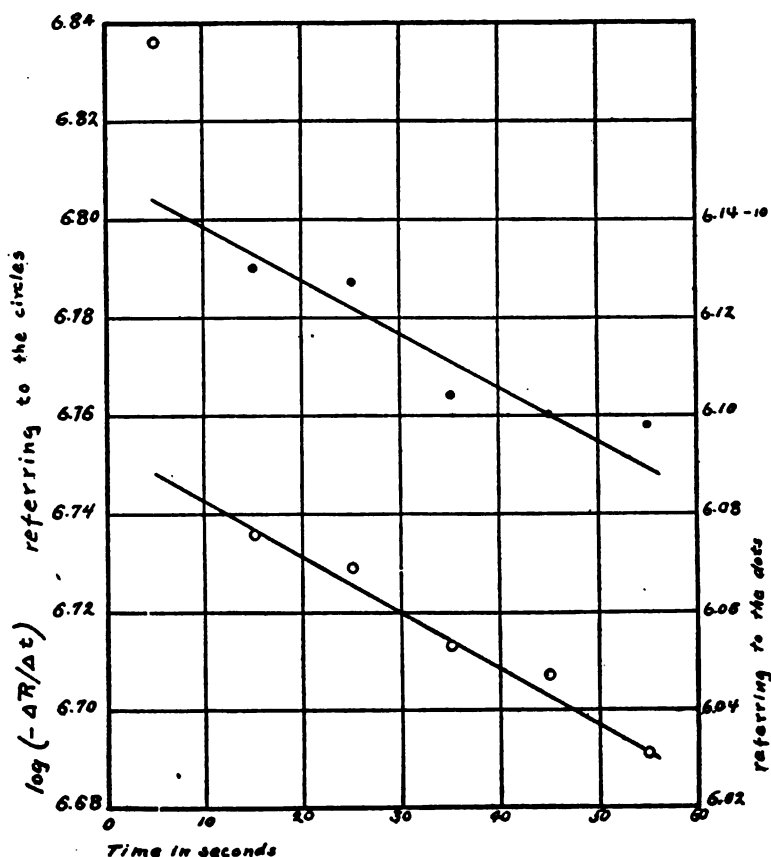


FIG. 4.—Method of plotting heating and cooling curves

The plot of $\log \left(\frac{-\Delta R}{\Delta t} \right)$ versus t is given by circles in Fig. 4.

Columns 6 and 7 of the above table are obtained from the values given by this plot. On account of the lag (in this case a lead) the

observed value of $\Delta R/\Delta t$ in the interval 0 to 10 seconds is far too great. After the first 10 seconds, however, the observations lie very satisfactorily on the straight line of Fig. 4. The coefficient E determined by the slope of the straight line is $E = 0.0027$. From columns 2 and 3 it is seen that the heating curve is similar in type to the cooling curve as demanded by the theory developed in Section IV. This series was picked at random from a large number. Some of the series obtained showed exact agreement of the heating and cooling curves; others showed somewhat poorer agreement than the example cited. Since every point on the cooling curve except the point taken for time 0 to 10, mean $t = 5$, lies well upon the straight line of Fig. 4, it is clear that the error due to lag is negligible (see below) after 10 seconds. Hence, in the general use of the instrument it is sufficient to take readings 10 seconds and 60 seconds after the starting of a heating and of a cooling, thus giving a heating or cooling over a 50-second interval. To obtain the correction factor, f , to convert a 50-second heating or cooling into the equivalent δ 60-second interval, use is made of equation (12)

$$f = \frac{1 - e^{-60E}}{e^{-10E} - e^{-60E}} = 1.217$$

for $E = 0.00269$

TABLE 3.—Cooling and Heating Curve Data, Radiometric Calibration

	Time	($R_1 - R_0$) observed		Time	($R_1 - R_0$) observed
	Seconds	Ohms		Seconds	Ohms
Heating	0	0.00000	Cooling	10	0.00646
	10	112		20	511
	20	247		30	377
	30	381		40	250
	40	508		50	124
	50	634		60	000
	60	759			

TABLE 4.—Analysis of Heating and Cooling Curve, Radiometric Calibration

t	$-\Delta R/\Delta t$			$\log (-\Delta R/\Delta t)$		$-\Delta R/\Delta t$, computed
	Heating	Cooling	Mean	Observed	Curve	
Seconds						
5	0.000112	0.000113	0.000112	6.049	6.144	0.000139
15	135	135	135	.130	.133	136
25	134	134	134	.127	.122	132
35	127	127	127	.104	.111	129
45	126	126	126	.100	.100	126
55	125	124	124	.093	.089	123

The plot of $\log \left(\frac{-\Delta R}{\Delta t} \right)$ versus t is given by dots in Fig. 4. Columns 6 and 7 are obtained from the values given by this plot. On account of the lag effect the observed value of $\Delta R/\Delta t$ in the interval 0 to 10 seconds is too small (in the case of electrical heating it is too great). After the first 10 seconds, however, the observations lie satisfactorily on the straight line of Fig. 4. The coefficient E was found to be 0.0025. Columns 2 and 3 show that the heating and cooling curves are similar. The correction factor to convert a 50-second heating or cooling interval into the equivalent for a 60-second interval is $f = 1.215$.

Several series of experiments were made to determine the form of the heating and cooling curves as illustrated above, for various amounts of energy expended both electrically and radiometrically in the receiver. These are summarized in the following table. The first column gives the total energy supplied to the receiver in a one-minute interval.

TABLE 5.—Summary of Preliminary Calibrations

ELECTRICAL CALIBRATION						
Calories per minute	Reduced ΔR_{60}	$R_{60} - R_{10} = \Delta R_{50}$	f	$E \cdot 10^6$	Observed $\Delta R'_{60}$	$\Delta R_{60}/\Delta R'_{60}$
0.830	0.01756	0.01443	1.217	265	0.01956	0.898
.825	.01765	.01450	1.217	264	.01961	.900
1.475	.03152	.02592	1.216	269	.03292	.958
.365	.00782	.00642	1.218	288	.00827	.946
Mean.....						.938
RADIOMETRIC CALIBRATION						
0.135	0.00323	0.00265	1.219	272	0.00307	1.052
.197	.00470	.00389	1.208	154	.00452	1.040
.154	.00375	.00308	1.218	269	.00360	1.042
.253	.00622	.00510	1.220	299	.00598	1.040
.320	.00785	.00646	1.215	249	.00758	1.036
Final means.....			1.216 ₄	261	Mean.....	1.04

The above values were taken with the receiver in various conditions as to the blackening upon its surface. Alterations in the surface affect the values of f and E to some extent. In the later work $f = 1.217$ was employed, since this value has been used by the Weather Bureau for some time and is practically identical with the above determination. However, the exact value of f is of no material significance, since it simply cancels out in the final computations. The important point is whether f is the same for an electrical calibration as for a radiometric calibration. That the values are identical in the two cases is amply shown by the

above table. On account of the lag, with readings taken over a 60-second interval, instead of a 50-second interval, the true change in resistance, ΔR_{60} , is less than the observed change, $\Delta R'_{60}$, for the electrical calibration, in the ratio $\Delta R_{60}/R'_{60} = 0.93$. For the radiometric calibration the true change ΔR_{60} is greater than the observed change, $\Delta R'_{60}$, in the ratio 1.04. Hence, for one-minute exposures with observations taken over 60-second intervals the electrical and radiometric calibrations will differ by about 12 per cent. This again illustrates the necessity of correcting for the lag.

As shown above, the observed cooling curve follows Newton's law of cooling after the first 10 seconds. This is what would be expected if the lag is small. The change in temperature of the receiver in the 50-second period is equal to that of the thermometer, although the thermometer is actually at a higher temperature, during cooling, than the disk. The observed lag is the result of two causes—the insulation around the thermometer coil and the lag of the galvanometer. It will be shown later that in the case of the radiometric calibration, the measured temperature of the coil lags from one to two seconds behind the effective temperature of the disk. On the assumption that the temperature of the disk is uniform throughout, the effect of the lag can be estimated in the manner shown by Harper.⁸

u = temperature of disk at time t

θ = temperature of thermometer at time t

λ = lag in seconds.

$$\frac{\delta\theta}{\delta t} = \frac{1}{\lambda}(u - \theta) \dots \dots \dots (18)$$

$$u = A + Be^{-Et} \text{ Newton's law for the period of cooling.} \dots (19)$$

Suppose that at time $t = 0$, $u = u_0 = \theta = \theta_0$.

Substituting (19) in (18) and integrating.

$$\theta - u = \left(\theta_0 + \frac{E\lambda A - u_0}{1 - E\lambda} \right) e^{-t/\lambda} + \frac{E\lambda}{1 - E\lambda} (u - A) \dots \dots \dots (20)$$

For an actual case, when the pyrheliometer was sighted on the sun the following values of θ_0 , u_0 , and A were obtained by converting the resistances observed into equivalent temperatures.

$$\lambda = 2$$

$$E = 0.0026$$

$$u = 10.0139 + 23.8763e^{-.0026t}$$

$$A = 10.0139$$

$$u_0 - \theta_0 = 33.8902$$

⁸ This Bulletin, 8, p. 666; 1912 (Scientific Paper No. 185).

Substituting these values in (20) one finds:

$$\theta - u = -0.125e^{-.5t} + 0.00523 (u - 10.0139) \dots \dots \dots (21)$$

and evaluating equation (21) for various times, the following table is obtained:

TABLE 6.—Computation of Thermometric Lag

<i>t</i>	<i>u</i>	$0.125e^{-.5t}$	$0.00523 (u - A)$	$\theta - u$
Seconds	°C	°C	°C	°C
0	33.8902	0.125	0.1249	0.000
5	33.5822	.01026	.1233	.113
10	33.2766	.00084	.1217	.121
20	32.6797	.000006	.1185	.118
30	32.0995		.1155	.116
60	30.4425		.1068	.107

Thus, as shown by column 5 of the above table, the temperature of the thermometer coil after the first 10 seconds of cooling is higher than that of the disk by practically a constant amount,

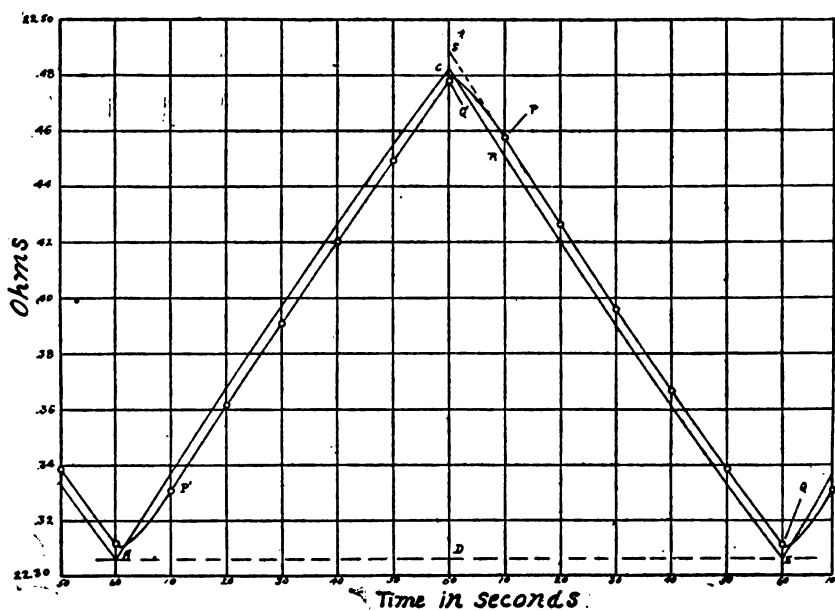


FIG. 5.—Typical radiometric heating and cooling curve

0.11 to 0.12°, for the series of observations made by sighting on the sun. The change in the temperature of the thermometer during cooling is therefore identical with the change in the temperature of the disk over the 50-second interval. If similar computations are made for a heating curve, it will be found that the

temperature of the thermometer is less than that of the disk by essentially a constant amount over the 50-second interval and that the change in temperature of the thermometer is again identical with the change in temperature of the disk.

A typical heating and cooling curve for a radiometric calibration is shown in Fig. 5. The small circles represent observed points. The line ACE represents the corrected heating and cooling curve. The total change in temperature of the disk in terms of the coil resistance is given by the distance CD . This is equal to the observed change in 50 seconds, PQ or $P'Q'$, multiplied by the

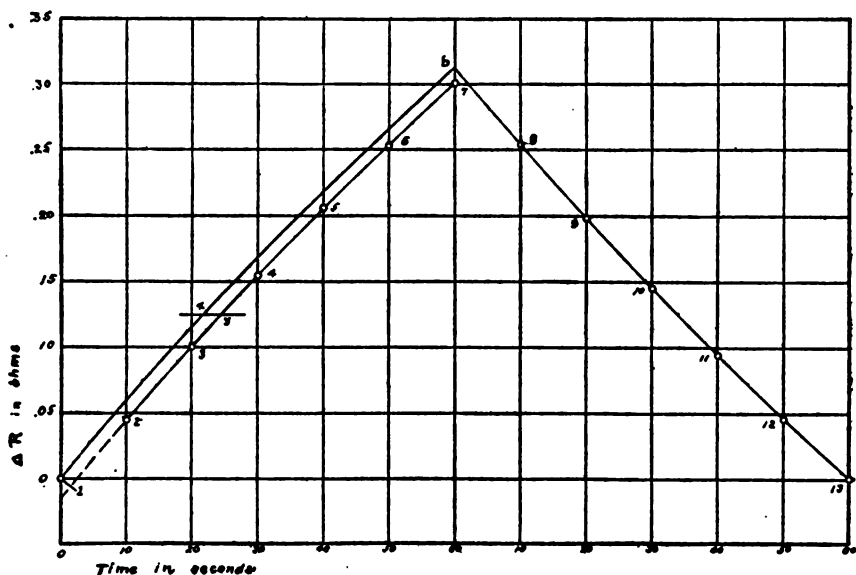


FIG. 6.—Method of determining lag

factor 1.217. It is also equal to the computed change RE in 50 seconds multiplied by the same factor. Another way to obtain the true change in 60 seconds is to extrapolate the observed curve PQ back to zero time (or to the time of the beginning of a cooling) and take the difference in the ordinates of the points S and Q .

The graphical method of obtaining the lag for a radiometric calibration is shown in Fig. 6.

The observed points are represented by circles. Through points 2, 3, 4, 5, 6, 7, and 8, 9, 10, 11, 12, 13 pass the best computed exponential curves. Extrapolate the cooling curve to the point b and move the heating curve up to the position $1 b$. The distance xy measured along the time axis is twice the lag of the thermometer.

In the case of electrical heating, the lag effect is somewhat more complicated. During the heating the thermometer coil is at a higher temperature than the disk. The thermometer "leads" the disk on this account. But the galvanometer on the bridge still lags. The net result may be effectively either a lag or a lead. It might be possible, also, although this was not observed, that the readings of the bridge represent the true temperature of the disk, the galvanometer lag exactly balancing the thermometer lead. In cooling, the temperature of the thermometer rapidly falls behind that of the disk, and the galvanometer and thermal insulation of the coil act together to produce a lag. The lag λ was taken as half the distance xy , Fig. 6, even for the electrical heatings. This is not exactly correct, but the main object of the following table is to show that the effective lag was small. The lag due to the galvanometer alone was about one second. Hence, at least for the radiometric heating, it would appear that the lag due to the thermometer were negligible. The data, however, can not be considered quantitative.

TABLE 7.—Thermometric Lag

Method	λ (lag)	Method	λ (lag)
	Seconds ^a		Seconds ^a
	1.5		3.4
	1.2		3.2
	1.7	Electrical.....	3.0
Radiometric.....	1.3		1.4
	1.0		1.7
	1.2		
	1.1		
	0.9		

^a Mean of all = 1.7 seconds.

That the lag can not be large is also clearly demonstrated every time a pyrheliometer is used. If the lag were large, the resistance would continue to increase, after an exposure, for some time after the shutter was closed. Actually the galvanometer reverses direction showing decreasing resistance almost at the instant the shutter is closed. In the derivation above showing that the lag effect is inappreciable after a few seconds, it was assumed that both the coil and the disk were at the same temperature at the instant of shifting the shutter; that is, when $t=0$. Actually, since the previous heating or cooling has an effect, the two are at a slightly different temperature at $t=0$, but at some

later instant, as is seen from Fig. 5, the cooling curve of the disk crosses the cooling curve of the thermometer, and thus the two show the same temperature at that point. Since the time that the disk and thermometer show the same temperature is so nearly $t=0$, the computations in the above discussion would be not materially altered. Accordingly it has been proven that the actual lag is small and that its effect is negligible after 10 seconds. Cooling and heating curves were made for exposures of 120 seconds, but in the actual use of the instrument it does not appear that anything is gained by doubling the time of exposure.

Blackening of the Receiver.—The choice of a proper method of blackening the receiver is of considerable importance. An improperly blackened receiver may show a reflection coefficient as high as 10 per cent. One pyrheliometer examined had a poorly blackened receiver. The result was that the radiometric

calibration ($\text{observed value of } F = \frac{\Delta Q}{\Delta T}$) was far too high and the

electrical calibration too low. The most satisfactory black surface was prepared in the manner specified by Coblenz.⁹ The receiver is first given a thin coat of a mixture of fine lampblack and platinum black in alcohol with sufficient turpentine to make it adhesive. The disk is then smoked with soot from a sperm candle. The smoke is produced by holding a small iron funnel over the candle. When properly prepared, such a surface absorbs about 98.8 per cent of the radiation falling upon it. The loss by reflection, 1.2 per cent,¹⁰ can be corrected for in working up the calibration data.

VIII. FINAL CALIBRATION OF MARVIN PYRHELIOMETER— RECEIVER NO. 3, CAREFULLY BLACKENED

The following table summarizes the results of various experiments with this instrument. Each experiment represents a series of observations which in most cases extended over an hour. The first column gives the total energy supplied to the disk during each minute of heating. It is clearly shown that the calibration constant F' , determined electrically, is independent of the amount of energy supplied although this latter extended over a considerable range.

⁹ Coblenz, this Bulletin, 11, p. 140; 1915 (Scientific Paper No. 229).

¹⁰ Coblenz, this Bulletin, 9, p. 322; 1913 (Scientific Paper No. 191).

TABLE 8.—Final Calibration of Receiver No. 3

ELECTRICAL CALIBRATION

Cal./minute	F'	Cal./minute	F'
0.2704	α 2.104	0.2755	β 2.141
0.2713	α 2.176	0.4925	β 2.116
0.4871	α 2.089	0.4945	β 2.099
0.7659	α 2.098	0.7772	β 2.089
0.7697	α 2.100		
3.136	α 2.100		2.111
7.211	α 2.098		
13.181	α 2.115		
21.321	α 2.101		
	α 2.109		
	2.110		

^a Data taken in November, 1915.^b Supposedly more accurate data taken March, 1916.

RADIOMETRIC CALIBRATION

Cal./minute	F'	Temperature of furnace	Distance between limiting diaphragms
		Degrees abs.	cm
0.1983	2.200	1601	94.0
0.3351	2.173	1602	72.4
0.4631	2.186	1731	71.9
0.3138	2.214	1649	79.2
	2.200		

The above calibrations were made under widely different conditions of room temperature, viz, warm, cold, rising, falling, and steady temperature. Rising and falling room temperatures were corrected for by the methods described above.

The constant F' showed no systematic changes with changes in the room temperature or rate of change of room temperature. The final uncorrected values of F' , the means of the above data, are electrical, $F' = 2.110$; radiometric, $F' = 2.200$. Corrections must be applied to these means as follows:

1. A correction of 1.2 per cent must be made for the reflection of radiation from the surface of the disk. The measurements of Coblentz establish this value fairly accurately and indicate that the reflection is about the same for solar radiation and for radiation from a furnace at 1600° abs. Since this loss by reflection enters alike in the general use and radiometric calibration of the pyrheliometer, it may be omitted. In order to reduce an electrical calibration, however, to the same basis as the radiometric calibration, the constant F' determined electrically must be increased by 1.2 per cent.

2. The black body departs somewhat from the ideal condition of blackness. It has generally been assumed that a white porcelain radiator with diaphragms increasing in opening from the outside to the inner inclosure emits a lesser quantity of radiation than that corresponding to the temperature measured by the thermocouples. A black-body furnace having diaphragms decreasing in size of opening from the outside to the inner inclosure furnishes better black-body conditions, but this type of furnace would so reduce the cone of rays entering the pyrheliometer that the energy transfer would be too small to measure. Even with the system of diaphragms employed the usual energy received during solar measurements is some 30 times greater than the maximum energy which could be obtained in the laboratory. Since the present work was completed a systematic investigation of the furnace employed indicates that the usual assumption that such a furnace is 99 per cent black is incorrect. An optical pyrometer of the Holborn-Kurlbaum type was carefully calibrated by sighting into a black body immersed in crucibles of molten metal.¹¹ Melting and freezing point curves were obtained in the usual manner. The calibration of the pyrometer was thus expressed directly in terms of the melting or freezing points of metals which have been carefully investigated by means of the gas thermometer and resistance thermometer used as a transfer instrument. The black body employed was made of graphite, and when immersed in the metal bath could not depart from the ideal black-body conditions. A series of measurements was made by Mr. Fairchild and the writer by comparing the readings of the optical pyrometer when sighted into the Lummer-Kurlbaum furnace with the readings of the thermocouples corrected as usual to conform with the optical scale. Several different thermocouples standardized in terms of the melting points of zinc, antimony, and copper, and several different pyrometer lamps were employed. Fig. 7 represents the differences in temperature obtained by the two methods, plotted against the readings of the thermocouple. The variation in the readings can not be attributed to errors in the standardization of either the couples or the optical pyrometer. The errors in temperature measurement are considerably less than the differences indicated in this plot. These differences simply represent the deviation of the black body from the ideal condition of blackness, and are determined by various factors, such as adjustment of current in the two heating coils, rate of change or

¹¹ Kanolt, Bureau of Standards Technologic Paper No. 10.

temperature, etc. Referred to the thermocouple readings a furnace of this type radiates more energy than a black body at the same temperature. The side walls are necessarily hotter than the interior or back wall, the temperature of which is measured by the couples. Since the back wall is of white porcelain, probably a good diffuse reflector even at high temperatures, radiation from the hotter side walls is reflected from the back wall through the openings in the diaphragm to the optical pyrometer. In the temperature range used for work with the pyrheliometers (1350 to 1400° C), Fig. 7 would indicate that the thermocouples read too low by about 7°. It is quite possible that the radiator under these conditions is selective, and that while the difference between the thermocouple and optical pyrometer readings is 7°, the difference between the thermocouple and a pyrometer using the

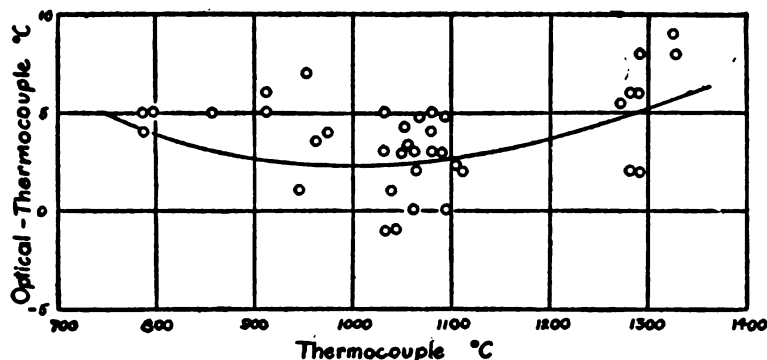


FIG. 7.—Departure of furnace from black body conditions

total radiation would not be this value. However, if we assume that the radiator is nonselective, the error in the total energy emitted, due to the error arising from reflected radiation, follows from the Stefan-Boltzmann law, thus:

$$\frac{\delta J}{J} = 4 \frac{\delta T}{T} = 1.7 \text{ per cent.}$$

This correction is additive to the experimentally observed value of F' as determined by the radiometric calibration. It may be noted that the constant σ has been determined with the Lummer-Kurlbaum furnace, but that the radiator was blackened. This blackening would of course do away with the question of stray-radiation

3. A correction which is subtractive from the value of F' determined radiometrically is due to the absorption of the radiation by the water vapor in the air. For the distances used, Coblentz's¹²

¹² This Bulletin, 12, p. 529; 1916 (Scientific Paper No. 261).

experiments would indicate that this correction may amount to from 1 to 3 per cent depending upon the humidity. Dr. Abbot¹³ states that "Mr. Fowle's¹⁴ recent work enables us to estimate very surely that in the conditions at the Bureau in November and March in 80 cm of air the absorption for rays from a 1600° source would be between 1 and 2 per cent."

4. A further subtractive correction which is small is due to the use of water-cooled diaphragms at a lower temperature than that of the room. This correction is discussed below.

Neglecting the fourth correction the total correction to be applied to the radiometric calibration is as follows:

Observed value of F'	2. 200
Correction for nonblackness of furnace.....per cent..	+1. 7
Correction for humidity.....do.....	-1 to -2
Corrected value of F' , about.....	2. 200

The correction to the electrical calibration is as follows:

Observed value of F'	2. 110
Correction for nonblack receiver.....per cent..	+1. 2
Corrected value of F'	2. 135

The agreement between these two final values is probably within the experimental errors of the radiometric calibration. It would appear that the electrical calibration should have the greater weight, because accidental errors are much smaller in an electrical calibration than in a radiometric calibration. The following considerations, however, based on more recent experimental evidence tend to support the higher value of F' determined radiometrically.

The convection and radiation losses from the receiver follow Newton's law of cooling which states that the energy lost in this manner is proportional to the difference in temperature of the surface of the receiver and the surroundings. Hence the total energy loss by convection and radiation is proportional to the mean surface temperature of the disk. Experiments were made with a disk 0.4 cm thick, having two coils inside instead of a single coil, to determine whether a measurable temperature gradient can exist in the silver disk. The disk employed for these tests should show a gradient much larger than that which would exist in the silver disk No. 3 under similar conditions because of the fact that it contained two insulated coils and was thicker in the ratio 4:3. Measurements made by a differential thermocouple with one junction soldered to the front face of

¹³ By letter.

¹⁴ Fowle, Smithsonian Misc. Coll., 68, No. 8; 1917.

the disk and the other junction soldered to the back face showed a temperature difference between the two faces amounting to 0.3°C during a 60-second exposure to solar radiation of about 0.8 cal./min cm^2 . The gradient dropped to zero within 5 seconds after closing the shutter, remaining zero until the end of the minute if cooling. Within 5 seconds after again exposing to the solar radiation the gradient returned to the value 0.3°C . It is impossible at the present time to attempt a careful study of the temperature differences established in the receiver employed in the work described in the present paper, and even if known their interpretation would be questionable, but the above measurements indicate that surprisingly large gradients may exist in spite of the fact that the disks are constructed of silver and that every precaution is taken to eliminate all air spaces in the interior. The fact that a temperature gradient of even 0.1° may exist, especially since one is not working with equilibrium conditions in an exposure of 60 seconds, shows that the gradient is exponential rather than linear through the disk. Hence, during an exposure to solar radiation, the thermometer is not measuring the mean surface temperature of the disk, but a temperature somewhat less than the mean value. During cooling the gradient through the disk is at least greatly diminished if not negligible. That this condition is not impossible is readily seen. During heating all the energy entering the disk flows through the front face and that not taken up by successive sections of the disk in raising the temperature of the various sections is conducted through the disk and radiated at the back surface. During cooling, however, the energy flows outward in all directions and there is a smaller flow in any one direction. Practically no difference in temperature exists between the front and back face during cooling and it seems probable that the gradient from the thermometer to the outside is small. However small this gradient be, the thermometer must indicate a temperature higher than that of the surface. Now, if exactly the same type of gradient existed in the disk during heating and cooling and if the lag of the thermometer were less than 10 seconds, as experimentally shown, the change during a 50-second period as described above, in the temperature of the thermometer, would be equal to the change in mean temperature of the surface. But when the gradient is shifting during heating from the exponential type to the linear type the change in temperature measured by the thermometer is less than the change in the mean temperature of the surface of the disk. The disk acts exactly as though

its front surface reflected a portion of the incident energy (besides the normal reflection coefficient). Actually, of course, this energy is lost by convection and radiation.

Since the measured ΔT is too small the calibration constant $F' \propto \frac{1}{\Delta T}$ should be higher when determined radiometrically than when determined electrically, as found experimentally. It is impossible to predict what correction should be applied to the electrical calibration to take account of this effect of change in type of temperature gradient, so that in the further work the constant determined radiometrically has been chosen. Another cause for the existence of a nonsymmetrical temperature gradient is the thick coat of lampblack, a poor thermal conductor, upon the front surface of the receiver. It is possible that the changing gradient effect depends to some extent upon the rate of energy supply, so that while the constant is 2.200 for an energy flow of about 0.03 cal./cm² min. it might be greater for solar measurements involving 50 times this flow of energy.

It is of interest to note that during a radiometric or electrical calibration in the laboratory, when the shutter is open the receiver radiates through a small solid angle to the walls of the room or to a water-cooled diaphragm at room temperature, while during exposures to the solar radiation the receiver radiates into space. The difference in these two conditions can not account for an error in F' greater than 1 part in 2000.

A further question which may be raised is in regard to the loss of heat from the receiver by conduction along the lead wires. The lead wires were of No. 28 copper, and a length of about 15 cm of each wire was inside the metal shell incasing the receiver, the object being to reduce the temperature gradient along each wire from the receiver to the outside room temperature. The heating of the lead wires developed by the measuring current was extremely small, and was negligible for all the heating currents used in the electrical calibration. Since, however, the type of gradient in the lead wires may be slightly different in the radiometric and electrical calibrations, and in the case of the electrical calibration may be different for heating and cooling, there is a possibility of some small error on this account. However, several preliminary experiments with lead wires of different sizes gave negative results, and a rough computation of the magnitude of the effect to be expected indicated that it was very small relative to the total energy expended in the receiver.

IX. SOLAR OBSERVATIONS WITH MARVIN PYRHELIOMETER RECEIVER NO. 3 AND SMITHSONIAN STANDARDIZED SILVER-DISK PYRHELIOMETER NO. 1

Using the above value of $F' = 2.200$ for the Marvin pyrheliometer solar measurements were made and comparison observations were taken at the same place and time with the Smithsonian standard as represented by the silver-disk pyrheliometer "S-I. No. 1." The following table summarizes the results. Each value represents a half-hour series of readings, and the solar intensity was remarkably constant during every run here recorded. The observations were taken at the Solar Radiation Laboratory of the Weather Bureau by Prof. Kimball, Mr. Hand, and the writer.

TABLE 9.—Data on Solar Observations

Date	Marvin pyrheliom- eter	Smith- sonian pyrheliom- eter	Marvin/ Smith- sonian
	Cal./cm ² min.	Cal./cm ² min.	
Nov. 10, 1915.....	1.162	1.189	0.978
Do.....	1.352	1.368	.974
Do.....	1.262	1.302	.971
Nov. 26, 1915.....	1.160	1.169	.991
Nov. 27, 1915.....	1.230	1.233	.980
Mean.....			.98

The Marvin pyrheliometer independently calibrated thus agrees with the Smithsonian instrument within 2 per cent. However, it is believed that the error in the calibration of the Marvin pyrheliometer may amount to 5 per cent, so that the two instruments are probably in agreement within the errors of experimental observation. A pyrheliometer is designed for the measurement of energy flow amounting to 1.5 cal./cm² min. To adapt such an instrument to the accurate measurement of quantities 1/50 of this magnitude is extremely difficult. Yet in a radiometric calibration made in the laboratory the constant of the instrument has to be determined for this small energy supply. A more satisfactory method of procedure would be to radiometrically calibrate an instrument which was designed for small energy flow and to compare this instrument with a pyrheliometer by solar measurements, using some system of diaphragms, sector disk, etc., for decreasing the amount of radiation entering the more sensitive instrument.

X. SUGGESTIONS FOR FURTHER WORK

With the present experimental arrangement it is not feasible to extend the work further. Certain improvements other than the one suggested above must be made in the apparatus if higher accuracy is to be obtained.

(a) The entire system, furnace, diaphragms, and pyrheliometer, must be mounted in a chamber in which the humidity can be controlled and reduced to a minimum when so desired. In designing this chamber attention must be given to the elimination of any possible means for radiation to be reflected, by the side walls, from the furnace into the pyrheliometer.

(b) The shutter on the pyrheliometer should be replaced by a shutter at the furnace.¹⁸ This new shutter should be water-

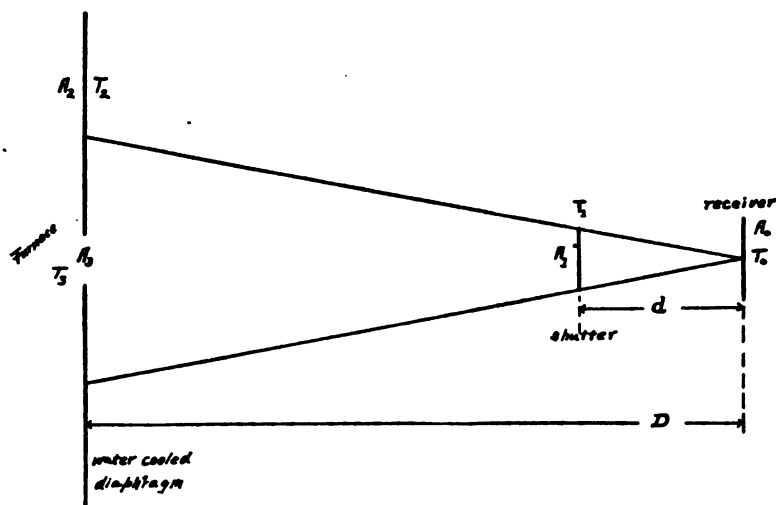


FIG. 8.—Graphical representation of experimental conditions

cooled, electromagnetically operated, and must be located between the furnace and the water-cooled diaphragm in front of the furnace. In this way all external conditions affecting the pyrheliometer remain the same during a heating as during a cooling. The only difference, then, radiometrically between a heating and a cooling is the exchange of the opening of a definite size, given by the water-cooled diaphragm, into the furnace, for a black water-cooled diaphragm of the same size at a measured temperature. Fig. 8 illustrates the conditions which were effective in the present calibration but which could be remedied by the use of a properly located shutter.

¹⁸ The importance of this has been noted by Coblenz. *This Bulletin*, 12, p. 523; 1916 (*Scientific Paper*, No. 261.)

The temperature of the pyrliometer receiver of area A_0 is T_0 . The temperature of the walls and shutter of the pyrliometer is T_1 and the area of the closed shutter, exposed to the receiver is A_1 . When the shutter is open the pyrliometer receives radiation from the large area A_2 of the water-cooled diaphragm at a temperature T_2 and from the small area A_3 opening into the furnace at a temperature T_3 .

Let J_1 = radiation received by the disk from the closed shutter.

J_2 = radiation received by the disk from the large water cooled diaphragm, when the shutter is open.

J_3 = radiation received by the disk, shutter open, from the furnace.

The net radiation measured is $J = J_2 + J_3 - J_1$

$$J_1 = \frac{\sigma}{\pi} (T_1^4 - T_0^4) \frac{A_1 A_0}{d^2}$$

$$J_2 = \frac{\sigma}{\pi} (T_2^4 - T_0^4) \frac{A_2 A_0}{D^2}$$

$$J_3 = \frac{\sigma}{\pi} (T_3^4 - T_0^4) \frac{A_3 A_0}{D^2}$$

hence, since T_0^4 is always negligible compared with T_2^4 and A_2 compared with A_3 ,

$$J = \frac{\sigma}{\pi} A_0 \left[\left(\frac{T_2}{100} \right)^4 \frac{A_2}{D^2} + \left\{ \left(\frac{T_3}{100} \right)^4 - \left(\frac{T_1}{100} \right)^4 \right\} \frac{A_1}{d^2} \right] 10^8 \quad (22)$$

In the calibration of the pyrliometer the water-cooled diaphragm was always at a slightly lower temperature than that of the shutter.

If we take a very extreme case as an example, suppose:

$$T_3 = 1600^\circ \text{ abs.}$$

$$T_2 = 290$$

$$T_1 = 300$$

$$T_0 = 300$$

$$D = 70 \text{ cm}$$

$$A_2 = 0.93 \text{ cm}^2$$

$$A_1 = 16 \text{ cm}^2$$

$$d = 30 \text{ cm}$$

Then equation (22) above becomes:

$$J = \frac{\sigma}{\pi} A_0 10^8 (12.44 - 0.18) \quad (23)$$

Thus, for this extreme example where the temperature of the diaphragm is 10° lower than that of the pyrheliometer shutter and second term of equations (22) and (23) amounts to 1.4 per cent of the first term. The second term was always neglected in the calibration. In general, it would only amount to a few tenths of 1 per cent. It is always a subtractive correction to J and hence would reduce slightly the observed value of F' .

(c) As a further check upon the behavior of the dynamic type of pyrheliometer of which the Marvin is an example, it would be highly interesting to study at the same time a pyrheliometer of the static type. The Marvin dynamic pyrheliometer readily lends itself to this modification. The entire tube and case could be jacketed and accurately maintained at a constant temperature. An ice bath, while furnishing the desired constancy of temperature, could not be easily used on account of its being usually below the dew point, allowing water vapor to condense upon the receiver, when the instrument was used for solar comparisons. But it would not be difficult to secure sufficient constancy of temperature by use of flowing water thermostated to, say, 30° C. Such a pyrheliometer would not be very satisfactory for ordinary field work on account of the inconvenience of operation, but as a laboratory instrument its behavior, so different in principle from that of the dynamic type, would permit interesting comparisons of solar measurements by two distinct methods.

SUMMARY

For the first time, it is believed, a pyrheliometer has been calibrated by two methods, the usual electrical method and a radiometric method. In the radiometric method a known quantity of radiation from a black body was allowed to fall upon the pyrheliometer receiver in exactly the same manner as when employed for solar measurements. The calibrations by the two methods agreed within limits of experimental error, if the Stefan-Boltzmann constant were chosen, as $\sigma = 5.7 \times 10^{-12}$ watts cm^{-2} deg^{-4} , the latest and most accurate determination of this constant of total radiation. Or conversely, the constant has been observed as 5.7×10^{-12} within an accuracy of possibly 5 per cent.

The behavior of the Marvin pyrheliometer has been carefully investigated. A lag, part of which is due to the galvanometer of the bridge, has been found to exist, and, for the silver disk No. 3, was experimentally shown to be less than 2 seconds. Both theoretically and experimentally it was shown that the effect of this lag is negligible after 5 to 10 seconds. The cooling and heating of the receiver follows Newton's law of cooling.

In order to completely eliminate errors due to a lag effect, readings should be made at 10 seconds and 60 seconds following the beginning of a heating or cooling. The factor for converting readings of temperature or resistance change over this 50-second interval to corresponding changes over a complete 60-second period is 1.217. This factor is the same for both electrical and radiometric heating and was determined with an accuracy of 0.1 per cent. There is no advantage in making the periods of heating and cooling 120 seconds in duration. Periods of 60 seconds are sufficient. The method of blackening the receiver is of great importance. The best method used for blackening is that used by Coblenz and is described above. The calibration constant F' of Section IV appears independent of the rate at which energy is supplied to the receiver, at least for an electrical calibration.

A Marvin pyrheliometer calibrated by two methods, electrical and radiometric, was compared by solar observations with the United States Weather Bureau Smithsonian standardized pyrheliometer "S. I. No. 1," calibrated by comparison with the Smithsonian primary standard water-flow pyrheliometer. The Marvin pyrheliometer thus calibrated gave 2 per cent lower values of the solar radiation than the Smithsonian pyrheliometer. This difference is within the errors of observation of the present calibration.

The above work was suggested by Dr. G. K. Burgess and Prof. H. H. Kimball. The writer is indebted to Dr. Waidner and Dr. Burgess, of the Bureau of Standards, and Prof. Marvin and Prof. Kimball, of the Weather Bureau, for many valuable suggestions. The writer is especially indebted to Prof. Kimball for cooperating in much of the earlier work, for performing the experiments necessary to obtain the temperature-resistance relations of the thermometer coils, and for the direction of the solar observations which were made in his laboratory. The pyrheliometers were furnished by the Weather Bureau through the courtesy of Prof. Marvin and Prof. Kimball.

WASHINGTON December 11, 1917.

WAVE LENGTHS IN THE RED AND INFRA-RED SPECTRA OF IRON, COBALT, AND NICKEL ARCS

By W. F. Meggers and C. C. Kiess

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I. INTRODUCTION

The scarcity of accurate data for wave lengths longer than those of yellow light has led the Bureau of Standards to make extensive application of photographic sensitometry to the study of long wave-length regions in the spectra of chemical elements. Wave-length measurements in spectra from 5600 Å to 9600 Å¹ have been published for 11 of the chemical elements, viz, lithium, sodium, potassium, rubidium, caesium, copper, beryllium, calcium, strontium, barium, and magnesium. The wave lengths in the arc spectra of iron, cobalt, and nickel, presented in the present paper, represent further investigations in standard wave lengths and spectrum analysis. It was thought especially desirable to know to what extent photographic methods with a large diffraction grating are applicable to the infra-red spectral regions. Furthermore, it was hoped that investigations of this kind would lead to the discovery of some spectrum which is more satisfactory than that of the iron arc for standard wave lengths in the red and infra-red. Before such standards can be accurately measured by interferometer methods it is necessary to have the approximate values represented by measurements from grating spectra. The values given below represent the long waves of iron, cobalt, and nickel to the extreme limit which it is practicable to reach with our apparatus and photographic method.

¹ This Bulletin, 14, p. 371; 1917.

II. APPARATUS AND METHODS

The spectral regions studied were photographed with a concave grating spectrograph which has already been described.³ The source of light was the electric arc of the type recommended by the international wave-length committee.³ The iron electrodes were of Norwegian or of electrolytic iron; those of cobalt were cut from an anode which contained not over 4.2 per cent impurities, chiefly iron, nickel, and carbon. Small lumps of "pure" fused cobalt obtained from Eimer and Amend were burned in an Acheson graphite arc for two of the spectrum photographs. Electrodes of nickel were cut from an anode which contained about 4 per cent of iron as well as small quantities of cobalt, copper, and a trace of manganese.

The photographs were made on Seed 27 plates which were made sensitive to the long waves by staining with pinacyanol, dicyanin, or dicyanin A. The preparation of the sensitizing bath has already been described.⁴ It was found that each bath of dicyanin could be used five or six times before losing its effectiveness, while the pinacyanol could be used many times. Plates bathed in pinacyanol were used for the region 5500 Å to 7000 Å; for wave lengths greater than 7000 Å the dicyanin or dicyanin A plates were used. Exposure times of 10 minutes sufficed to photograph the spectrum up to 7000 Å; between 7000 Å and 9000 Å, 20 to 30 minutes exposure were sufficient; beyond 9000 Å the exposure times ranged from 5 hours to 10 hours. The spectral region to be measured was photographed in the grating spectrum of the first order, with standard reference lines photographed adjacent, in either the first, second, or third orders, according to the region under investigation. During the exposures to the first order, the overlapping orders were absorbed either by a potassium bichromate cell or by a screen of Jena red glass.

The comparison spectra were photographed before and after the long exposures in the first-order spectrum so as to detect any shift due to temperature changes or displacements of the apparatus. In general, no appreciable shift was observed, even when exposures of 10 hours duration were made.

A few of the plates were measured on the measuring machine at the Johns Hopkins University. The remainder were measured on a large engine constructed by Gaertner and kindly loaned

³ This Bulletin, 14, p. 371; 1917.

³ *Astroph. Jr.*, 29, p. 93; 1914.

⁴ This Bulletin, 14, p. 371; 1917.

to the Bureau of Standards by the Weather Bureau. The plates were measured directly in the sense increasing wave lengths with increasing scale readings; that is, with the red ends to the right, and then in the reversed position, with red ends to the left. From two to six settings were made on a line, and the mean of the readings was recorded as the scale reading for the line. In all cases the arc spectrum of iron served as the reference spectrum, and in the reductions the values of the wave lengths determined here^a by interference methods were used. The probable errors in the wave lengths of iron, cobalt, and nickel given below are less than two or three hundredths of an angstrom in most cases where the line was measured two or more times. Lines indicated as having been measured only once may contain larger errors.

III. DESCRIPTION OF THE PLATES

Tables 1, 2, and 3 contain a description of the plates that were measured. In the first column is given the laboratory number of the plate; the second gives the region covered in the first order spectrum. In the third and fourth columns, respectively, are given the exposure times required and the slit widths used. In the fifth column is recorded the current strength, which was regulated to conform approximately to the region photographed. The sixth column contains the initials of the observers: M for Meggers and K for Kiess. "Observers" refers to the measurement and reduction of a plate as well as the securing of it. To Mr. Burka credit is due for assistance given in securing the plates for which long exposures were required.

TABLE 1.—Photographs of the Iron Spectrum

Plate No.	Region	Exposure	Slit width	Current	Observer
	A	Min.	Mm	Amp	
131	8400-10 400	30	0.05	6.0	M
208	7800- 9800	30		7.0-8.0	M
214	7800- 9800	30	.04	8.0	M
221	6400- 8400	10			M
426	7800- 9800	120	.05	7.0	K
467	7800- 9800	360	.07	7.0	K
469	8400-10 400	600	.07	7.5	K
474	8600-10 600	600	.07	8.0	K
476	9400-11 400	600	.07	8.0	K
499(a) ^a	7000- 9000	30	.03	3.0	M, K
499(b) ^a	6400- 8400	15	.03	3.0	M, K
500 ^a	8200-10 200	360	.07	4.0	K
505	9000-11 000	420	.07	8.5	K

^a This Bulletin, 18, p. 245; 1915.

^a Electrolytic iron.

TABLE 2.—Photographs of the Cobalt Spectrum

Plate No.	Region	Exposure	Slit width	Current	Observer
	A	Min.	Mm	Amp	
129	8400-10 400	120		7.0	M
197	6500- 8500	20		7.0-8.0	M
206	7600- 9600	25	0.02	8.0	M
214	7800- 9800	30	.04	8.0	M
221 a	6400- 8400	10			M
224 a	5500- 7500	2			M
412	7000- 9000	25	.02	7.5	M, K
419	7600- 9600	120	.02	7.5	K
478	8400-10 400	330	.07	8.0	K
480	8800-10 800	390	.07	9.0	K
486	9400-11 400	600	.07	9.0	K
492	9700-11 700	420	.07	9.0	K
493	6000- 8000	30	.04	9.0	K
495	5500- 7500	20	.03	7.0	K
497	9700-11 700	480	.07	9.0	K

a E. & A. Co. in graphite.

TABLE 3.—Photographs of the Nickel Spectrum

Plate No.	Region	Exposure	Slit width	Current	Observer
	A	Min.	Mm	Amp	
209	7600- 9600	30	0.02	7.0	M
214	7600- 9600	30	.04	7.0	M
221	6400- 8400	10	.04	6.0	M
224	5500- 7500	2	.04	6.0	M
493	6000- 8000	30	.04	9.0	K
495(a)	7200- 9200	60	.04	9.0	K
495(b)	5500- 7500	20	.03	7.0	K
519	8600-10 600	405	.07	9.0	K
520	9000-11 000	480	.07	10.5	M

IV. RESULTS

In the final wave length Tables 4, 5, and 6 an attempt is made to give the arc spectra of "spectroscopically pure" iron, cobalt, and nickel. It is practically impossible to obtain metals that are absolutely pure and it is especially difficult to separate chemically the ferrous metals from each other and to prevent contamination by closely related elements such as copper, chromium and manganese. As far as possible, this separation is made spectroscopically in this work, by intercomparison of the intensities and wave lengths of all the lines observed in the spectra of these elements. Faint lines are considered as due to impurities whenever their measured wave lengths agree within the errors of observation with those of intense lines in the spectrum of a particular element. In some cases the relative intensities or the wave-length comparisons

do not seem to warrant the removal of the lines, so they are allowed to remain in the tables and marked as questionably belonging to some probable element present as an impurity. Some of these cases may represent coincidences of lines with approximately the same intensities and wave lengths in different elements but most of them are probably due to some unrecognized impurity contained in common by these elements.

Table 4 contains the iron lines measured in this investigation. The adopted wave lengths are, for the most part, the means of the values derived from two to six plates. Those wave lengths for which only one determination is available are specially indicated in the table. By using electrodes of electrolytic iron it was hoped to free the table from the lines of impurities. The spectrum of electrolytic iron was photographed from 6750 to 9118, and all lines in the table not measured in it are specially indicated. Beyond 9118 it was not possible to use the electrodes of electrolytic iron for the long exposures and at the current strength necessary. The lines in this region are those given by Norwegian iron. Most of the lines shorter than 9000 have been observed by Geiger,⁶ or by Burns.⁷ Those longer than 9000 are new. In his table Geiger gives four lines with wave lengths greater than 9000 Å which he designates as doubtful. These lines as well as several more of Geiger's doubtful lines, shorter than 9000 Å, were not measured on any of our plates. The intensities given in the second column of the table are included between the limits 1 and 10, intensity 1 indicating the weakest line capable of accurate measurement. After 10 000 Å no intensities are assigned, as the photographic action of the lines is too weak to permit of relative estimates. Most of the lines representing the longer waves are near the limit of measurability when the plate is viewed through the measuring engine microscope which magnifies about six diameters.

⁶ *Annalen der Physik*, 4te Folge, **39**, p. 782; 1912.

⁷ *Lick Observatory Bulletin*, **8**, p. 27; 1913.

TABLE 4.—The Arc Spectrum of Iron

[In the column headed "Remarks" the letters have the following significance: a=measured only once; b=broad; d=perhaps double; e=not measured in the spectrum of electrolytic iron; f=not given by Burns (6750 Å to 8824 Å); g=not given by Geiger (6750 Å to 8999 Å); h=hazy or diffuse; l=shaded to red; and v=shaded to violet)

λ Å.	I	Remarks	λ Å.	I	Remarks	λ Å.	I	Remarks
6750.15	4		90.41	6	d	7389.43	7	
52.71	2		91.83	1	a, l, g	7401.71	2	
55.62	1	a, g	92.99	1	a, f	11.21	8	
86.94	2		7095.43	2		18.67	3	
93.32	1	a, g	7107.46	2		21.60	1	f
6796.06	1	a	12.17	2		30.73	1	b, f
6804.11	2	b	30.96	10		41.05	2	b, f
06.85	2	g	32.98	3		43.05	1	
10.27	3		42.55	2		45.80	9	
20.43	2	f	45.34	2	a	47.40	1	g
28.62	4		51.49	1	a, Mn?	53.98	1	a, a, g
37.00	1	l, g	55.62	2	a, g	61.55	1	
38.85	1		64.49	9		73.56	1	a, a, l, g
39.88	1	g	75.94	1		76.30	1	a, a, f
41.35	5		76.88	1	g	82.00	1	f
42.68	1	g	80.06	1	a, g	83.43	3	a, l, g
43.69	4		81.20	3	g	91.68	2	
55.15	6	l	81.94	1	a, NIP	95.12	8	
57.23	1	g	87.36	10		7498.51	1	a, a, g
58.19	2		89.13	2	g	7507.32	2	
62.46	2	a	91.66	1	a, f	11.09	9	
85.78	4		7194.92	1	a	31.20	4	
6898.27	1		7207.41	10		46.18	2	
6902.84	1		12.47	1		59.68	1	a, f
16.72	4		19.70	2		63.03	1	f
33.59	2	a	21.23	1		68.94	4	a
45.22	7		23.66	3		73.53	1	f
47.48	1		28.69	1	a	83.81	4	a
51.29	4		39.88	4		7586.08	7	
75.46	1	a	44.83	2		7605.32	1	a, l, g
77.42	1	a	54.62	1		20.54	3	
78.86	7		61.51	2	v	53.80	2	f
88.52	2		82.43	1		58.97	2	a, a, l, g
6999.92	4		84.83	2	l	61.24	3	
7000.56	1	g	88.79	4		7664.31	4	
06.02	2		93.01	6		7710.40	3	
10.33	1	g	7295.02	1		23.20	1	f
11.35	1		7300.47	2	a, b, a, f	42.71	1	a, l, h
14.99	1	a, l, g	06.60	2		48.30	4	
16.30	9	d (Burns gives two lines)	07.95	3		51.18	1	a, f
			11.13	4		7780.62	5	
22.98	5	d	20.72	3		7808.04	1	a, b, f
24.09	1	a, g	33.60	1		32.24	6	
24.65	2	a, NIP	51.42	3	b	44.66	1	a, a, l, g
38.23	4		53.50	1		55.48	1	a, a, f
38.73	1	a, l, g	63.96	1	b, l, g	69.65	1	a, a, l, g
44.60	1	a, f	66.37	1	f	7879.84	1	a, a, l, g
68.42	5	a	70.16	1	f	7912.85	1	f
71.88	1		76.38	1	f	37.19	9	
83.38	1		82.89	1		41.09	2	f
86.73	1		86.41	4		45.91	7	

TABLE 4.—The Arc Spectrum of Iron—Continued

λ I. A.	I	Remarks	λ I. A.	I	Remarks	λ I. A.	I	Remarks
55.81	1	a, b, e, f	8611.73	3	f	25.55	1	
59.21	1	a, b, e, f, g	16.09	1	f, g	42.32	1	a
94.48	1	f	21.55	2	f, g	46.54	1	
7999.00	6		61.85	6		58.49	3	
8024.50	1	a, e, f	74.69	3	f	9259.17	1	a
28.31	1		88.58	7	f	9307.84	1	a
46.08	5		8699.43	1	f, g	18.09	1	
47.60	1	e, f, g	8710.28	1	f, g	24.07	1	a
75.13	1	e, f, g	13.15	1	f, g	43.23	1	a
80.62	1	e, f	57.12	2	f	44.64	1	
85.19	5		63.97	2	f	50.52	2	
8096.85	1	e, f, g	84.39	1	a, e, f, g	54.93	1	a
8133.38	1	a, e, f, g	90.55	1	a, e, f, g	59.36	1	
45.47	1	a, e, f, g	8793.37	2	f	62.29	1	
49.59	1	a, e, f, g	8804.55	1	f, g	72.82	1	
79.03	1	a, e, f, g	24.18	6		9382.83	1	a
86.80	2	e, f, g	38.35	2	g	9401.03	1	
8198.95	2		46.67	1	a, e, g	14.12	1	
8207.75	1		8866.94	3		9443.88	1	a
20.41	7		8919.83	1	e, g	9507.39	1	a
32.33	2		29.02	1	e, g	13.22	1	
39.09	1	e, f, g	45.13	1	g	9569.85	1	
48.09	3	d, e, g	75.33	1	g	9649.85	1	a
74.28	1	f, g	8999.52	4		9653.14	1	
75.91	1	f, g	9012.05	1		9738.69	1	
8293.47	2	f, g	24.26	1	e	9765.53	1	a
8327.04	8		62.29	1	a	9961.23	1	a
31.94	6		70.39	1	e	9978.67	1	a
39.41	4	f	79.64	4	e	10 025.39		
60.79	2	f	80.58	1	e	026.24		a
65.61	2		88.21	4		057.38		
87.74	8		9089.40	4		085.45		
95.10	1	a, f, g	9100.48	2	e	222.02		a
98.27	1	a, f, g	03.66	1	a, e	234.11		
8399.86	1	a, f, g	16.18	1	e	244.35		
8401.42	1	a, f, g	17.17	1	e	246.24		a
22.95	1	a, f, g	18.85	4		257.18		a
24.14	1	a, f, g	21.12	1	a	293.46		
39.58	3	b, f	46.08	1		331.63		a
46.42	1	f, (oxygen)	47.94	1		370.48		a
68.35	7		55.84	1	a	374.52		
71.75	1	a, f	56.93	1	a	375.69		a
8497.00	2	f	60.32	1		396.35		
8514.01	3		64.44	1		478.42		
15.08	2	f	73.48	1		488.09		
26.66	2	f	9183.85	1	a	510.37		
82.20	2	f	9209.99	2		553.06		
92.97	1	f, g	14.42	1		10 689.33		
8598.79	1	f, g	17.55	1				

Wave lengths in the arc spectrum of cobalt are given in Table 5 of which the arrangement is the same as that of Table 4. No intensities are assigned to the lines beyond 10 000 Å for reasons stated above, although it is to be noted that the cobalt lines in the infra-red are relatively much stronger than the iron lines in the same region.

The red region of the cobalt arc spectrum has been measured by Stütting⁸ whose lines are included between the wave lengths 5905 Å and 7054 Å. A number of his lines were not measured by us, and vice versa. Lines in our table not given by Stütting are specially designated. Lines of impurities, chiefly iron, copper, chromium, nickel, and manganese have been eliminated as far as possible.

Three lines 7771 Å, 7774 Å, and 7775 Å deserve special remark. These lines are given by Meggers⁹ in his table of copper lines. We have since observed them not only in the cobalt spectrum but also in the spectra of iron, nickel, silicon, and copper. The mean values of their wave lengths derived from all our measurements, including those not yet published, are 7771.928 Å, 7774.138 Å, and 7775.433 Å. These lines constitute the well-known triplet which, according to Runge and Paschen,¹⁰ belongs to the principal triplet series of oxygen. The chief interest here lies in the fact that they appear among the arc lines of various metals. They have been observed as absorption lines in the solar spectrum by St. John¹¹ and by Meggers.¹² In this connection, attention is called to the wave length given as 8446.42 Å in Table 4 ("The Arc Spectrum of Iron"). This wave length represents a strong line in the spectrum of oxygen as observed by Runge and Paschen.¹³ It also exists as an absorption line in the spectrum of the sun,¹⁴ and it was found in the spectrum of the copper arc¹⁵ in air.

⁸ Zeit. für Wiss. Phot., 7, p. 73; 1909.

⁹ This Bulletin, 14, p. 371; 1917.

¹⁰ Wied. Annalen, 61, p. 41; 1897.

¹¹ See Abbot, The Sun, p. 92; 1911.

¹² Unpublished work on the solar spectrum.

¹³ Wied. Annalen, 61, p. 641; 1897.

¹⁴ Unpublished work on the solar spectrum.

¹⁵ This Bulletin, 14, p. 371; 1917.

TABLE 5.—The Arc Spectrum of Cobalt

[In the column headed "Remarks" the letters have the following significance: a=measured only once
b=broad; d=perhaps double; g?=perhaps ghost; h=hazy or diffuse; and s=not given by Stütting
(5935; A to 7054 Å)]

λ I. A.	I	Remarks	λ I. A.	I	Remarks	λ I. A.	I	Remarks
5503.23	1	a	34.25	1	a	90.46	1	
08.29	1	a	36.52	1	a	91.89	10	
15.93	4		37.95	2		93.48	1	
23.27	8		39.59	1	a	95.79	2	a, s
24.93	7		40.99	2		5996.78	5	Ni?
27.19	1	a	43.80	1	a, b	6000.70 ₆	8	
28.77	1	a	50.96	3		02.48	3	
30.76	8		52.89	2		05.02	3	
33.09	1	a	53.99	1	a	06.31	8	
36.45	1	a	60.39	1	a	07.67	6	
45.85	6		68.71	1	a	11.43	3	
46.96	3		70.37	6		13.62	3	
58.83	4		74.33	3		15.34	3	
63.69	1	a	82.09	5	Cu?	29.89	2	
65.87	1	a	90.02	4		49.05	10	
73.79	1	a	93.09	1	a	51.35	1	a, s
75.97	1	a	5793.92	1	a, Fe?	58.27	4	
78.78	1	a Ni?	5806.34	1	a	70.61	7	
81.28	2	a	09.39	1	a	82.49	10	
89.27	1	g?	18.09	2	a	83.37	2	a
90.72	8		26.30	4		86.68	7	
5994.79	2		30.05	7		6093.15	6	
5602.40	1	a	33.44	1	a	6100.76	5	
06.83	2	a	34.59	2	a	03.33	1	s
15.70	2		46.57	3		05.49	4	
16.00	2		76.06	3		07.93	9	
22.72	1	a	77.42	3		14.26	1	s
27.84	2	a	78.03	2		17.00	6	
31.69	1		81.05	3		22.68	10	
36.11	6	b	83.42	3		25.74	1	s
37.76	5		86.52	1	a	28.26	3	
40.04	2		90.48	7		29.15	3	
42.98	4		96.02	1	a	30.41	1	
47.20	8		5898.77	1	a	32.44	3	
51.70	4		5905.59	3		43.78	4	
54.14	1	a	15.55	8		46.38	3	
56.10	1	a	16.88	1	a	51.22	2	s
59.07	4		22.35	1		58.53	2	
63.88	1	a	23.13	1		60.04	1	s
69.83	1	a	33.53	1	a, b, h, s	68.86	1	
75.39	3		35.38	7		71.43	1	s
76.49	2		40.49	1		75.08	2	
79.57	3		46.51	7		81.00	5	
84.69	1	a	51.73	2	a	81.90	1	s
86.96	3		53.81	1	s	88.99	7	
88.51	3		63.99	1	a, s	93.58	6	
5696.75	1	a	65.02	3		6197.83	2	
5703.03	2		65.57	2		6203.50	5	
06.07	3		82.01	3		05.50	3	
20.80	1	a	83.36	3		08.75	1	
30.45	1	a	84.25	10		11.15	8	
33.28	1	a	89.58	1		22.32	1	

TABLE 5.—The Arc Spectrum of Cobalt—Continued

λ L. A.	I	Remarks	λ L. A.	I	Remarks	λ L. A.	I	Remarks
23.43	6		47.00	2		56.50	2	s
28.45	1	s	50.24	10		58.08	2	
30.98	7		51.14	5		67.60	4	NIP
32.45	5		55.03	10		71.05	10	
37.13	2		63.02	5		78.06	1	s
42.48	2	s	70.16	3		84.89	4	
46.42	2		74.57	5		89.28	3	s
47.26	5		77.93	9		91.14	1	s
49.50	6		90.33	7		92.35	1	s
50.80	1	s	6496.18	1	a	6799.39	2	s
53.99	3		6502.29	1	a, s	6807.43	1	s
57.56	10		04.25	4		09.01	5	
62.84	4		08.76	4		14.99	10	
65.97	1		17.06	3		19.57	3	s
71.40	10		28.63	1		26.99	2	b, s
73.06	7		35.16	3		29.92	1	s
75.16	4		38.40	1	b, h, s	38.20	3	s
76.62	5		40.64	1		45.66	2	s
78.19	1		51.45	6		46.99	3	s
82.66	10		54.57	3		58.44	5	s
91.89	3		63.42	9		64.94	3	s
6296.96	2		67.10	2	s	72.42	7	
6302.50	1	a, s	79.29	8		6878.50	2	s
11.29	7	s	88.02	1	s	6901.51	2	s
13.07	6		91.80	3		06.39	1	s
14.50	7		6595.91	9		08.11	5	s
15.76	1		6610.78	1	a, s	10.02	1	s
20.35	10		14.52	1	a, s	10.84	2	s
22.94	2		17.21	10	d	13.31	1	a, s
33.68	1		20.01	1		13.98	1	s
37.98	3		23.70	7		22.23	2	s
40.82	4		32.42	8	s	37.85	7	
47.72	10		35.12	4	NIP	46.33	2	s
51.37	6		38.40	1	s	48.43	1	a, s
52.80	2		43.76	3	NIP	50.99	1	a, s
54.80	1	a, b, s	45.33	2	s	72.70	1	a, s
66.20	1	b, s	49.97	3		77.02	1	s
68.86	1	a, s	52.32	3		78.50	2	b, a, Cr?
74.53	1		63.68	2		88.24	1	s
81.86	1	a, b, s	65.28	2		6997.30	7	s
84.49	2	s	72.96	2		7004.82	5	
86.68	6		78.84	6		15.13	2	s
95.19	7		80.35	1		16.65	10	
6396.52	4		82.30	1	b, s	27.86	8	
6403.63	1	s	84.05	3		32.56	4	s
07.36	2		84.88	3		42.61	2	s
08.39	2		6692.89	2	s	46.90	1	a, s
17.79	7		6700.99	1	s	52.84	10	
21.71	5		03.94	3		54.08	8	
25.12	4		12.70	3	s	55.94	3	
29.97	7		17.61	3		57.91	2	
30.28	1		20.97	2	s	58.65	1	
35.17	1	s	22.82	2	b, s	65.43	1	
39.83	2		42.12	3	s	70.45	4	
44.75	6		51.16	1	s	79.21	2	

TABLE 5.—The Arc Spectrum of Cobalt—Continued

λ I. A.	I	Remarks	λ I. A.	I	Remarks	λ I. A.	I	Remarks
84.99	10		33.52	5		7885.21	2	
94.64	4		54.04	8		7907.14	1	a
7097.84	1		59.68	3		08.75	10	
7101.77	1		61.08	4		12.90	1	a
02.57	4		64.98	5		19.50	2	
13.74	9		78.34	1	a	26.59	8	
14.09	1	a	80.95	1		57.77	3	
17.91	2		86.72	4		60.55	2	
22.26	5	NiP	88.71	1	a	62.40	1	a
24.45	5		90.60	6		66.12	2	
34.37	8		7594.18	1		80.48	1	a
54.71	8		7600.11	1	a	84.22	1	a
59.23	8		04.30	1		87.38	7	
73.37	1		06.30	1		96.83	1	
85.63	1	CrP	10.29	6		7998.12	1	
7193.63	8		16.13	1	a	8007.34	10	
7202.07	1	a	18.66	1		13.02	2	
04.54	1	a	34.56	5		16.59	1	
17.36	2		37.63	4		17.88	1	
21.06	1	a	41.43	1		22.15	7	
27.16	1	a	48.19	4		24.75	4	
50.09	3		64.89	1	a	29.29	7	
63.57	2		85.65	1		31.07	1	a
85.29	7		95.97	2		32.41	1	
7297.39	1	a	7698.95	1	a	37.63	1	
7305.35	1	a	7701.88	2		41.33	2	
07.86	1		04.90	2		43.33	8	
15.72	3		12.68	9		50.60	1	
41.61	1		25.92	2		53.50	1	
46.32	1	a	28.59	1	a	56.03	8	
49.68	1		34.25	6		62.98	2	
51.55	1		35.47	1		66.50	7	
53.48	2		43.27	5		80.23	5	
54.61	6		64.07	1		82.60	1	
65.77	1		71.95	1		85.47	2	
88.66	7		74.16	1	Oxygen triplet	8094.08	10	
7398.72	1		75.43	1		8112.13	1	a
7406.23	1		79.06	1	a	14.03	1	
14.64	1		86.63	1		16.43	7	
17.40	8		7794.15	1	a	37.10	5	
29.03	2		7809.25	1		40.42	2	
37.15	4		10.39	1		50.23	2	
43.48	2		17.15	1	a	52.08	6	
57.43	8		18.25	1	a	54.31	1	
71.21	1		22.12	1	a	60.68	2	
74.35	1	a	38.18	8		67.97	2	
77.28	1		40.05	7		89.29	1	a
78.78	1		43.61	1		8193.05	8	
84.00	1	a	55.88	7		8208.67	8	
89.41	3		59.41	1		43.38	1	a
7495.09	1		66.10	1	a	46.57	1	a
7502.74	2		69.92	6		59.10	1	a
15.28	1		71.43	6		69.99	8	
24.07	1	a	73.36	1		72.34	1	
26.32	1		77.44	2		75.55	1	

TABLE 5.—The Arc Spectrum of Cobalt—Continued

λ I. A.	I	Remarks	λ I. A.	I	Remarks	λ I. A.	I	Remarks
83.49	5		50.74	10		020.68		a
96.85	5		56.63	1	a	189.16		a
8299.02	5		62.69	2	Ni?	194.38		a
8301.44	1		66.96	1		206.05		
12.92	2		70.79	4		210.83		
15.32	2		78.30	1		213.32		
18.55	1		86.25	1	a	236.37		
31.70	2		888.70	2		272.89		a
34.71	1	a	8904.65	8		284.63		a
42.66	4		26.24	10		303.08		b
45.59	2		39.20	1		310.95		a
72.82	10		58.46	6		324.71		Fe, sec. ord?
78.37	7		8972.91	1		332.47		
8379.54	3		9025.96	1	a	339.09		
8409.03	1		32.70	1	a	366.64		a
54.71	1		37.92	8		368.94		a, b, h
78.45	2		9095.36	6		379.28		
8489.41	2		9178.03	1		435.30		a
8504.54	1		9280.71	1	a	505.88		
13.48	1		9344.89	1		557.90		
49.04	1	a, Cr?	9357.02	10		570.55		a
59.04	2		9474.77	1		616.54		
60.01	1	a	9530.99	1	a	838.19		
69.72	2		36.23	1		885.87		
74.49	3		44.52	2		10 908.52		
75.32	4		9597.89	2		11 000.41		
86.71	3		9616.55	1		012.76		a
89.70	3		9619.41	1	a	231.24		Fe, sec. ord?
8596.09	1		9745.98	1		265.64		a
8648.81	1	a	9818.39	2		275.45		a
55.76	1	a	34.19	1		289.33		
58.18	1	a	42.06	1	a	293.46		a
61.04	2		69.23	2		316.92		
8675.02	1		70.84	1		340.76		
8733.22	1		9892.90	1		453.42		a
8750.13	1		9914.12	3		601.93		a
8819.15	10		37.98	1		610.54		a, Ni, sec. ord?
85.22	8		54.24	1		11 623.63		a
37.83	2		9983.20	1				
38.41	2	a	10 017.97					

Table 6 contains the wave lengths found in the arc spectrum of nickel. Stütting¹⁶ and Hamm¹⁷ have measured a portion of this spectrum in the red but recorded only three lines (6842, 6914, and 7122) whose wave lengths exceeded 6772 Å. It is hoped that most of the lines due to iron, cobalt, copper, chromium, and manganese have been omitted from Table 6. As in the preceding tables no intensities are assigned to the lines whose wave lengths exceed 10 000 Å.

¹⁶ Zeit. für Wiss. Phot., 7, p. 75; 1909.¹⁷ Zeit. für Wiss. Phot., 12, p. 105; 1913.

TABLE 6.—The Arc Spectrum of Nickel

[The letters in the "Remarks" column have the following significance: a=measured only once; b=broad; d=perhaps double; g?=perhaps ghost; h=hazy or diffuse; v=shaded to violet; l=shaded to red; s=not given by Stütting (5892 Å to 6772 Å); and m=not given by Hamm (5504 Å to 6772 Å)]

λ I. A.	I	Remarks	λ I. A.	I	Remarks	λ I. A.	I	Remarks
5504.14	1		12.21	3		6414.63	5	
09.97	4		25.73	1	m, s	21.45	5	s
14.80	1	m	39.35	1	m	24.93	4	s
21.44	1	a, m	53.71	5		32.06	1	a, m, s
37.11	1	a, m	86.36	9		51.59	2	m, s, Fe?
53.74	3		90.27	1	a, m, s	52.77	1	m, s
78.71	5		6095.38	1	m	55.06	1	m, Co?
87.91	5		6105.72	1	m, s	6482.83	6	
89.36	3		08.13	5		6502.27	2	Co?
92.21	6	d	10.94	8	v	32.90	5	
5593.74	5		12.86	1	a, m, s, g?	63.42	1	a, m, s
5600.09	4		16.13	8		76.34	2	
07.05	1	a, m	18.06	1	m, s	80.16	4	
14.81	5		19.80	3		86.33	6	
25.27	6		28.96	3		92.52	6	
28.37	3		30.18	4		6598.53	6	
31.72	1	m	34.03	1	m	6610.82	1	a, m, s
37.19	5		41.96	1	s	21.24	1	m, s
38.82	1	m	63.31	8	v	29.67	1	m, s
41.80	4	m	70.55	3		35.16	6	s
43.13	2		75.46	9		43.67	8	
49.75	3		76.83	10		47.80	1	a, m, s
64.06	5		80.06	2	m	61.45	3	m
69.99	3		83.14	1	a, m, s	6690.82	2	m, s
82.21	7		83.92	1		6700.90	1	a, m, s
91.52	1	Fe? m	86.80	6		42.64	1	m, s
5694.95	7		91.24	7		59.41	1	m, s
5703.65	1	m	6198.62	1	m, s	67.80	10	
09.57	7		6204.65	5		72.40	9	
11.92	6		23.95	6		82.43	2	
15.11	6		30.07	4		91.19	1	
48.35	3		56.46	6		6798.44	1	
49.28	1	m	58.60	3		6813.60	3	
54.67	6		59.68	3		42.10	6	
60.87	5		71.84	1	m, s	50.46	2	
80.77	1	m	6272.69	1		61.20	3	
96.10	2	m Fe?	6300.31	1	a, s	70.13	3	
5798.20	1	m	14.63	9	v	72.38	1	a
5805.20	8	v	16.61	1	h, m, s, g?	6876.77	3	
31.62	6		22.15	4		6901.89	1	
47.02	3		27.60	4		02.88	2	Fe?
57.77	8		39.17	8		14.60	7	
63.97	2	m	50.39	1	m, s	28.36	2	
5892.88	6		60.75	6		55.10	5	
5906.50	2		64.58	1		6973.51	2	
23.98	2		66.41	6		7001.55	4	
73.66	1	a, m, s	70.45	2		04.32	1	h, a
96.80	4	Co? s	75.32	1	m, s	08.56	1	h, a
97.61	4		78.22	7		16.51	1	Co?
5998.86	1	a, m	80.99	1	m, s	23.82	1	
6007.33	3		6894.70	6		24.76	8	

TABLE 6.—The Arc Spectrum of Nickel—Continued

λ I. A.	I	Remarks	λ I. A.	I	Remarks	λ I. A.	I	Remarks
28.79	2		14.51	6		7994.50	2	
30.10	5		19.28	2		8012.94	2	
32.16	1	a	22.34	9		28.33	2	Fe?
34.42	4		33.40	3		34.55	1	a
37.21	2		38.53	1	a	8095.93	1	a
49.56	1	b	58.92	1	a	8248.17	1	
63.05	4		81.49	5		8417.22	2	
63.52	2		7488.72	2	l	8501.81	2	
67.38	2		7501.81	1	b	8606.43	1	
68.27	1		21.09	2		8637.05	2	
82.22	1	h, a	22.87	8		8702.46	2	
7095.47	4	Fe?	25.18	8		8809.46	3	
7101.98	1		34.57	1	a	62.60	4	
10.98	6		37.27	1	h	8877.07	1	a
22.31	10		45.69	2		8965.96	2	
26.73	2		47.65	1	a	8968.16	2	
29.15	1	a	52.25	2	b	9005.05	1	a
41.55	1		55.67	9		11.97	1	a Fe?
46.69	1		59.63	3		58.55	2	
50.81	2		67.35	1	h, a	76.85	1	a
67.04	4		7574.10	7		78.67	2	
70.10	2		7610.02	3		9085.15	1	a
73.70	2		17.02	10		9106.33	3	
82.06	9		19.24	9		9150.64	1	a
7197.07	7		24.75	2		9296.42	1	
7220.74	3	b	27.39	1	a	9470.23	1	
25.13	1	a	43.74	1	a	9519.99	2	
26.97	1	a	7657.28	3		9675.56	1	a
56.72	1	a	7709.05	1	a	9966.19	3	
61.94	8		11.39	1	a	9989.94	1	
66.14	4		14.27	8		10 049.85		a
86.59	2		15.64	7		346.08		
90.88	1	a	27.68	10		368.08		a
91.30	8		35.99	1	a	386.22		
7297.75	3		48.94	10		526.76		a
7309.57	2		88.95	6		530.09		a
27.69	4		7797.66	8		551.83		a
33.57	2	Fe?	7826.84	4		572.74		a
48.49	1	a	55.05	3	h	598.38		a
51.35	3	Fe?	61.10	4		634.31		a
81.93	5		63.70	5		658.57		a
85.23	7		69.81	1	a	661.66		a?
86.24	7		7890.18	3		832.93		a
7393.67	10		7917.47	7		839.80		a?
7401.12	4		53.13	1	a	10 843.17		a
09.35	9		60.77	1	a			

The large number of lines photographed in the infra-red spectra of these arcs shows the importance of using and developing photographic methods as far as possible. The photographic method has great advantages over the radiometric method in being able to record fainter lines and resolve complex or close lines in addition to allowing greater possible accuracy in the wave-

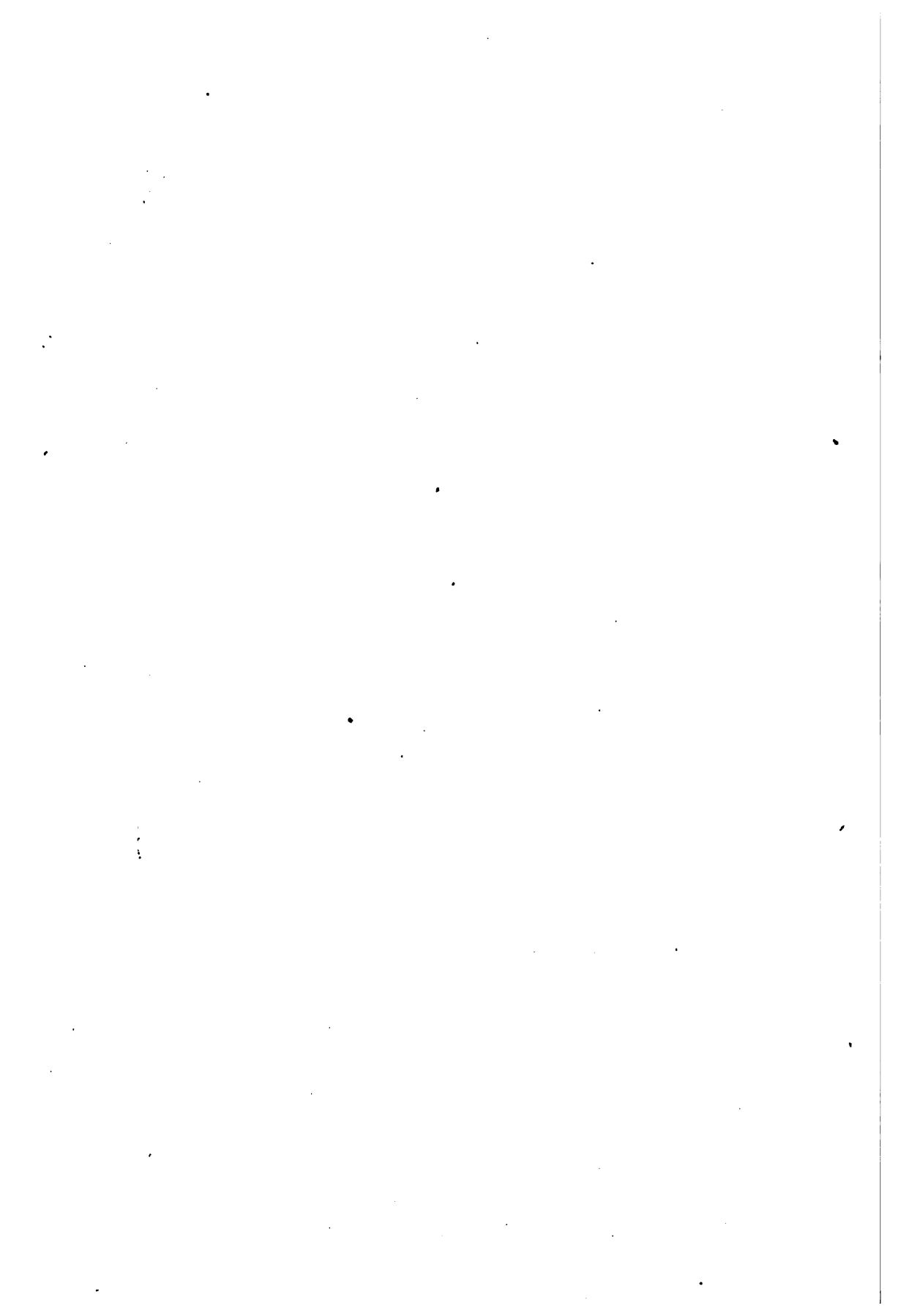
length measurements. In the spectral region in which our photographs overlap radiometric observations, for the elements in question, our tables contain from 5 to 10 times as many lines as were observed by means of the radiometer. However, for waves longer than about 10 000 Å the sensitivity of the bolometer so enormously exceeds that of photographic plates at the present time that it is necessary to rely upon the radiometer for practically all spectroscopic data. Extensive and careful investigations in the region to which both methods apply should assist in removing some of the difficulties and uncertainties in studying the longer waves.

V. SUMMARY

The arc spectra of iron, cobalt, and nickel were photographed in the red and infra-red regions on plates stained with pinacyanol and dicyanin. A large concave grating was used, and exposures up to 10 hours duration registered many lines with wave lengths greater than 10 000 Å, or 1 micron. In the arc spectrum of iron, 298 lines were measured between the wave-length limits 6750 Å and 10 689 Å; 606 lines were measured between 5503 Å and 11 623 Å in the arc spectrum of cobalt; and 290 lines between 5504 Å and 10 843 Å in the arc spectrum of nickel. As far as possible, impurities were eliminated and lines assigned to their proper element. Four lines due to oxygen were found in these arc spectra and are included in the wave-length tables. These results demonstrate that an invisible long-wave interval as large as the entire visible spectrum is accessible to photography with dicyanin-stained plates. The incompleteness of spectroscopic data for these longer light waves invites extensive application of this method of spectrum photography.

WASHINGTON, February 8, 1918.

NOTE (April 16, 1919).—Since the above article was written, a number of the lines of wave length greater than 9000 Å in the foregoing tables have been found to be "ghosts" of the type discovered by Lyman. A preliminary investigation shows that they may be calculated by multiplying the parent lines (lines ranging in intensity from 7 to 10) by the factors 1.39931, 1.40267, 1.59732, and 1.60066, respectively. A study of the false spectra produced by gratings is being made and will be published later.



SPECTRORADIOMETRIC INVESTIGATION OF THE TRANSMISSION OF VARIOUS SUBSTANCES

Obblentz, W. B. Emerson, and M. B. Long

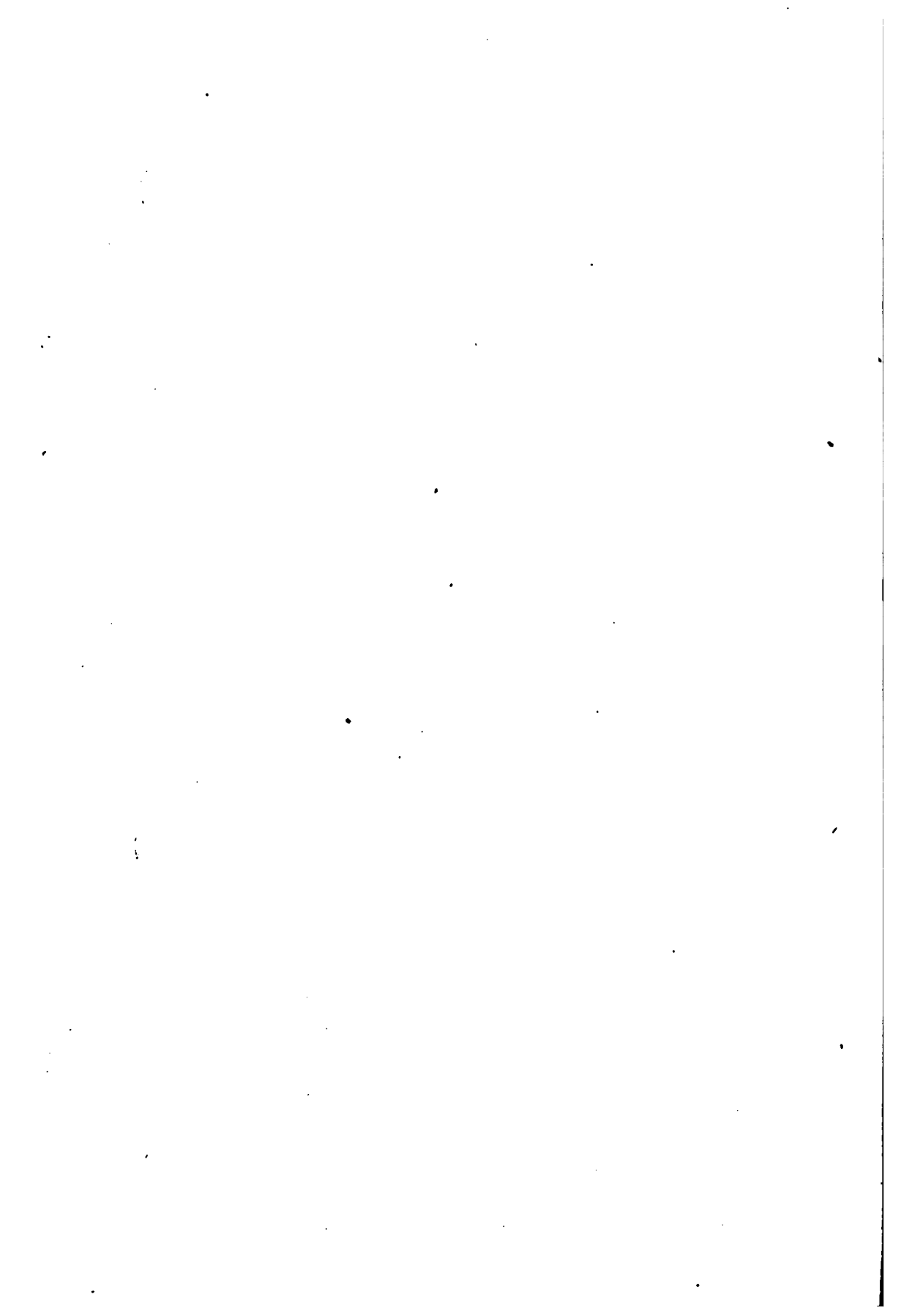
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I. INTRODUCTION

For many years radiometric determinations were made of the transmission of radiant energy through various substances. The data are published in the present paper.

A practical application of these data may be made in isolating narrow spectral bands of radiation without the use of a spectrometer. In this manner it may be possible to obtain the spectral-energy distribution of stars which are too weak in energy for measurement after dispersion by a prism. The data are useful in eliminating scattered light in spectrophotographic and spectroradiometric work. In this laboratory some of the substances



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I. INTRODUCTION

During the past few years radiometric determinations were made of the spectral transmission of radiant energy through various substances. These data are published in the present paper.

A practical application of these data may be made in isolating narrow spectral bands of radiation without the use of a spectrometer. In this manner it may be possible to obtain the spectral-energy distribution of stars which are too weak in energy for measurement after dispersion by a prism. The data are useful in eliminating scattered light in spectrophotographic and spectroradiometric work. In this laboratory some of the substances

described have been employed as screens for transmitting bands of spectral radiations in an investigation of the photoelectric sensitivity of various substances.

When it is possible to sacrifice spectral purity in order to obtain high intensity, some of these substances, which have bands of transmission which are 70 to 80 per cent at the maximum, may be more efficient than a spectrometer in producing stimuli of spectral radiation of high intensity and covering a large area.

The apparatus used for determining the transmissions in the visible (yellow to red) and infra-red parts of the spectrum consisted of a mirror spectrometer, a fluorite prism, and a vacuum bismuth-silver thermopile, described in previous papers.

Some of the data in the violet part of the spectrum (indicated by dotted lines in the illustrations) were taken from papers published by Luckiesh¹ and by Gage.² The rest of the data in the blue and violet were obtained by means of a lens spectrometer, a glass or quartz prism, a potassium photo-electric cell (constructed by Dr. Kunz), and a high-resistance Thomson galvanometer.³

II. GROUP 1.—VARIOUS SUBSTANCES

1. PURPLE FLUORITE

In view of the increased scarcity of fluorite for use as prisms, lenses, thin plates, etc., it was of interest to examine the transmission of colored fluorites.⁴ The samples used in the present examination were of the purple variety. They contained streaks of material which were of light amethyst color; but as will be noticed presently in glasses, the effect of this coloring matter does not extend very far into the infra-red.

The transmission curve of two superposed plates (thickness 2.32 and 2.63 mm, respectively) is given in Fig. 1. The curve, *A*, is broken at 2μ ($\mu = 0.001$ mm), owing to the fact that the plates were reset. The surfaces did not take a uniformly high polish (appeared etched in spots), which accounts for part of the low transmission, which at best could not be much higher than about 80 per cent for the two plates.

¹ Luckiesh, *Trans. Illum. Eng. Soc.*, 9, p. 472; 1914.

² Gage, *Trans. Illum. Eng. Soc.*, 11 (2), p. 1050; 1916.

³ Coblenz, this *Bulletin*, 14, p. 507; 1918. It is relevant to add that the photo-electric cell was under test for accuracy in comparison with a bismuth-silver thermopile. The latter requires more skill in handling, but it is nonselective and its responses are proportional to the incident energy. The quartz lens spectrometer was constructed for ultra-violet radiometric work several years ago; this *Bulletin*, 7, p. 245, 1911; 10, p. 38 (Fig. 5), 1912. The mirror spectrometer and fluorite prism is described in this *Bulletin*, 10, p. 1, 1913; and the thermopile is described in this *Bulletin*, 11, p. 132, 1914.

⁴ In this *Bulletin*, 9, p. 116, 1912, data are given on the transmission of green fluorites, some of which had marked absorption bands.

2. GELATIN LIGHT FILTERS

In photographing the infra-red spectrum using plates sensitized with dicyanin (which makes it possible to photograph beyond 1μ) it is necessary to absorb the scattered visible rays by means of deep red glass or some other filter which is highly transparent to the infra-red.

In Fig. 1, curves *B* and *C* give the transmission of two gelatin filters (No. 724A and No. 740A; thickness 0.8 mm, made by the research laboratory of the Eastman Kodak Co.) which appear well adapted as a screen for absorbing the visible rays when photographing infra-red lines.

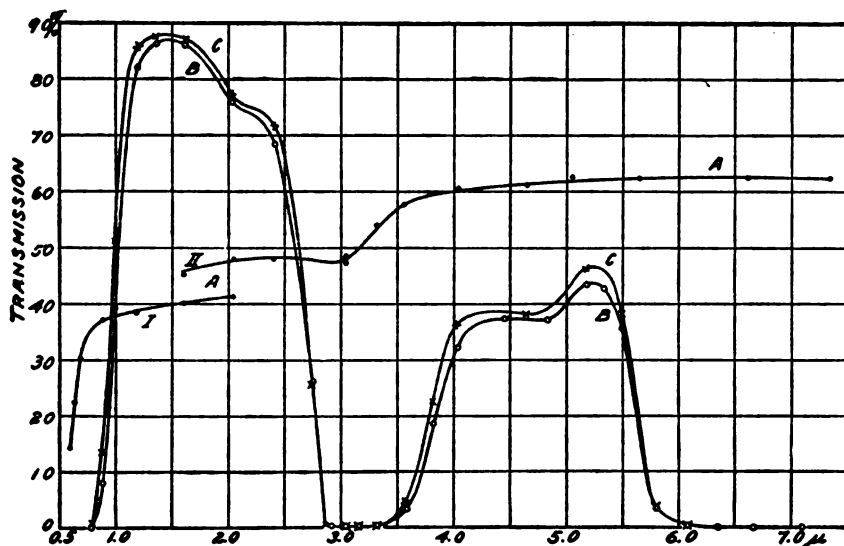


FIG. 1.—Transmission of purple fluorite, *A*, and of light filters, *B* and *C*

A comparison of these curves with those of transparent gelatin⁶ and of nitrocellulose⁶ shows that the absorption of the dye does not extend far into the infra-red.

The Wratten and Wainright filters, F 29 and H 45, are also well adapted for infra-red filters.⁷

3. GOLD-PLATED GLASS

It is well known that metals are opaque to infra-red rays, and that some of them have narrow bands of high transmission in the visible or ultra-violet spectrum. Gold has a band of low reflectivity and great transparency in the region of 0.5μ .

⁶ This Bulletin, 7, p. 648; 1911.

⁶ Publication No. 97, p. 42, Carnegie Institution of Washington, 1908.

⁷ Paschen, *Ann. der Phys.*, 43, p. 858; 1914.

The samples examined (Fig. 2) were thin films of gold deposited upon Crookes's neutral-tint glass (Fig. 17) by means of cathode disintegration.⁸ The small absorption band at 0.58μ is caused by the glass (see Fig. 17). Curve *B* gives the transmission of a thinner film of gold mounted upon crown glass. The data are of

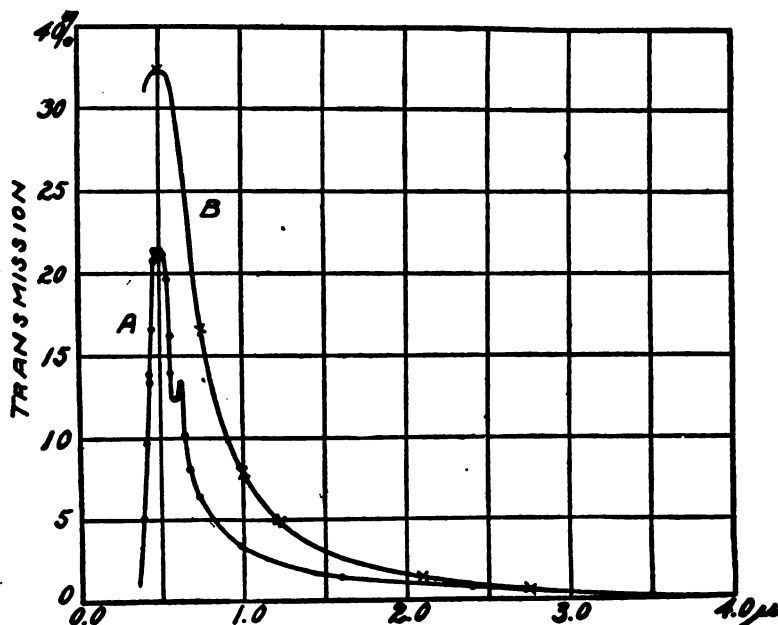


FIG. 2.—Transmission of a thin film of gold on glass

interest in connection with the question of absorbing the infra-red rays in eye-protective glasses.⁹

4. MOLYBDENITE

The transmission spectrum of a thin lamina ($t=0.007$ mm) of molybdenite, MoS_2 , is given in curve *A*, Fig. 3. This sample¹⁰ was examined in connection with the photoelectric properties of this mineral. The wavy character of the transmission curve, beyond 2.5μ , is attributable to interference,¹¹ which did not occur in the thick samples previously examined.¹²

These data are of interest in connection with the photoelectric sensitivity of molybdenite, to be published in a subsequent paper.

⁸ Taken from goggles made by the American Optical Co., Southbridge, Mass.

⁹ See this Bureau's Technologic Paper No. 93; 1917.

¹⁰ National Museum sample No. 53046, from Wakefield, Ottawa Co., Canada.

¹¹ See further illustrations by Crandall, *Phys. Rev.*, 8 (2), p. 543; 1923.

¹² Publication No. 97, p. 41, Carnegie Institution of Washington, 1908.

5. CHROMIUM SULPHATE

The sample of chromium sulphate examined, $\text{Cr}_2(\text{SO}_4)_3 + 18\text{H}_2\text{O}$, was a large flat crystal, 0.24 mm in thickness, which appeared green in transmitted light. The presence of the large amount of water of crystallization¹³ renders this substance very opaque to infra-red rays, as illustrated in curve *B*, Fig. 3. The transmission of a 1 cm layer of 1 gr. of CrSO_4 in 100 cc of water is given in curve *A*, Fig. 4.

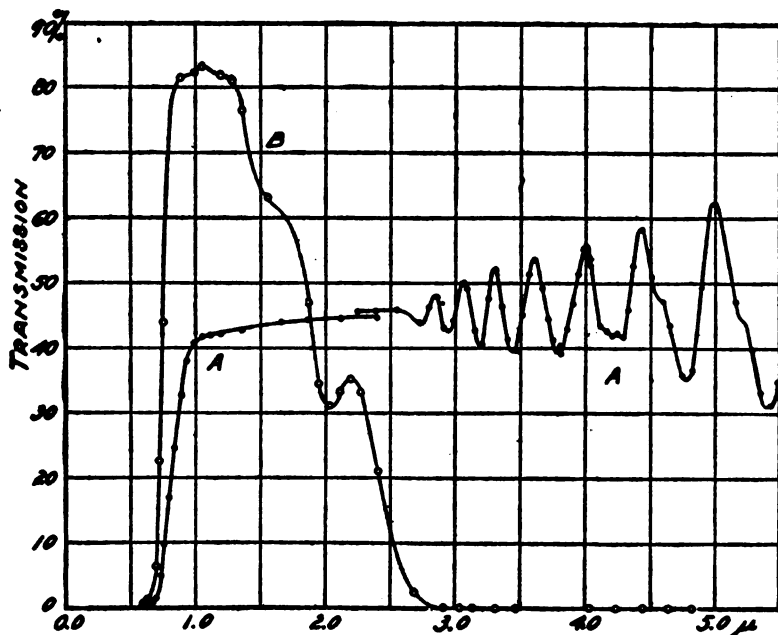


FIG. 3.—Transmission of, *A*, Molybdenite and, *B*, chromium sulphate

6. CHROME ALUM

The transmission of solution of 10 gr. of potassium chromium sulphate $\text{CrK}(\text{SO}_4)_2$ in 100 cc of water is given in curve *B*, Fig 4. The cell containing glass windows was 1 cm in thickness.

7. COBALT CHLORIDE

The transmission of a 1 cm layer of 10 gr. of cobalt chloride, $\text{CoCl}_2 + 6\text{H}_2\text{O}$ in 100 cc of water is given in curve *C*, Fig. 4. The transmission spectra of these salts are of interest in connection with the question of making screens for transmitting narrow regions of the spectrum. Unfortunately, none are more efficient than some of the glasses discussed on a subsequent page.

¹³ This Bulletin, 7, p. 619; 1922.

8. SOLUTIONS OF NICKEL SALTS

In a previous search for a substance which absorbs all the infra-red rays, nickel chloride was examined.¹⁴ In the present work the transmissions of a 1 cm layer of nickel sulphate, $\text{NiSO}_4 + 7\text{H}_2\text{O}$, nickel nitrate, $\text{Ni}(\text{NO}_3)_2$, and nickel acetate, $\text{Ni}(\text{C}_2\text{H}_3\text{O}_2)_2$, were determined, the concentration in all cases being 10 gr. of salt in 100 cc of water. The transmission curves are given in Fig. 5. They are of interest in showing a sharp absorption band at 0.7μ , followed by complete opacity at 1.4μ caused by the absorption of water.

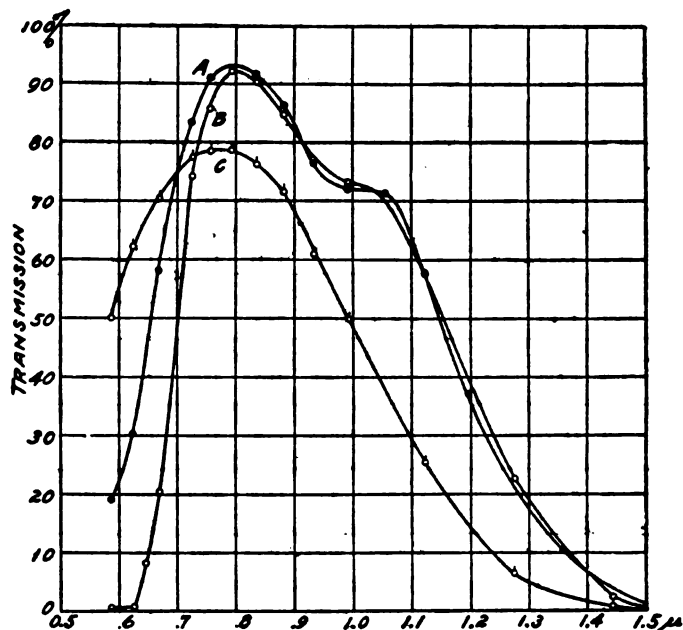


FIG. 4.—Transmission of solutions: A= CrSO_4 , B= $\text{CrK}(\text{SO}_4)_2$, C= CoCl_2

None of these salts are as suitable as cupric chloride for absorbing the infra-red rays.

III. GROUP 2.—VARIOUS GLASSES

It would be futile to attempt to determine the transmissive properties of all of the numerous glasses which are obtainable under different trade names but which have a characteristic color—red, yellow, blue-green, etc.

The color of the same kind of glass may be different for different melts and for different parts of the same melt; and it depends upon

¹⁴ This Bulletin, 7, p. 658; 1911.

the length of time of the heat treatment.¹⁵ However, while this has a marked effect in the visible spectrum, as will be noticed presently, the effect of the coloring matter usually does not extend far into the infra-red.

In view of the fact that, for most of the glasses herein described, the transmission in the visible and ultra-violet had already been determined,¹⁶ the present data relate principally to the infra-red. The application of some of these glasses in eye-protective goggles has already been described.¹⁷

1. RED GLASSES

Red glasses include (1) copper ruby, which is most commonly found as flashed ruby, consisting of a thin layer of intensely

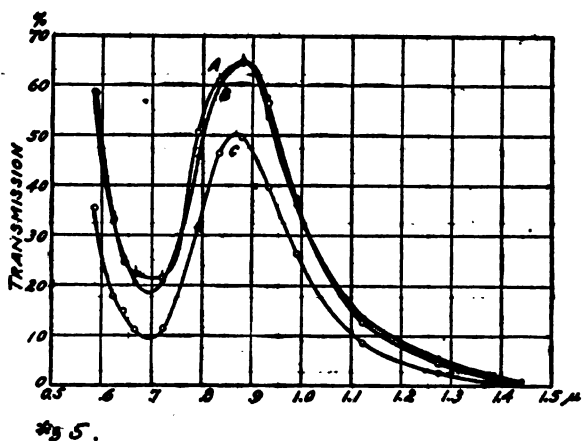


FIG. 5.—Transmission of nickel salts in solution: A, sulphate, B, nitrate, C, acetate

colored glass covering a layer of colorless glass, (2) gold ruby, and (3) selenium red, in which selenium is the essential coloring agent.

In Fig. 6, curves C and D ($t=2.48$ and 1.95 mm), give the transmission of two copper ruby glasses, the latter being a flashed ruby glass.

Curve A, Fig. 6, gives the transmission of a Corning¹⁸ high-transmission selenium glass, G 24; thickness, $t=5.90$ mm. It is conspicuous for its absorption band at 1.1μ , which is found in ordinary glass containing iron as an impurity. (See curve A, Fig. 20.) This property makes it useful as an absorption glass in

¹⁵ Gage, Trans. Illum. Eng. Soc., 11 (2), p. 1050; 1916.

¹⁶ Luckiesh, Trans. Illum. Eng. Soc., 9, p. 472, 1914; Gage, Trans. Illum. Eng. Soc., 11 (2), p. 1050, 1916.

¹⁷ This Bureau's Technologic Paper No. 93, 1917.

¹⁸ Kindly supplied by the Corning Glass Works, Corning, N. Y.

the eyepiece of an optical pyrometer. Combined with the light blue-green glass, Corning G 124JA (Fig. 11, C), to limit more effectively the deep red end of the spectrum, a narrower monochromatic red line is produced. This combination has been found useful and well adapted for optical pyrometer measurements.

Curve *B*, Fig. 6, gives the transmission of a sample of Schott's red glass, No. 2745 ($t=3.18$ mm), which is conspicuous for its extraordinary transparency of 85 to 90 per cent, extending from

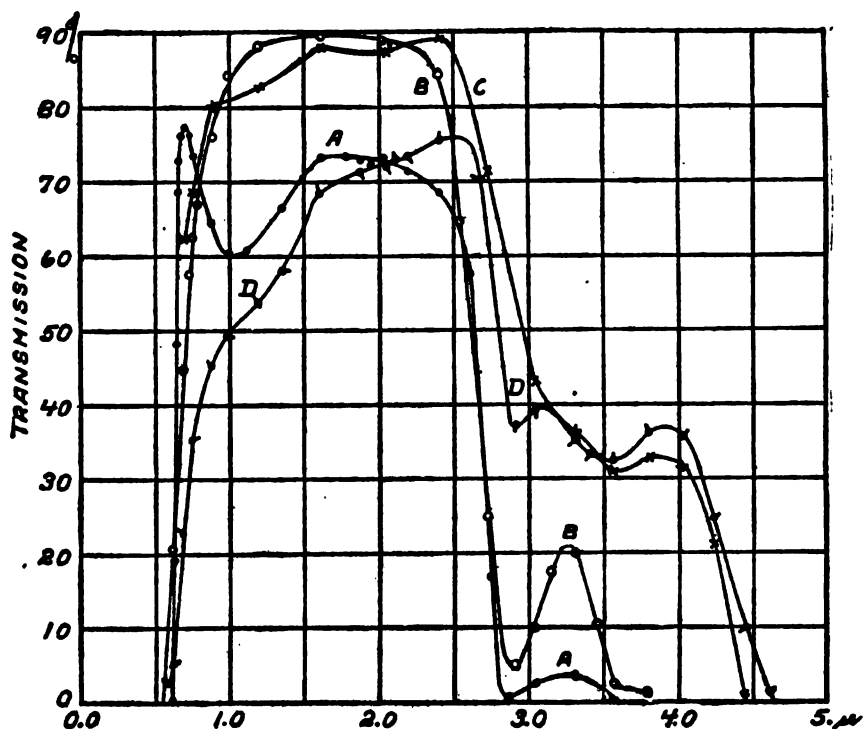


FIG. 6.—Transmission of red glasses

1 to 2.4μ . Using a 1 cm cell of water and a red glass, a strong transmission band is obtained at about 1μ .

2. ORANGE GLASSES

The samples examined were Corning G 34 and G 36 ($t=3.55$ mm and 5.65 mm, respectively), curves *B* and *A*, Fig. 7. They have the characteristic absorption band of the selenium red glass, Fig. 6, but probably contain a substance which increases the transparency at 3 to 4μ . In the visible, the transmission terminates rather abruptly at about 0.55μ . Chemically, these glasses belong to the group of selenium red and noviol yellow glasses.

3. YELLOW GLASSES

In Fig. 8, curves A and B, give the transmission of two fluorescent yellow glasses, Corning G 371 ($t=4.90$ mm) and G 311 Y ($t=4.97$ mm). The coloring matter is presumably uranium, which produces the absorption bands at 0.92μ and 1.55μ .

Corning noviol (shade B, $t=2.88$ mm), curve C, Fig. 8, (also Fig. 19) is a light-colored, yellow glass used for protecting the eyes from ultra-violet light. This glass is opaque¹⁹ to ultra-violet

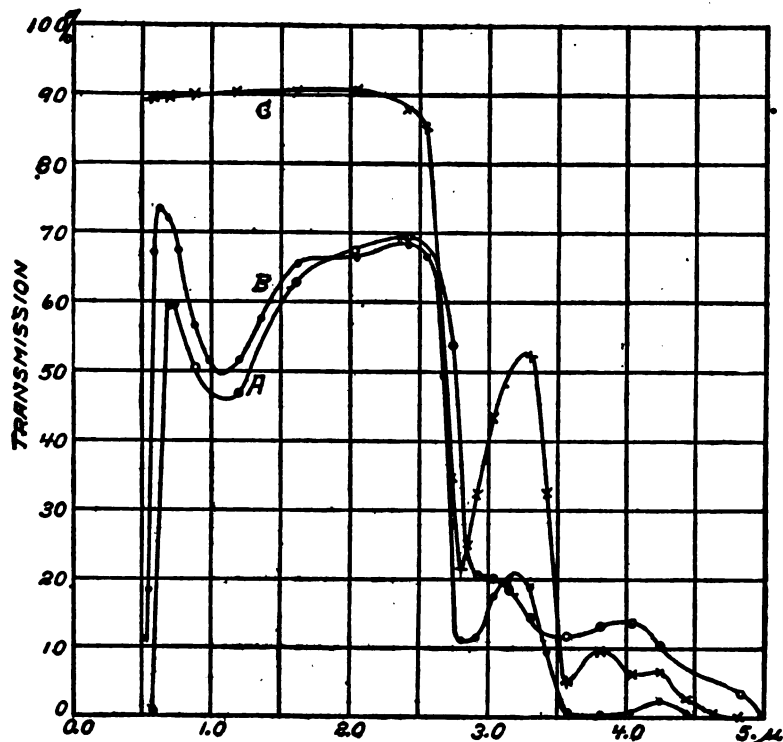


FIG. 7.—Transmission of Corning orange-colored glasses: A, G 36; B, G 34; C, Corning Pyrex glass ($t=1.55$ mm)

radiations (as indicated by the dotted line in Fig. 8, which represents data taken from Gage's paper). An examination of various shades of this glass shows that the coloring matter does not have a marked effect upon the infra-red transmissions.²⁰

"Noviweld" is a dark-yellow, eye-protective glass which absorbs the blue and violet, and transmits red, yellow, and green. The samples examined were kindly provided for examination by Dr.

¹⁹ Gage, *Trans. Illum. Eng. Soc.*, 11 (2), p. 1050; 1916.

²⁰ This Bureau's Technologic Paper No. 93; 1917. In this paper, Fig. 1, curve B, is noviol "Shade A" and curve C, is noviol "shade C."

H. P. Gage, of the Corning Glass Works. The maker's number is Corning G 391DM. The transmissions of various shades of this glass are given in Fig. 9, the thickness in all cases being very close to 2.2 mm. Curves A = shade 30 per cent; B = shade 3; C = shade $4\frac{1}{2}$; D = shades 6 and 7. These glasses are unique for their great opacity to infra-red rays.

4. GREEN GLASSES

The infra-red transmission of a yellowish-green glass, Schott's copper oxide, No. 431 III ($t=3.43$ mm), is given in curve A, Fig. 10. Curve B gives the transmission of a sample of slightly

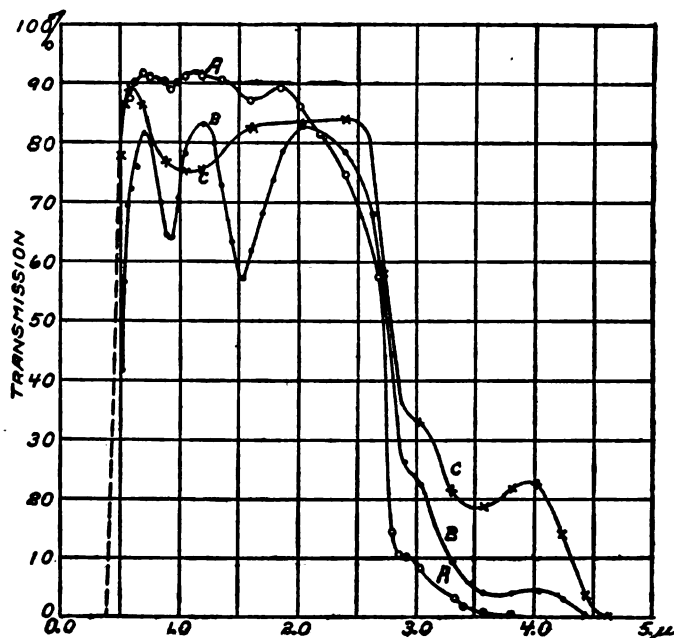


FIG. 8.—Transmission of various yellow glasses (A and B, fluorescent)

bluish-green glass; Corning high-transmission green G 171 ON ($t=5.11$ mm). The infra-red transmission of Corning signal green G 40 (curve C, Fig. 10, $t=4.93$ mm.) differs but little from the preceding sample. The dotted part of the curve, illustrating the transmission in the green and blue, was taken from the paper published by Gage.²¹

The transmission of Crookes's sage-green glass, ferrous No. 30, also of a blue-green glass, is given in Fig. 17, while in Fig. 18, curve A, is given the transmission of a light yellowish-green glass which is conspicuous for its high transmission in the infra-red.

²¹ Gage, Trans. Illum. Eng. Soc., 11 (2), p. 1050; 1916.

Combined with a water-cell and noviol glass, these glasses give narrow transmission bands in the visible spectrum.

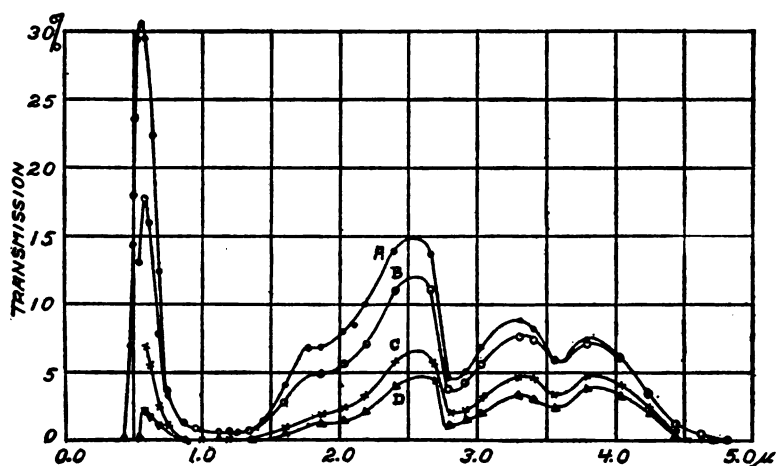


FIG. 9.—Transmission of Corning "Noviwell" glasses

5. BLUE-GREEN GLASSES

The transmission curves of several glasses, made by the Corning Glass Works and having marked opacity for infra-red rays, are given in Fig. 11. The light greenish-blue glass, Corning G 124J

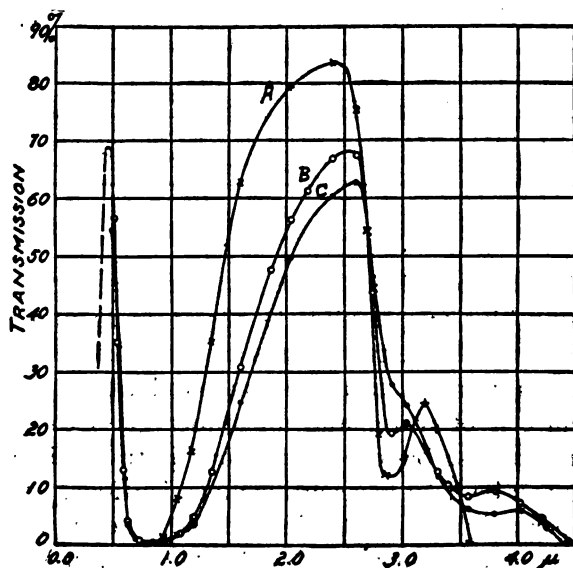


FIG. 10.—Transmission of green glasses

($t = 2.6$ mm), curve B, is obtainable under the trade name, "Heat absorbing glass." The transmission of a sample of another melt, marked G 124J" ($t = 2.90$ mm), is given in curve A.

Curve *C* gives the transmission of a very light blue-green glass, G 124JA ($t=1.5$ mm), which has a high transmission (50 per cent) in the visible spectrum. (See also Fig. 17.) Curve *D* gives the transmission of a very dark yellowish-brown glass, G 124IP ($t=2.0$ mm), which is one of the most opaque glasses yet examined.

6. BLUE GLASSES

A number of blue glasses were available for examination. In Fig. 12, curve *A* gives the transmission of a cobalt blue glass ($t=2.43$ mm). Curve *B*, Fig. 12, gives the transmission of a high transmission blue, Corning G 401Z ($t=6.56$ mm). Curve *C*, Fig. 12, gives the transmission of Schott's blue F 3086 ($t=2.58$ mm).

Curve *D*, Fig. 12, gives the transmission of an interesting blue glass (probably of foreign make $t=2.09$ mm) obtained from the

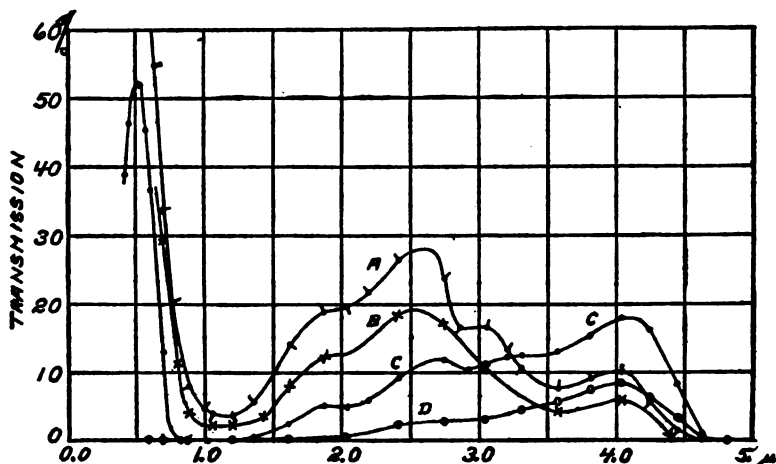


FIG. 11.—Transmission of blue-green glasses

Bausch & Lomb Optical Co. The transmission curve of a similar glass, Corning blue-purple ultra G 585 ($t=3.13$ mm) is given in curve A, Fig. 13. These two glasses are remarkable for their extraordinarily high, narrow, transmission band at 0.82μ . Combined with a 1 cm cell of water and an amber glass to eliminate the blue, a very intense band of radiations is transmitted at 0.7 to 0.8μ , which can be used in scientific investigations. Using a solution of copper sulphate or cupric chloride, to eliminate the transmission band at 0.7μ , leaves a band of high transmission in the violet.

Curve *B*, Fig. 13, gives the transmission of a pale blue-green glass, Corning, G 584 ($t=3.70$ mm). Curve *C* gives the transmission of a light blue-green glass, Corning, G 171 IZ ($t=3.23$ mm).

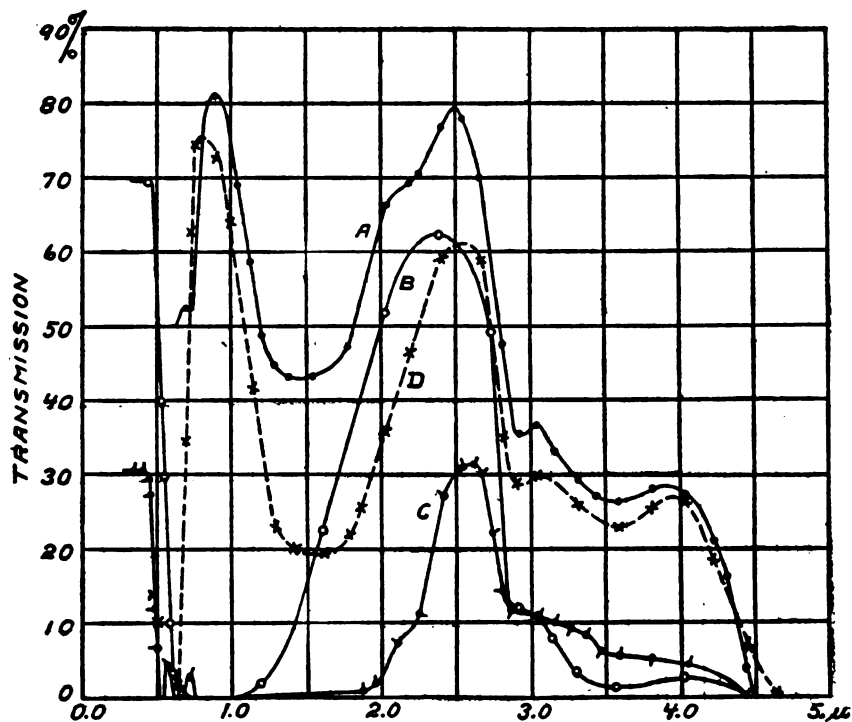


FIG. 12.—Transmission of blue glasses.

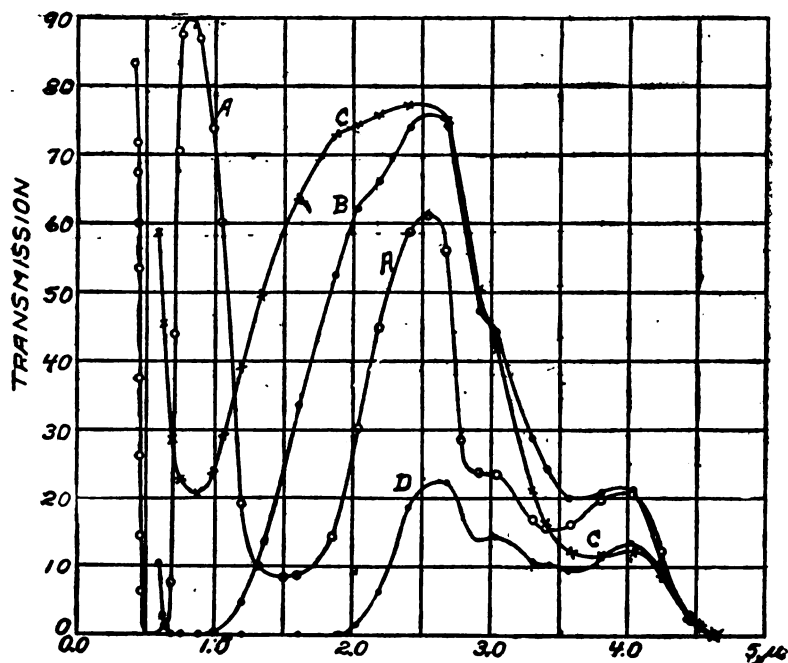


FIG. 13.—Transmission of various Corning glasses: A, blue purple ultra, G 585; B, pale blue-green, G 584; C, light blue-green glass, G 171 12; D, dark blue, G 53

Curve D, Fig. 13, gives the transmission of a dark blue glass, Corning, G 53 ($t=2.40$ mm).

All these glasses are conspicuous for their great opacity in the region from 1 to 2 μ .

7. PURPLE GLASSES

Curve A, Fig. 14, gives the transmission of a deep purple glass ("electric smoke," purple: $t=1.90$ mm) made by the American Optical Co. Curve B gives the transmission of a thicker ($t=2.46$ mm), darker sample of the same material which transmits only the deep red of the visible spectrum. This glass has an unusually

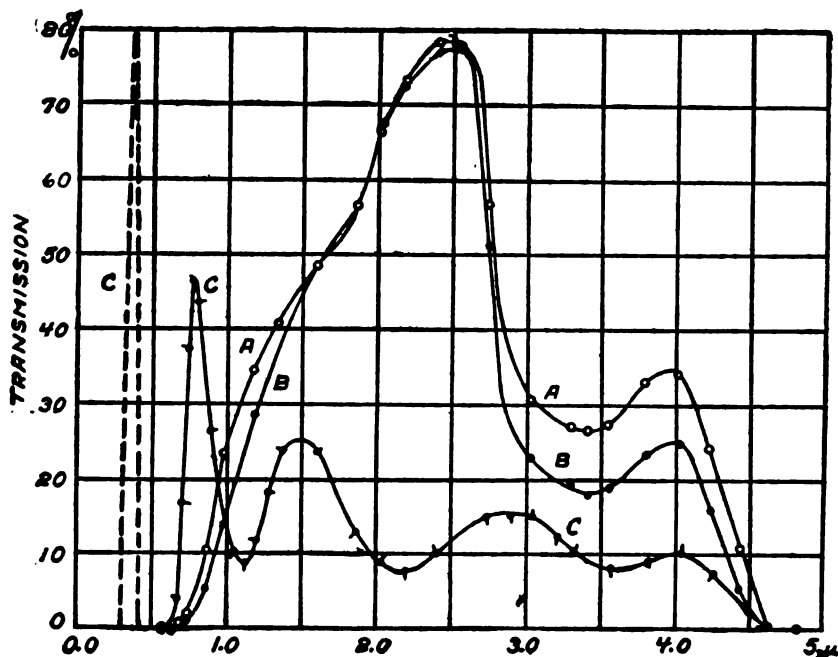


FIG. 14.—Transmission of purple glasses

high transmission at 2.5 μ which makes it valuable as a screen for transmitting these radiations. Curve C, Fig. 14, gives the transmission of an interesting purple glass, Corning, G 55 A 62 (the new number is G 586 A; $t=2.85$ mm), which has a narrow band of high transmission in the violet,²² 0.36 μ , and another band of high transmission at 0.77 μ . A third transmission band occurs at 1.5 μ . It would be interesting to know what substance causes these absorption bands, which occur at the roughly harmonic wave lengths (0.27 μ ?), 0.55 μ , 1.1 μ , and 2.2 μ . The band at 3.6 μ is commonly found in glasses and hence is not to be consid-

²² Data supplied by Dr. K. S. Gibson.

ered as belonging to the coloring matter. Using a cupric chloride solution and a purple glass, the 0.38μ band is transmitted. Using a cell of water and a noviol glass, the band at 0.77μ is transmitted.

Another Corning glass, G 586 J transmits only the ultra-violet band, 0.36μ , the band at 0.77μ being suppressed to an immeasurable value.

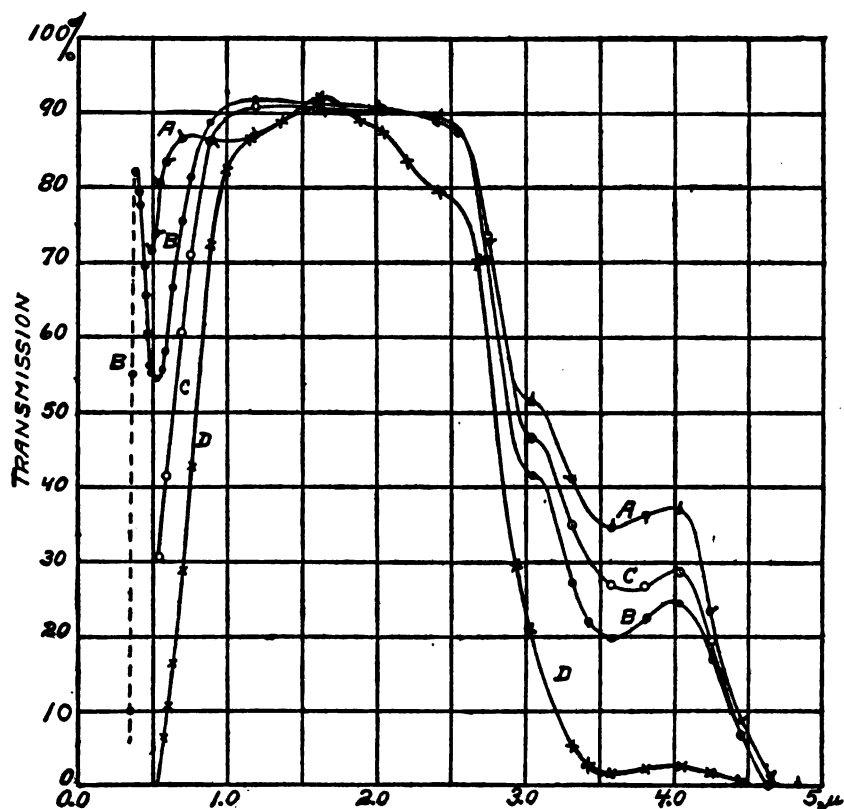


FIG. 15.—Transmission through amethyst glasses. (A, is a colorless glass, A. O. C. Lab. No. 58)

8. AMETHYST GLASSES

The amethyst color in glasses is the result of absorption in the yellow-green. As shown in Fig. 15, the absorption in the infra-red is the same as that of white crown glass. (See Fig. 17.) Curve B, Fig. 15, gives the transmission of a dark sample of amethyst (shade C, from A. O. C.;²³ $t=2.11$ mm). The data in the ultra-violet are from Luckiesh.²⁴

²³ American Optical Co., Southbridge, Mass.

²⁴ Luckiesh, Trans. Illum. Eng. Soc., 9, p. 475; 1914.

Curve *C* gives the transmission of a deep wine-colored sample of unknown origin ($t=1.92$ mm). Curve *D*, Fig. 15, gives the transmission of a Corning red-purple glass, G 172 B. W. 5 ($t=4.43$ mm), which is conspicuous for its high transmission between 1 and $2.5\ \mu$ (see the ruby glasses) and its abrupt terminations at 0.5 and $3.5\ \mu$.

9. BLACK GLASSES

Black, or "smoke," glasses are commonly used for eye protection. The different kinds vary greatly in their infra-red transmission. In Fig. 16, curve *A* gives the transmission of a very dark spectacle glass (of unknown origin, $t=1.86$ mm) which has been in use in the laboratory. Curve *B* gives the transmission of Schott's black glass, No. 444 III ($t=3.6$ mm), which is unusually opaque to infra-red radiations.

Curve *D*, Fig. 16, gives the transmission of a sample of smoke glass, shade D ($t=2.45$ mm), obtained from the Bausch & Lomb Optical Co. The sample, curve *C* ($t=2.91$ mm), evidently has the same composition, although its source is unknown. It is probably a Jena glass. These two samples are conspicuous for their high, narrow, transmission band at $0.75\ \mu$.

10. CROOKES'S GLASSES²⁵

As a result of an extensive investigation, Crookes²⁶ has produced glasses designed, respectively, for (1) absorption of infra-red ("heat") rays, (2) absorption of ultra-violet rays, (3) high transmission of luminous rays, and (4) low transmission of luminous rays for the reduction of the glare of the sun on expanses of snow or reflected from water.

Neutral-tinted glass has a smoky, neutral tint which enables one to perceive objects in their natural colors, as is true of the smoke, or black, glasses just described. The shades *A* and *B* ("light" and "dark") of Crookes's neutral-tint glass have a high transmission in the visible. In the infra-red the lighter shade, curve *A*, Fig. 17 ($t=1.96$ mm), absorbs but little more than ordinary white crown glass (curve *E*, $t=2.18$ mm). Curve *B*, Fig. 17, gives the transmission of a sample of dark neutral tint ($t=2.00$ mm).

The complex absorption band at 0.50 and $0.58\ \mu$ is found in didymium. This part of the curve was obtained with a glass prism, using either a thermopile or photoelectric cell as a radiometer.

²⁵ Kindly furnished by the American Optical Co.

²⁶ Crookes, Phil. Trans. Roy. Soc., 218, p. 25; 1914.

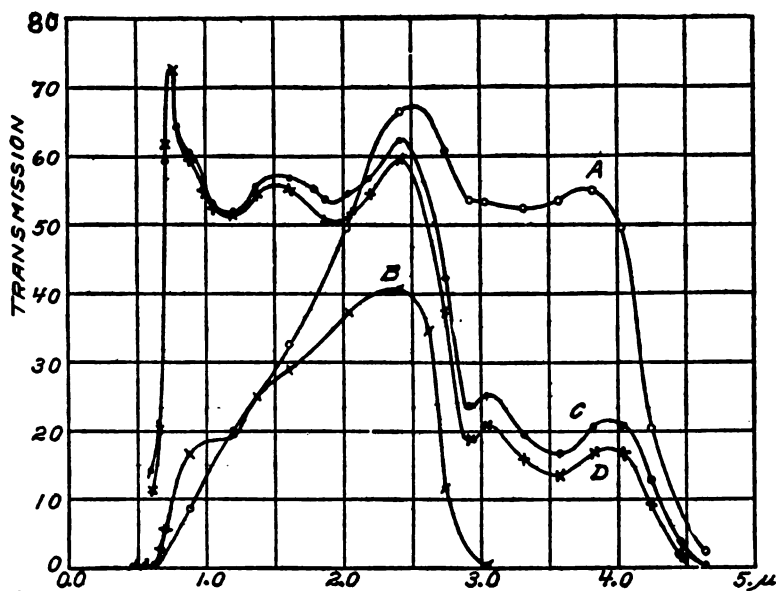


FIG. 16.—Transmission of black glasses

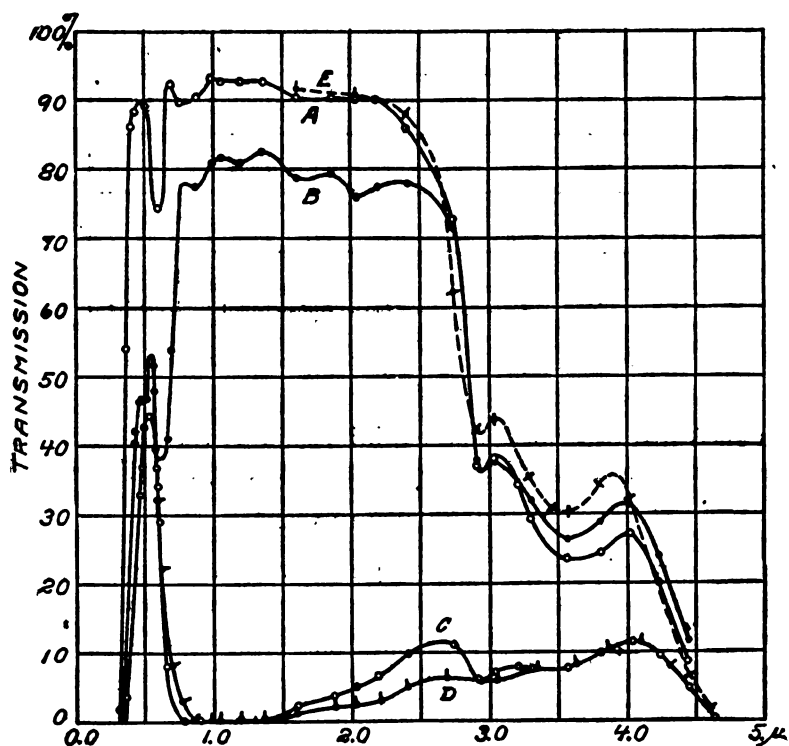


FIG. 17.—Crookes's glasses: A, light ($t=1.96$ mm.); B, dark ($t=2.00$ mm.); C, ferrous No. 30, sage-green ($t=1.98$ mm.); D, blue-green glass (A. O. C. Lab. No. 59; $t=1.93$ mm.). (A and B are Crookes's neutral-tint glass.) E, white crown glass ($t=2.18$ mm)

According to Gage,²⁷ this glass absorbs the 0.359μ band of carbon, but transmits the 0.388μ band of the carbon arc.

Crookes's sage-green (A. O. C., Ferrous No. 30) is another interesting glass, having a high absorption in the infra-red, curve C, Fig. 17 ($t=1.98$ mm), and a band of high transmission (45 per cent) at 0.53μ in the green. Combined with a water cell (or better still, cupric chloride solution), this glass enables one to obtain visible radiations free from infra-red rays.

A *blue-green glass* (A. O. C., lab. No. 59, $t=1.93$ mm), which transmits about 40 per cent in the visible, is illustrated in curve D, Fig. 17. In the infra-red it is more opaque than the sage-

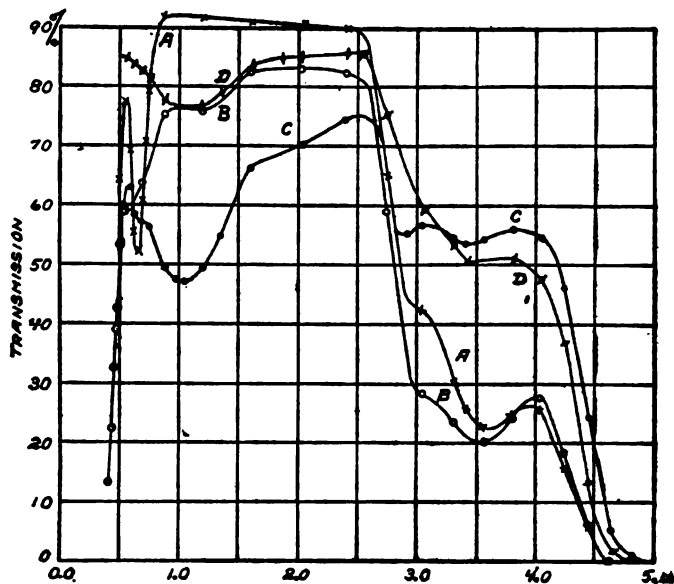


FIG. 18.—A, Lab. No. 61, A. O. C.; B and D, Fiesul glass, shade B; C, Hallauer glass

green glass just described, and exhibits some of the characteristics of the bluish-green Corning glass, G 124JA, Curve C, Fig. 11.

11. COLORLESS GLASSES

The transmission curve of a white crown glass is given in Fig. 17, curve E ($t=2.18$ mm). It is of interest in comparison with yellow and amber-colored glasses which have a high absorption in the violet, but which show little or no absorption in the infra-red, caused by the coloring matter.

Pyrex glass, from the Corning Glass Works (colorless, thickness 1.55 mm), has a marked absorption band at 2.8μ , as is true of

²⁷ Gage, Trans. Illum. Eng. Soc., 11 (2), p. 1050; 1916.

all glasses containing a high percentage of silica. (See curve C, Fig. 7.)

Window glass contains iron, which causes a greenish color. This is produced by an absorption band that has its maximum at 1.1μ . (See curves A and C, Fig. 20; $t=2.11$ and 3.3 mm.) Curve C gives the transmission of a sample (battery jar) which had a faint bluish tinge.

Lab. No. 58, of the American Optical Co., is a colorless glass which has a marked absorption in the ultra-violet. As shown in curve A, Fig. 15, the absorption (thickness of sample 2.04 mm)

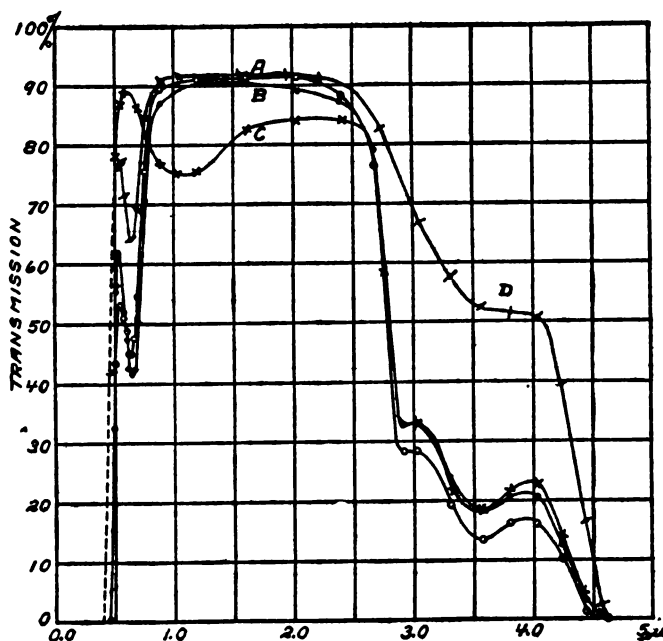


FIG. 19.—Euphos, shade B, B, from Bausch & Lomb ($t=3.1$ mm); A (B. S. scrap material, $t=3.30$ mm). Corning Noviol, shade B, C. Akopos green, curve D

in the infra-red is the same as that of white crown glass. Viewed edgewise, this glass shows a light brownish color.

12. UNCLASSIFIED GLASSES

Lab. No. 61, of the American Optical Co., is a light yellowish-green glass having a narrow absorption band at 0.65μ . (See curve A, Fig. 18; thickness 2.09 mm).

Fieuzal glass (shade B, curve B, Fig. 18; thickness 2.04 mm), from the American Optical Co., is a dark greenish-yellow glass of French origin. It has the characteristic absorption band at 1.1μ , of glasses, containing iron, and the didymium band at

0.6μ . Curve *D*, Fig. 18, gives the transmission of a sample ($t=1.98$ mm) purporting to be a Fieuzal glass, obtained from an optician. Its transmission curve has properties in common with the Fieuzal and the Hallauer glass.

Hallauer glass (thickness of sample 1.41 mm) is an eye protective glass of German origin. It has practically the same tint as the Fieuzal glass. However, as shown in curve *C*, Fig. 18, its transmission is quite different in the infra-red.

Euphos glass is an eye protective glass of German origin. It has a slightly more greenish tint than the Fieuzal glass. As

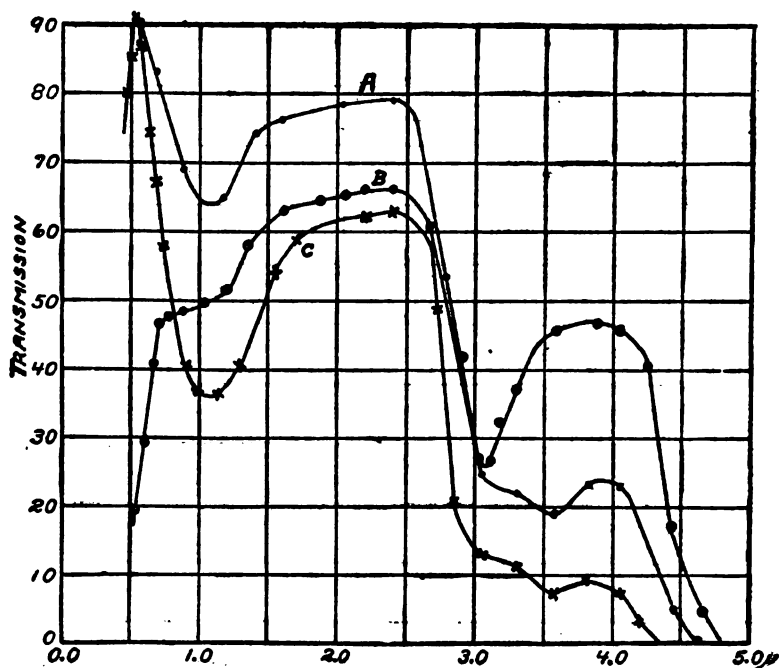


FIG. 20.—Transmission of plate window glass, *A* and *C* and of amber bottle glass, *B*

shown in curve *A*, Fig. 19 (thickness 3.30 mm), the absorption band at 1.1μ is absent and the band at 0.62μ is more sharply defined than in the Fieuzal glass.

Curve *B*, Fig. 19, gives the transmission of a sample of Euphos glass (thickness 3.1 mm), which was obtained from the Bausch & Lomb Optical Co. The transmission is practically the same as that of the preceding sample, the source of which is unknown.

Amber-colored bottle glass.—The sample examined, curve *B*, Fig. 20 ($t=2.2$ mm), was a fragment from an ordinary brown beer bottle. Its absorption in the ultra-violet is quite as effective as some of the new, highly advertised protective glasses.

Akopol green ($t = 1.58$ mm), curve *D*, Fig. 19, is another eye-protective glass. It is to be noticed that all of these glasses protect the eye from ultra-violet rays, but they have little or no protective value for infra-red rays, other than that which obtains in colorless glass.

IV. SUMMARY

This paper gives the spectral transmission of various substances, especially colored fluorite, light filters, and colored glasses.

Some of the substances provide a simple means for obtaining narrow spectral bands of radiant energy of high intensity and large area, without the employment of a spectroscope. By properly combining them one can obtain a screen having a narrow band of high transmission at 0.38μ , 0.5μ , 0.55μ , 0.7μ , 0.8μ , 1μ , and 2.2μ .

The data on glasses are also useful in giving information as to their applicability for protecting the eyes from injurious radiations. (See Table 1.)

WASHINGTON, January 31, 1918.

APPENDIX

TRANSMISSION DATA OF VARIOUS GLASSES ²⁸

The quartz mercury vapor lamp is one of the most common sources of injury to the eye, from ultra-violet rays. Injurious effects result even after these rays are reflected from fabrics, etc. The ultra-violet rays are very insidious in that the inexperienced and unsuspecting experimenter does not feel the effects until several hours after exposure to these rays. The milder cases of injury consist of granulation of the eyelids and pain through the optic nerve. This may continue for several days. Plotnikoff (*Photochemische Versuchstechnik*) says that repeated exposure to the ultra-violet rays from these lamps causes permanent injury to the eyesight. Reports have come to the writers' notice of the deleterious effects of ozone and nitrous oxide upon the bronchial tubes, etc., which gases are formed by the ultra-violet rays from quartz mercury vapor lamps.²⁹

Mercury vapor lamps having glass tubes are not known to be injurious.

It is extremely important to protect the eyes from ultra-violet rays by means of colored glasses, the most efficient ones being amber, yellow, and greenish-yellow glasses. The most efficient glass, used singly, for absorbing the infra-red and the ultra-violet is the Corning new "Noviweld," as shown in Table 1, which gives the per cent of total radiant energy transmitted by various glasses, some of whose spectral transmission curves are given in the foregoing pages.

The radiations from the quartz mercury vapor lamp are mostly in the visible and in the ultra-violet, only a few weak emission lines being in the infra-red. A new Cooper-Hewitt, also an R. U. V. quartz mercury lamp, were used in making these determinations. These lamps become red-hot. Hence, in order to eliminate the radiations of hot quartz from these transmission data, the bismuth silver thermopile, which was used to measure the radiation intensities, was covered with a 1 cm cell, having thin quartz windows and containing distilled water, both of which substances are very transparent to ultra-violet rays, but which are very opaque to infra-red rays. As shown at the bottom of Table 1, white crown glass and mica absorb much of the extreme ultra-violet, but they are not sufficiently opaque to protect the eye from these injurious rays.

The transmission of the radiation from a magnetic arc lamp, through some of these protective glasses, is given in the seventh column of Table 1. A 1 cm cell of water (just described) eliminated the radiation from the electrodes. In making these measurements the electrodes were mounted in a hand-operated mechanism to control the steadiness of the arc. These data are of interest to workmen doing arc welding.

The transmission of these glasses (Table 1, column 5) was determined also for the radiations from a (500-watt stereopticon) gas-filled tungsten lamp which emits considerable visible and infra-red radiation. In this test no water cell intervened between the thermopile and the sample under examination. The data are instructive in comparison with the spectral transmission curves already described.

²⁸ See Technologic Paper No. 93, revised edition, which gives Table 1 of the present paper without the data on the transmission of solar radiation.

²⁹ For a thorough discussion of the injurious effects of radiant power upon the eye, the following papers should be consulted: Verhoeff and Bell, *Proc. Amer. Acad. Arts and Sci.*, 51, p. 630, 1916. Birge, *Trans. Hum. Eng. Soc.*, 10, p. 932, 1915. *Amer. Jour. Physiol.*, 26, p. 21, 1914; 29, p. 335, 1916.

The last column of Table 1 gives the transmission of solar radiation through various glasses. In making these measurements a single thermojunction was exposed, through a rock salt window, to the solar radiation, on a very clear day (Mar. 16, 1918, $Q=1.1$ gr. cal. per cm^2 of horizontal surface).

Instructive data on the ultra-violet component in various artificial lights have been published by Bell,³⁰ who concluded that no commercial illuminant radiates, for any ordinary working value of illumination, enough ultra-violet energy to be at all harmful, provided one exercises ordinary discretion in keeping unpleasantly bright light out of the eyes.

TABLE 1.—Transmission of the Radiations from a Gas-Filled Tungsten Lamp, the Sun, a Magnetite Arc, and from a Quartz Mercury Vapor Lamp (no Globe) Through Various Substances, Especially Colored Glasses

Color	Trade name	Source *	Thick- ness in mm	Transmission, per cent			
				Gas- filled tung- sten	Quartz mer- cury vapor	Mag- netite arc	Solar radia- tion
Greenish-yellow.....	Fleuzal, B.....	A. O. C.	2.04	71.6	26.9	46.0	63
Do.....	Fleuzal, 63.....	F. H. E.	1.80	75.5	34.3	55.0	72
Do.....	Fleuzal, 64.....	F. H. E.	1.65	50.7	22.0		
Do.....	Euphos.....	B. S.	3.27	78.9	25.0		
Do.....	Euphos, B.....	B. & L.	3.12	78.8	24.7	53.0	64
Do.....	Akapos green.....	J. K.	1.58	84.6	29.5	59.0	74
Do.....	Hallauer, 65.....	B. S.	2.36	70.3	17.7		
Do.....	Hallauer, 64.....	F. H. E.	1.35	58.7	25.9		55
Smoky green.....	G 124, IP.....	C. G. W.	2.81	.4	.2		
Yellow-green.....	Noviweid, 30%.....	C. G. W.	2.14	5.1	7.8		9
Do.....	Noviweid, shade 3.....	C. G. W.	2.20	3.4	4.2	2.7	
Do.....	Noviweid, shade 4½.....	C. G. W.	2.20	1.6	1.2	.8	
Do.....	Noviweid, shade 6.....	C. G. W.	2.17	.9	.4	.2	.9
Do.....	Noviweid, shade 7.....	C. G. W.	2.17	.8	.2		
Amber.....	No. 213.....	P. W. G.	5.57				43
Do.....		B. S.	3.12	51.6	15.2		
Do.....	Bottle.....	B. S.	2.2				49
Do.....	Saniweid, dark.....	J. K.	1.32	78.1	10.6	43.0	30
Orange.....	G 34.....	C. G. W.	3.57	56.9	17.0		47
Yellow.....	Noviol, shade A.....	C. G. W.	2.00				81
Do.....	Noviol, shade B.....	C. G. W.	2.88	74.1	32.2	56.0	75
Do.....	Noviol, shade C.....	C. G. W.	2.00				72
Sage green.....	Ferrous No. 30.....	A. O. C.	1.95	5.3	17.5		17
Yellow-green.....	No. 61.....	A. O. C.	2.10	82.7	28.6		72
Blue-green.....	Lab. No. 59.....	A. O. C.	1.93	3.7	17.3	11.5	
Do.....	G 124JA.....	C. G. W.	1.53	5.3	21.5	12.5	19
Black.....	Smoke, C.....	B. & L.	2.26	65.3	31.2	52.0	60
Do.....	Smoke, D.....	B. & L.	2.45	50.9	16.0	39.0	43
Neutral tint.....	Crookes, A.....	A. O. C.	1.97	85.3	46.1		89
Do.....	Crookes, B.....	A. O. C.	2.00	75.7	32.0	64.0	69
Gold plate.....	Pfund.....	A. O. C.		2.6	7.2	1.2	12
Do. (darker).....	Pfund.....	A. O. C.			1.3		

³⁰ Bell, Proc. Amer. Acad. Arts and Sci., 48, p. 1, 1912.

* A. O. C., Amer. Optical Co., Southbridge, Mass.; C. G. W., Corning Glass Works, Corning, N. Y.; B. & L., Bausch & Lomb, Rochester, N. Y.; J. K., Julius King Optical Co., New York City; F. H. E., F. H. Edmonds, Optician, Washington, D. C.; P. W. G., Penna. Wire Glass Co., Philadelphia, Pa.; B. S. Bureau of Standards; scrap material, source unknown.

TABLE 1—Continued

Color	Trade name	Source	Thick- ness in mm	Transmission, per cent			
				Gas- filled tung- sten	Quartz mer- cury vapor	Mag- netic arc	Solar radia- tion
Colorless.....	Lab. No. 58.....	A. O. C.	1.58	83.3	40.0	66.0	88
Do.....	Lab. No. 57.....	A. O. C.	2.00		51.9		
Amethyst.....	Shade C.....	A. O. C.	2.11	82.8	44.3		79
Purple.....	Electric smoke.....	A. O. C.	1.89	36.6	2.2		11
Do.....	G 55 A 62.....	C. G. W.	2.85	17.4	17.0		16
Blue.....	Shade D.....	B. & L.	2.09	37.6	20.7	39.0	
Blue, dark.....	G 53.....	C. G. W.	2.51	2.9	3.9		
Blue-green.....	G 171-LZ.....	C. G. W.	3.21	46.6	41.7		
Blue, pale.....	G 504.....	C. G. W.	3.75	24.9	25.2		
Red-purple.....	G 172 BW 5.....	C. G. W.	4.93	72.4	26.5		
Blue-purple.....	G 585.....	C. G. W.	3.13	35.8	34.0		41
Red.....	Selenium.....	C. G. W.	2.90	67.8	7.9	48.0	48
Do.....	Schott's ^a		3.22	69.4			46
Do.....	Flashed.....	B. S.			4.8		
Colorless.....	Window.....	B. S.	1.85		59.5		82
Do.....	Crown.....	B. S.	1.56		64.9		92
Do.....	Aqueduct.....	P. W. G.	4.75				81
Brown.....	Mica.....	B. S.	1.30		35.4		
Colorless.....	do.....	B. S.	.09		43.1		
Clear.....	Water.....	B. S.	10.00	34.2	^a 56.0		
Do.....	do.....	B. S.			^b 83.0		^b 76

^a Transmission of 1 cm cell having glass windows.

^b Using a 1 cm. cell having thin quartz windows. The transmission of this cell, for the total radiation from the quartz mercury vapor lamps, no water cell intervening, was 36 per cent.

The results of the foregoing investigation show that, in glasses which have a high absorption in the violet and ultra-violet (thus producing yellow and amber colored glasses), the effect of the coloring matter does not, as a rule, extend into the infra-red. Such glasses usually absorb but little more than colorless glass in the infra-red.

Glasses which have a wide absorption band in the red and yellow (blue-green glasses) usually have a marked absorption in the infra-red.

In view of the diversity of glasses on the market and the uncertainty of their protective properties, the foregoing brief summary will give the reader a rough estimate of the protective properties that may be expected when purchasing untested glasses whose origin is unknown.

ELECTRICAL OSCILLATIONS IN ANTENNAS AND INDUCTANCE COILS

By John M. Miller

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I. INTRODUCTION

A modern radiotelegraphic antenna generally consists of two portions, a vertical portion or "lead-in" and a horizontal portion or "aerial." At the lower end of the lead-in, coils or condensers or both are inserted to modify the natural frequency of the electrical oscillations in the system. When oscillating, the current throughout the entire lead-in is nearly constant and the inductances, capacities, and resistances in this portion may be considered as localized or lumped. In the horizontal portion, however, both the strength of the current and the voltage to earth vary from point to point and the distribution of current and voltage varies with the frequency. The inductance, capacity, and resistance of this portion must therefore be considered as distributed throughout its extent and its effective inductance, capacity, and resistance will depend upon the frequency. On this account the mathematical treatment of the oscillations of an antenna is not

as simple as that which applies to ordinary circuits in which all of the inductances and capacities may be considered as lumped.

The theory of circuits having uniformly distributed electrical characteristics such as cables, telephone lines, and transmission lines has been applied to antennas. The results of this theory do not seem to have been clearly brought out; hazy and sometimes erroneous ideas appear to be current in the literature, text-books, and in the radio world in general so that the methods of antenna measurements are on a dubious footing. It is hoped that this paper may clear up some of these points. No attempt has been made to show how accurately this theory applies to actual antennas.

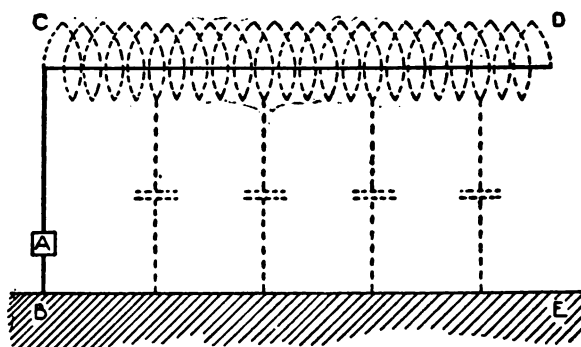


FIG. 1.—Antenna represented as a line with uniform distribution of inductance and capacity

The aerial-ground portion of the antenna, or aerial for short (*CD* in Fig. 1), will be treated as a line with uniformly distributed inductance, capacity, and resistance. As is common in the treatment of radio circuits the resistance will be considered to be so low as not to affect the frequency of the oscillations or the distribution of current and voltage. The lead-in, *BC* in Fig. 1, will be considered to be free from inductance or capacity excepting as inductance coils or condensers are inserted at *A* to modify the oscillations.

An inductance coil, particularly if a long single-layer solenoid, may also be treated from the standpoint of the transmission-line theory. The theoretical results obtained furnish an interesting explanation of certain well-known experimental results.

II. CIRCUIT WITH UNIFORMLY DISTRIBUTED INDUCTANCE AND CAPACITY

The theory, generally applicable to all circuits with uniformly distributed inductance and capacity, will be developed for the case of two parallel wires. The wires (Fig. 2) are of length l and of low resistance. The inductance per unit length L_1 is defined by the flux of magnetic force between the wires per unit of length that there would be if a steady current of 1 ampere were flowing in opposite directions in the two wires. The capacity per unit length C_1 is defined by the charge that there would be on a unit length of one of the wires if a constant emf of 1 volt were impressed between the wires. Further, the quantity $L_0 = l L_1$ would be the total inductance of the circuit if the current flow were the same at all parts. This would be the case if a constant or slowly

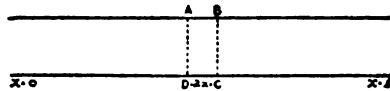


FIG. 2

alternating voltage were applied at $x=0$ and the far end ($x=l$) short-circuited. The quantity $C_0 = l C_1$ would represent the total capacity between the wires if a constant or slowly alternating voltage were applied at $x=0$ and the far end were open.

Let it be assumed, without defining the condition of the circuit at $x=l$, that a sinusoidal emf of periodicity $\omega = 2\pi f$ is impressed at $x=0$ giving rise to a current of instantaneous value i at A and a voltage between A and D equal to v . At B the current will be $i + \frac{\delta i}{\delta x} dx$ and the voltage from B to C will be $v + \frac{\delta v}{\delta x} dx$.

The voltage around the rectangle $ABCD$ will be equal to the rate of decrease of the induction through the rectangle, hence

$$\left(v + \frac{\delta v}{\delta x} dx\right) - v = -\frac{\delta}{\delta t} (L_1 i dx)$$

$$\frac{\delta v}{\delta x} = -L_1 \frac{\delta i}{\delta t} \quad (1)$$

Further, the rate of increase of the charge q on the elementary length of wire AB will be equal to the excess in the current flowing in at A over that flowing out at B .

Hence

$$\begin{aligned}\frac{\delta q}{\delta t} - \frac{d}{dt}(C_1 v dx) &= i - \left(i + \frac{\delta i}{\delta x} dx\right) \\ -\frac{\delta i}{\delta x} &= C_1 \frac{\delta v}{\delta t}\end{aligned}\quad (2)$$

These equations (1) and (2) determine the propagation of the current and voltage waves along the wires. In the case of sinusoidal waves, the expressions

$$v = \cos \omega t (A \cos \omega \sqrt{C_1 L_1} x + B \sin \omega \sqrt{C_1 L_1} x) \quad (3)$$

$$i = \sin \omega t \sqrt{\frac{C_1}{L_1}} (A \sin \omega \sqrt{C_1 L_1} x - B \cos \omega \sqrt{C_1 L_1} x) \quad (4)$$

are solutions of the above equations as may be verified by substitution. The quantities A and B are constants depending upon the terminal conditions. The velocity of propagation of the waves, at high frequencies is $V = \frac{1}{\sqrt{C_1 L_1}}$.

III. THE ANTENNA

1. REACTANCE OF THE AERIAL-GROUND PORTION

Applying equations (3) and (4) to the aerial of an antenna and assuming that $x=0$ is the lead-in end while $x=l$ is the far end which is open, we may introduce the condition that the current is zero for $x=l$. From (4)

$$\frac{A}{B} = \cot \omega \sqrt{C_1 L_1} l \quad (5)$$

Now the reactance of the aerial, which includes all of the antenna but the lead-in, is given by the current and voltage at $x=0$. These are, from (3), (4), and (5),

$$v_0 = A \cos \omega t = B \cot \omega \sqrt{C_1 L_1} l \cos \omega t$$

$$i_0 = -\sqrt{\frac{C_1}{L_1}} B \sin \omega t.$$

The current leads the voltage when the cotangent is positive and lags when the cotangent is negative. The reactance of the aerial, given by the ratio of the maximum values of v_0 to i_0 , is

$$X = -\sqrt{\frac{L_1}{C_1}} \cot \omega \sqrt{C_1 L_1} l$$

or in terms of $C_o = lC_1$ and $L_o = lL_1$

$$X = -\sqrt{\frac{L_o}{C_o}} \cot \omega \sqrt{C_o L_o} \quad (6)$$

or since

$$V = \frac{I}{\sqrt{L_1 C_1}}$$

$$X = -L_1 V \cot \omega \sqrt{C_1 L_1} l \text{ as given by J. S. Stone.}^1$$

At low frequencies the reactance is negative and hence the aerial behaves as a capacity. At the frequency $f = \frac{I}{4\sqrt{C_o L_o}}$

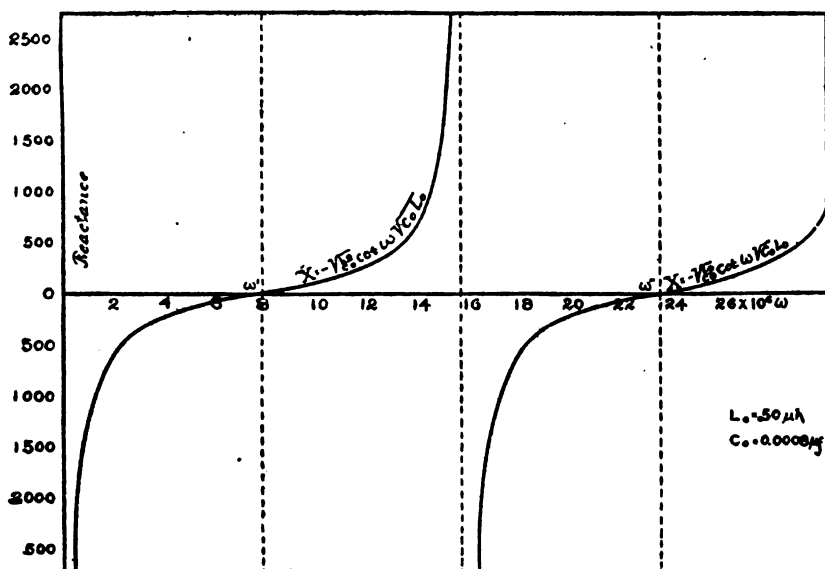


FIG. 3.—Variation of the reactance of the aerial of an antenna with the frequency

the reactance becomes zero and beyond this frequency is positive or inductive up to the frequency $f = \frac{I}{2\sqrt{C_o L_o}}$ at which the reactance becomes infinite. This variation of the aerial reactance with the frequency is shown by the cotangent curves in Fig. 3.

2. NATURAL FREQUENCIES OF OSCILLATION

Those frequencies at which the reactance of the aerial, as given by equation (6), becomes equal to zero are the natural frequencies of oscillation of the antenna (or frequencies of resonance) when the

¹ J. S. Stone, Trans. Int. Congress, St. Louis, 8, p. 555; 1904.

lead-in is of zero reactance. They are given in Fig. 3 by the points of intersection of the cotangent curves with the axis of ordinates and by the equation

$$f = \frac{m}{4\sqrt{C_0 L_0}}; m = 1, 3, 5, \text{ etc.}$$

The corresponding wave lengths are given by

$$\lambda = \frac{V}{f} = \frac{l}{f\sqrt{C_0 L_0}} = \frac{4l}{m}$$

that is, $4/1$, $4/3$, $4/5$, $4/7$, etc., times the length of the aerial. If, however, the lead-in has a reactance X_X , the natural frequencies

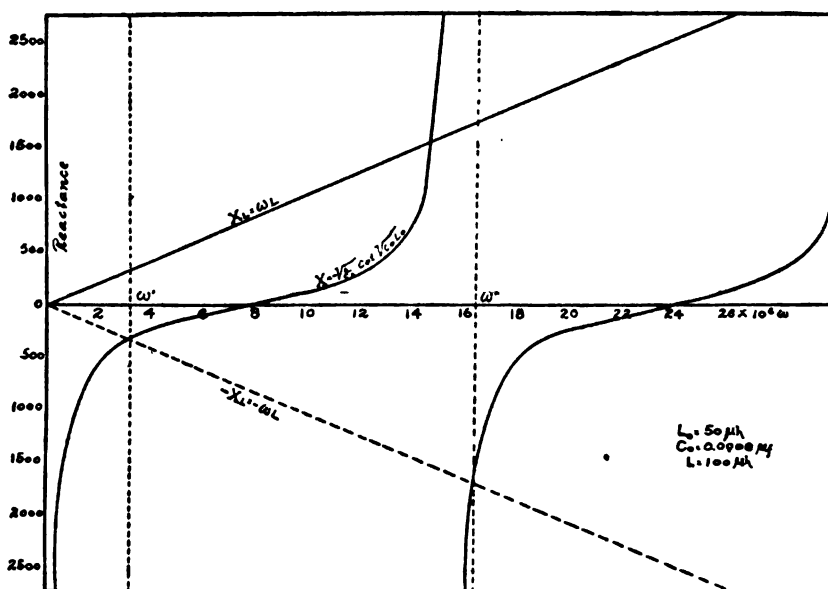


FIG. 4.—Curves of aerial and loading coil reactances

of oscillation are determined by the condition that the total reactance of lead-in plus aerial shall be zero; that is,

$$X_X + X = 0$$

provided that the reactances are in series with the driving emf.

(a) *Loading Coil in Lead-in.*—The most important practical case is that in which an inductance coil is inserted in the lead-in. If the coil has an inductance L , its reactance $X_L = \omega L$. This is a positive reactance increasing linearly with the frequency and represented in Fig. 4 by a solid line. Those frequencies at which the reactance of the coil is equal numerically but opposite in sign

to the reactance of the aerial, are the natural frequencies of oscillation of the loaded antenna since the total reactance $X_L + X = 0$. Graphically, these frequencies are determined by the intersection of the straight line $-X_L = -\omega L$ (shown by a dashline in Fig. 4) with the cotangent curves representing X . It is evident that the frequency is lowered by the insertion of the loading coil and that the higher natural frequencies of oscillation are no longer integral multiples of the lowest frequency.

The condition $X_L + X = 0$, which determines the natural frequencies of oscillation, leads to the equation

$$\omega L - \sqrt{\frac{L_0}{C_0}} \cot \omega \sqrt{C_0 L_0} = 0.$$

TABLE 1.—Data for Loaded Antenna Calculations

$\frac{L}{L_0}$	$\omega \sqrt{C_0 L_0}$	$\frac{1}{\sqrt{\frac{L}{L_0} + \frac{1}{3}}}$	Difference, per cent	$\frac{L}{L_0}$	$\omega \sqrt{C_0 L_0}$	$\frac{1}{\sqrt{\frac{L}{L_0} + \frac{1}{3}}}$	Difference, per cent
0.0	1.571	1.732	10.3	3.2	0.532	0.532	0.1
.1	1.429	1.519	6.3	3.3	.524	.525	.1
.2	1.314	1.369	4.2	3.4	.517	.518	.1
.3	1.220	1.257	3.0	3.5	.510	.511	.1
.4	1.142	1.168	2.3	3.6	.504	.504	.0
.5	1.077	1.095	1.7	3.7	.4977	.4979	.0
.6	1.021	1.035	1.4	3.8	.4916	.4919	.0
.7	.973	.984	1.1	3.9	.4859	.4860	.0
.8	.931	.939	.9	4.0	.4801	.4804	.0
.9	.894	.900	.7	4.5	.4548	.4549	.0
1.0	.860	.866	.7	5.0	.4330	.4330	.0
1.1	.831	.835	.5	5.5	.4141		
1.2	.804	.808	.5	6.0	.3974		
1.3	.779	.782	.4	6.5	.3826		
1.4	.757	.760	.4	7.0	.3693		
1.5	.736	.739	.4	7.5	.3574		
1.6	.717	.719	.3	8.0	.3465		
1.7	.699	.701	.3	8.5	.3366		
1.8	.683	.685	.3	9.0	.3275		
1.9	.668	.669	.3	9.5	.3189		
2.0	.653	.655	.3	10.0	.3111		
2.1	.640	.641	.2	11.0	.2972		
2.2	.627	.628	.2	12.0	.2850		
2.3	.615	.616	.2	13.0	.2741		
2.4	.604	.605	.2	14.0	.2644		
2.5	.593	.594	.2	15.0	.2556		
2.6	.583	.584	.2	16.0	.2476		
2.7	.574	.574	.2	17.0	.2402		
2.8	.564	.565	.1	18.0	.2336		
2.9	.556	.556	.1	19.0	.2277		
3.0	.547	.548	.1	20.0	.2219		
3.1	.539	.540	.1				

or

$$\frac{\cot \omega \sqrt{C_0 L_0}}{\omega \sqrt{C_0 L_0}} = \frac{L}{L_0} \quad (8)$$

This equation has been given by Guyau² and L. Cohen.³ It

² A. Guyau, *Lumiere Electrique*, 18, p. 135, 1911. ³ L. Cohen, *Electrical World*, 65, p. 286, 1915.

determines the periodicity ω and hence the frequency and wave length of the possible natural modes of oscillation when the distributed capacity and inductance of the aerial and the inductance of the loading coil are known. This equation can not, however, be solved directly; it may be solved graphically, as shown in Fig. 4, or a table may be prepared indirectly which gives the values of $\omega\sqrt{C_0L_0}$ for different values of $\frac{L}{L_0}$, from which then ω, f , or λ may be determined. The second column of Table 1 gives these values for the lowest natural frequency of oscillation, which is of the major importance practically.

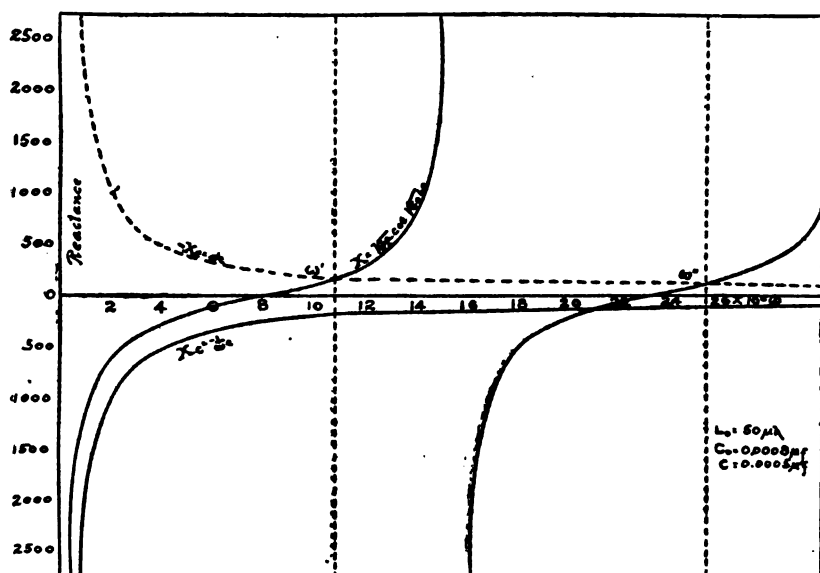


FIG. 5.—Curves of aerial and series condenser reactances

(b) *Condenser in Lead-in.*—At times in practice a condenser is inserted in the lead-in. If the capacity of the condenser is C , its reactance is $X_c = -\frac{1}{\omega C}$. This reactance is shown in Fig. 5 by the hyperbola drawn in solid line. The intersection of the negative of this curve (drawn in dash line) with the cotangent curves representing X gives the frequencies for which $X_c + X = 0$, and hence the natural frequencies of oscillation of the antenna. The frequencies are increased (the wave length decreased) by the insertion of the condenser and the oscillations of higher frequencies are not integral multiples of the lowest.

The condition $X_o + X = 0$ is expressed by the equation

$$-\frac{\tan \omega \sqrt{C_o L_o}}{\omega \sqrt{C_o L_o}} = \frac{C}{C_o} \quad (9)$$

which has also been given by Guyau. Equation (9) may be solved graphically as above or a table similar to Table 1 may be prepared giving $\omega \sqrt{C_o L_o}$ for different values of $\frac{C}{C_o}$. More complicated circuits may be solved in a similar manner.

3. EFFECTIVE RESISTANCE, INDUCTANCE, AND CAPACITY

In the following the most important practical case of a loading coil in the lead-in and the natural oscillation of lowest frequency alone will be considered. The problem is to replace the antenna

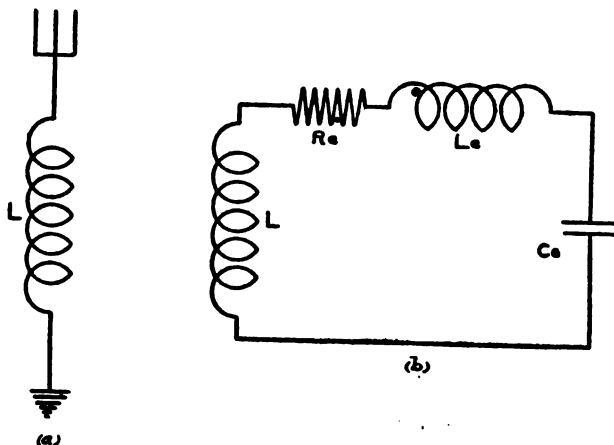


FIG. 6.—(a) Antenna with loading coil; (b) artificial antenna with lumped constants to represent antenna in (a)

of Fig. 6, (a), which has a loading coil L in the lead-in and an aerial with distributed characteristics, by a circuit (Fig. 6, (b)) consisting of the inductance L in series with lumped resistance R_o , inductance L_o , capacity C_o , which are equivalent to the aerial. It is necessary, however, to state how these effective values are to be defined.

In practice the quantities which are of importance in an antenna are the resonant wave length or frequency and the current at the current maximum. The quantities L_o and C_o are therefore defined as those which will give the circuit (b) the same resonant frequency as the antenna in (a). Further the three quantities L_o , C_o , and R_o must be such that the current in (b) will be the same

as the maximum in the antenna for the same applied emf whether undamped or damped with any decrement. These conditions determine L_0 , C_0 , and R_0 uniquely at any given frequency and are the proper values for an artificial antenna which is to represent an actual antenna at a particular frequency. In the two circuits the corresponding maxima of magnetic energies and electrostatic energies and the dissipation of energy will be the same.

Zenneck⁴ has shown how these effective values of inductance capacity and resistance can be computed when the current and voltage distributions are known. Thus, if at any point x on the oscillator the current i and the voltage v are given by

$$i = I f(x); v = V \phi(x)$$

where I is the value of the current at the current loop and V the maximum voltage, then the differential equation of the oscillation is

$$\frac{\partial I}{\partial t} \int R_1 f(x)^2 dx + \frac{\partial^2 I}{\partial t^2} \int L_1 f(x)^2 dx + \frac{I}{\frac{\int C_1 \phi(x) dx}{\int C_1 \phi(x)^2 dx}} = 0$$

where the integrals are taken over the whole oscillator. If we write

$$R_0 = \int R_1 f(x)^2 dx \quad (10)$$

$$L_0 = \int L_1 f(x)^2 dx \quad (11)$$

$$C_0 = \frac{\{\int C_1 \phi(x) dx\}^2}{\int C_1 \phi(x)^2 dx} \quad (12)$$

the equation becomes

$$R_0 \frac{\partial I}{\partial t} + L_0 \frac{\partial^2 I}{\partial t^2} + \frac{I}{C_0} = 0$$

which is the differential equation of oscillation of a simple circuit with lumped resistance, inductance, and capacity of values R_0 , L_0 , and C_0 and in which the current is the same as the maximum in the distributed case. In order to evaluate these quantities, it is necessary only to determine $f(x)$ and $\phi(x)$; that is, the functions which specify the distribution of current and voltage on the oscillator. In this connection it will be assumed that the resistance is not of importance in determining these distributions.

⁴ Zenneck, *Wireless Telegraphy* (translated by A. E. Selig), Note 40, p. 410.

At the far end of the aerial the current is zero; that is, for $x=l$; $i_l=0$. From equations (3) and (4) for $x=l$

$$v_l = \cos \omega t (A \cos \omega \sqrt{C_1 L_1} l + B \sin \omega \sqrt{C_1 L_1} l),$$

$$i_l = \sin \omega t \sqrt{\frac{C_1}{L_1}} (A \sin \omega \sqrt{C_1 L_1} l - B \cos \omega \sqrt{C_1 L_1} l);$$

and since $i_l=0$,

$$A \sin \omega \sqrt{C_1 L_1} l = B \cos \omega \sqrt{C_1 L_1} l.$$

From (3), then, we obtain

$$v = v_l \cos (\omega \sqrt{C_1 L_1} l - \omega \sqrt{C_1 L_1} x).$$

Hence

$$\phi(x) = \cos (\omega \sqrt{C_1 L_1} l - \omega \sqrt{C_1 L_1} x).$$

Now for $x=0$ from (4) we obtain

$$i_0 = -B \sqrt{\frac{C_1}{L_1}} \sin \omega t = -A \sqrt{\frac{C_1}{L_1}} \tan \omega \sqrt{C_1 L_1} l \sin \omega t,$$

whence

$$i = i_0 \frac{\sin (\omega \sqrt{C_1 L_1} l - \omega \sqrt{C_1 L_1} x)}{\sin \omega \sqrt{C_1 L_1} l}$$

and

$$f(x) = \frac{\sin (\omega \sqrt{C_1 L_1} l - \omega \sqrt{C_1 L_1} x)}{\sin \omega \sqrt{C_1 L_1} l}.$$

We can now evaluate the expressions (10), (11), and (12).

From (10)

$$\begin{aligned} R_0 &= \int_0^l R_1 \frac{\sin^2 (\omega \sqrt{C_1 L_1} l - \omega \sqrt{C_1 L_1} x) dx}{\sin^2 \omega \sqrt{C_1 L_1} l} \\ &= \frac{R_1}{\sin^2 \omega \sqrt{C_1 L_1} l} \left[\frac{l}{2} - \frac{\sin 2 \omega \sqrt{C_1 L_1} l}{4 \omega \sqrt{C_1 L_1}} \right] \\ &= \frac{R_0}{2} \left[\frac{1}{\sin^2 \omega \sqrt{C_0 L_0}} - \frac{\cot \omega \sqrt{C_0 L_0}}{\omega \sqrt{C_0 L_0}} \right] \end{aligned} \quad (13)$$

and from (11) which contains the same form of integral

$$L_0 = \frac{L_0}{2} \left[\frac{1}{\sin^2 \omega \sqrt{C_0 L_0}} - \frac{\cot \omega \sqrt{C_0 L_0}}{\omega \sqrt{C_0 L_0}} \right] \quad (14)$$

and from (12)

$$\begin{aligned}
 C_o &= \frac{\{\int_0^l C_1 \cos(\omega\sqrt{C_1 L_1} l - \omega\sqrt{C_1 L_1} x) dx\}^2}{\int_0^l C_1 \cos^2(\omega\sqrt{C_1 L_1} l - \omega\sqrt{C_1 L_1} x) dx} \\
 &= \frac{C_1 \frac{\sin^2 \omega\sqrt{C_1 L_1} l}{(\omega\sqrt{C_1 L_1})^2}}{C_1 \left(\frac{l}{2} + \frac{\sin 2 \omega\sqrt{C_1 L_1} l}{4 \omega\sqrt{C_1 L_1}} \right)} \\
 &= C_o \frac{1}{\left[\frac{\omega\sqrt{C_o L_o} \cot \omega\sqrt{C_o L_o}}{2} + \frac{\omega^2 C_o L_o}{2 \sin^2 \omega\sqrt{C_o L_o}} \right]} \quad (15)
 \end{aligned}$$

The expressions (14) and (15) should lead to the same value for the reactance X of the aerial as obtained before. It is readily shown that

$$X = \omega L_o - \frac{1}{\omega C_o} = -\sqrt{\frac{L_o}{C_o}} \cot \omega\sqrt{C_o L_o}$$

agreeing with equation (6).

It is of interest to investigate the values of these quantities at very low frequencies ($\omega=0$), frequently called the static values, and those corresponding to the natural frequency of the unloaded antenna or the so-called fundamental of the antenna. Substituting $\omega=0$ in (13), (14), and (15), and evaluating the indeterminate which enters in the first two cases, we obtain for the low-frequency values

$$\begin{aligned}
 R_o &= \frac{R_o}{3} \\
 L_o &= \frac{L_o}{3} \\
 C_o &= C_o
 \end{aligned} \quad (16)$$

At low frequencies the current is a maximum at the lead-in end of the aerial and falls off linearly to zero at the far end. The effective resistance and inductance are one-third of the values which would obtain if the current were the same throughout. The voltage is, however, the same at all points and hence the effective capacity is the capacity per unit length times the length or C_o .

At the fundamental of the antenna, the reactance X of equation (6) becomes equal to zero and hence $\omega\sqrt{C_o L_o} = \frac{\pi}{2}$. Substituting this value in (13), (14), and (15),

$$\begin{aligned}
 R_o &= \frac{R_o}{2} \\
 L_o &= \frac{L_o}{2} \\
 C_o &= \frac{8}{\pi^2} C_o.
 \end{aligned}
 \tag{17}$$

Hence, in going from low frequencies up to that of the fundamental of the antenna, the resistance (neglecting radiation and skin effect) and the inductance (neglecting skin effect) increase by 50 per cent, the capacity, however, decreases by about 20 per cent. The incorrect values $\frac{2}{\pi} L_o$ and $\frac{2}{\pi} C_o$ have been frequently given and commonly used as the values of the effective inductance and capacity of the antenna at its fundamental. These lead also to the incorrect value $L_o = \frac{L_o}{2}$ for the low-frequency inductance.⁵

The values for other frequencies may be obtained by substitution in (13), (14), (15). If the value L of the loading coil in the lead-in is given, the quantity $\omega\sqrt{C_o L_o}$ is directly obtained from Table I.

4. EQUIVALENT CIRCUIT WITH LUMPED CONSTANTS

In so far as the frequency or wave length is concerned, the aerial of the antenna may be considered to have constant values of inductance and capacity and the values of frequency or wave length for different loading coils can be computed with slight error using the simple formula applicable to circuits with lumped inductance and capacity. The values of inductance and capacity ascribed to the aerial are the static or low frequency; that is, $\frac{L_o}{3}$ for the inductance and C_o for the capacity. The total inductance in case the loading coil has a value L will be $L + \frac{L_o}{3}$ and the frequency is given by

$$f = \frac{1}{2\pi\sqrt{\left(L + \frac{L_o}{3}\right)C_o}}
 \tag{18}$$

⁵ These values are given by J. H. Morecroft in Proc. I. R. E., 5, p. 389; 1917. It may be shown that they lead to correct values for the reactance of the aerial and hence to correct values of frequency, as was verified by the experiments. They are not, however, the values which would be correct for an artificial antenna in which the current must equal the maximum in the actual antenna and in which the energies must also be equal to those in the antenna. The resistance values given by Prof. Morecroft agree with these requirements and with the values obtained here.

or the wave length in meters by

$$\lambda = 1884 \sqrt{\left(L + \frac{L_0}{3}\right) C_0} \quad (19)$$

where the inductance is expressed in microhenrys and the capacity in microfarads. The accuracy with which this formula gives the wave length can be determined by comparison with the exact formula (8). In the second column of Table I are given the values of $\omega\sqrt{C_0 L_0}$ for different values of $\frac{L}{L_0}$ as computed by formula (8).

Formula (18) may be written in the form $\omega\sqrt{C_0 L_0} = \frac{1}{\sqrt{\frac{L}{L_0} + \frac{1}{3}}}$ so

that the values of $\omega\sqrt{C_0 L_0}$, which are proportional to the frequency, may readily be computed from this formula also. These values are given in the third column and the per cent differences in the fourth column of Table I. It is seen that formula (18) gives values for the frequency which are correct to less than 1 per cent excepting when very close to the fundamental of the antenna; that is, for very small values of L . Under these conditions the simple formula leads to values of the frequency which are too high. Hence to the degree of accuracy shown, which is amply sufficient in most practical cases, the aerial can be represented by its static inductance $\frac{L_0}{3}$ with its static capacity C_0 in series, and the frequency of oscillation with a loading coil L in the lead-in can be computed by the ordinary formula applicable to circuits with lumped constants.

In an article by L. Cohen,* which has been copied in several other publications, it was stated that the use of the simple wave-length formula would lead to very large errors when applied to the antenna with distributed constants. The large errors found by Cohen are due to his having used the value L_0 for the inductance of the aerial, instead of $\frac{L_0}{3}$, in applying the simple formula.

5. DETERMINATION OF STATIC CAPACITY AND INDUCTANCE

In applying formula (8) to calculate the frequency of a loaded antenna, a knowledge of the quantities of L_0 and C_0 is required. In applying formula (18), $\frac{L_0}{3}$ and C_0 are required. Hence either

* L. Cohen, *Electrical World*, 65, p. 286; 1915.

formula may be used if the static capacity and inductance values are known. We will call these values simply the capacity C_a and inductance L_a of the antenna. Hence $C_a = C_0$, $L_a = \frac{L_0}{3}$ and the wave length from (19) is given by

$$\lambda = 1884 \sqrt{(L + L_a) C_a} \quad (20)$$

where inductance is expressed in microhenrys and capacity in microfarads, as before.

The capacity and inductance of the antenna are then readily determined experimentally by the familiar method of inserting, one after the other, two loading coils of known values L_1 and L_2 in the lead-in and determining the frequency of oscillation or wave length for each. From the observed wave lengths λ_1 and λ_2 and known values of the inserted inductances, the inductance of the antenna is given by

$$L_a = \frac{L_1 \lambda_2^2 - L_2 \lambda_1^2}{\lambda_1^2 - \lambda_2^2} \quad (21)$$

and the capacity of the antenna from either

$$\begin{aligned} \lambda_1 &= 1884 \sqrt{(L_1 + L_a) C_a} \\ \lambda_2 &= 1884 \sqrt{(L_2 + L_a) C_a} \end{aligned} \quad (22)$$

using, preferably, the equation corresponding to the larger valued coil. This assumes that formula (20) holds exactly.

As an example, let us assume that the antenna has $L_0 = 50$ microhenrys and $C_0 = 0.001$ microfarad and that we insert two coils of 50 and 150 microhenrys and determine the wave lengths, experimentally. We know from formula (8) and Table I that the wave lengths would be found to be 491 and 771 m. From the observed wave lengths and known inductances, the value of L_a would be found by (21) to be

$$L_a = 17.8 \text{ microhenrys}$$

and from (22)

$$C_a = 0.00099, \text{ microfarad.}$$

C_a is very close to the assumed value C_0 but L_a differs by 7 per cent from $\frac{L_0}{3}$. This accuracy would ordinarily be sufficient. We can, however, by a second approximation, derive from the experimental data a more accurate value of L_a . For, the observed value

of L_a furnishes rough values of $\frac{L_1}{L_o}$ and $\frac{L_2}{L_o}$, which in this example come out 0.96 and 2.88, respectively. But Table 1 gives the per cent error of formula (20) for different values of $\frac{L}{L_o}$ and shows that this formula gives a 0.7 per cent shorter wave length than 491 m (or 488 m) for $\frac{L}{L_o} = 0.96$ but no appreciable difference for $\frac{L}{L_o} = 2.88$. Recomputing L_a , using 488 and 771 m, gives

$$L_a = 0.0168$$

which is practically identical with the assumed $\frac{L_o}{3}$.

6. DETERMINATION OF EFFECTIVE RESISTANCE, INDUCTANCE, AND CAPACITY

When a source of undamped oscillations in a primary circuit induces current in a secondary tuned circuit, the current in the secondary, for a given emf, depends only upon the resistance of the secondary circuit. When damped oscillations are supplied by the source in the primary, the current in the secondary, for a given emf and primary decrement, depends upon the decrement of the secondary; that is, upon the resistance and ratio of capacity to inductance. The higher the decrement of the primary circuit relative to the decrement of the secondary the more strongly does the current in the secondary depend upon its own decrement. This is evident from the expression for the current I in the secondary circuit.

$$I^2 = \frac{N E_o^2}{4 f R^2 \delta' \left(1 + \frac{\delta'}{\delta}\right)}$$

where δ' is the decrement of the primary, δ that of the secondary, R the resistance of the secondary, f the frequency, E_o the maximum value of the emf impressed on the primary, and N the wave-train frequency.

These facts suggest a method of determining the effective resistance, inductance, and capacity of an antenna at a given frequency in which all of the measurements are made at one frequency and which does not require any alteration of the antenna circuit whatsoever. The experimental circuits are arranged as shown in Fig. 7, where S represents a coil in the primary circuit which may be thrown either into the circuit of a source of undamped

or of damped oscillations. The coil L is the loading coil of the antenna, which may be thrown over to the measuring circuit containing a variable inductance L' , a variable condenser C' , and variable resistance R' . The condenser C' should be resistance free and shielded, the shielded terminal being connected to the ground side. First, the undamped source is tuned to the antenna, and then the $L'C'$ circuit tuned to the source. The resistance R' is then varied until the current is the same in the two positions. The resistance of the $L'C'$ circuit is then equal to R_0 , the effective resistance of the aerial-ground portion of the antenna and $L'C' = L_0C_0$. Next, the damped source is tuned to the antenna and the change in current noted when the connection is thrown over to the $L'C'$ circuit. If the current increases, the value of C' is greater than C_0 , and vice versa. By varying both L' and C' ,

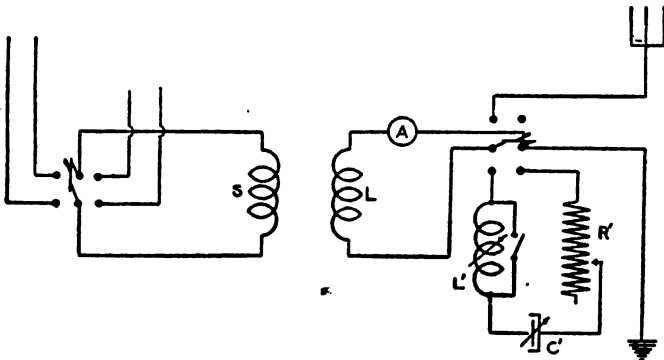


FIG. 7.—Circuits for determining the effective resistance, inductance, and capacity of an antenna

keeping the tuning and R' unchanged, the current can be adjusted to the same value in both positions. Then, since $L'C' = L_0C_0$ and $\frac{C'}{L'} = \frac{C_0}{L_0}$, the value of C' gives C_0 and that of L' gives L_0 . Large changes in the variometer setting may result in appreciable changes in its resistance so that the measurement should be repeated after the approximate values have been found. To eliminate the resistance of the variometer in determining R_0 , the variometer is shorted and, using undamped oscillations, the resonance current is adjusted to equality in the two positions by varying R' . Then $R' = R_0$. The measurement requires steady sources of feebly damped and strongly damped current. The former is readily obtained by using a vacuum-tube generator. A resonance transformer and magnesium spark gap operating at a low-spark fre-

quency serve very satisfactorily for the latter source, or a single source of which the damping can be varied will suffice. An accuracy of 1 per cent is not difficult to obtain.

IV. THE INDUCTANCE COIL

The transmission-line theory can also be applied to the treatment of the effects of distributed capacity in inductance coils. In Fig. 8, (a), is represented a single-layer solenoid connected to a variable condenser C . A and B are the terminals of the coil, D the middle, and the condensers drawn in dotted lines are supposed to represent the capacities between the different parts of the coil. In Fig. 8, (b), the same coil is represented as a line with uniformly distributed inductance and capacity. These

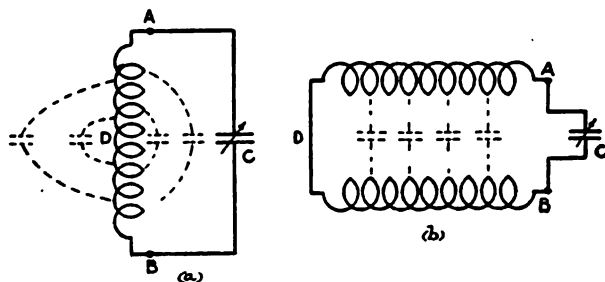


FIG. 8.—Inductance coil represented as a line with uniform distribution of inductance and capacity

assumptions are admittedly rough but are somewhat justified by the known similarity of the oscillations in long solenoids to those in a simple antenna.

1. REACTANCE OF THE COIL

Using the same notation as before, an expression for the reactance of the coil, regarded from the terminals AB ($x=0$) will be determined considering the line as closed at the far end D ($x=l$). Equations (3) and (4) will again be applied, taking account of the new terminal condition: that is, for $x=l$; $v=0$. Hence

$$A \cos \omega \sqrt{C_1 L_1} l = -B \sin \omega \sqrt{C_1 L_1} l$$

and for $x=0$

$$v_0 = A \cos \omega t = -B \tan \omega \sqrt{C_1 L_1} l \cos \omega t$$

$$i_0 = -\sqrt{\frac{C_1}{L_1}} B \sin \omega t$$

which gives for the reactance of the coil regarded from, the terminals A B,

$$X' = \sqrt{\frac{L_1}{C_1}} \tan \omega \sqrt{C_1 L_1} l$$

or

$$X' = \sqrt{\frac{L_o}{C_o}} \tan \omega \sqrt{C_o L_o} \quad (23)$$

2. NATURAL FREQUENCIES OF OSCILLATION

At low frequencies the reactance of the coil is very small and positive but increases with increasing frequency and becomes infinite when $\omega \sqrt{C_o L_o} = \frac{\pi}{2}$. This represents the lowest frequency of natural oscillation of the coil when the terminals are open. Above this frequency the reactance is highly negative, approaching zero at the frequency $\omega \sqrt{C_o L_o} = \pi$. In this range of frequencies the coil behaves as a condenser and would require an inductance across the terminals to form a resonant circuit. At the frequency $\omega \sqrt{C_o L_o} = \pi$ the coil will oscillate with its terminals short-circuited. As the frequency is still further increased the reactance again becomes increasingly positive.

Condenser Across the Terminals.—The natural frequencies of oscillation of the coil when connected to a condenser C are given by the condition that the total reactance of the circuit shall be zero.

$$X' + X_o = 0$$

From this we have

$$\sqrt{\frac{L_o}{C_o}} \tan \omega \sqrt{C_o L_o} = \frac{1}{\omega C}$$

or

$$\frac{\cot \omega \sqrt{C_o L_o}}{\omega \sqrt{C_o L_o}} = \frac{C}{C_o} \quad (24)$$

This expression is the same as (8) obtained in the case of the loaded antenna, excepting that $\frac{C}{C_o}$ occurs on the right-hand side instead of $\frac{L}{L_o}$, and shows that the frequency is decreased and wave length increased by increasing the capacity across the coil in a manner entirely similar to the decrease in frequency produced by inserting loading coils in the antenna lead-in.

3. EQUIVALENT CIRCUIT WITH LUMPED CONSTANTS

It is of interest to investigate the effective values of inductance and capacity of the coil at very low frequencies.

Expanding the tangent in equation (23) into a series we find

$$X' = \omega L_o \left(1 + \frac{\omega^2 C_o L_o}{3} + \dots \right)$$

and neglecting higher-power terms this may be written

$$X' = \frac{(\omega L_o) \left(-\frac{3}{\omega C_o} \right)}{\omega L_o - \frac{3}{\omega C_o}}$$

This is the reactance of an inductance L_o in parallel with a capacity $\frac{C_o}{3}$, which shows that at low frequencies the coil may be regarded as an inductance L_o with a capacity $\frac{C_o}{3}$ across the terminals and therefore in parallel with the external condenser C . Since at low frequencies the current is uniform throughout the whole coil, it is self-evident that its inductance should be L_o .

Now, the similarity between equations (24) and (8) shows that, just as accurately as in the similar case of the loaded antenna, the frequency of oscillation of a coil with *any capacity* C across the terminals is given by the formula

$$f = \frac{1}{2\pi \sqrt{L_o \left(C + \frac{C_o}{3} \right)}} \quad (25)$$

This, however, is *also* the expression for the frequency of a coil of pure inductance L_o with a capacity $\frac{C_o}{3}$ across its terminals and which is in parallel with an external capacity C . Therefore, in so far as frequency relations are concerned, an inductance coil with distributed capacity is closely equivalent at any frequency to a pure inductance, equal to the low-frequency inductance (neglecting skin effect), with a constant capacity across its terminals. This is a well-known result of experiment,⁷ at least in the case of single-layer solenoids which, considering the changes in current and voltage distribution in the coil with changing frequency, is not otherwise self-evident.

WASHINGTON, March 9, 1918.

⁷ G. W. O. Howe, Proc. Phys. Soc. London, 24, p. 251, 1912; F. A. Kolster, Proc. Inst. Radio Eng., 1, p. 19, 1913; J. C. Hubbard, Phys. Rev., 9, p. 529, 1917.

MEASUREMENTS ON THE INDEX OF REFRACTION OF AIR FOR WAVE LENGTHS FROM 2218 Å TO 9000 Å

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I. INTRODUCTION

The optical property of air, by virtue of which the velocity, wave length, and, in general, the direction of light is changed when passing from other media into air, has been of practical interest to astronomers and physicists for many centuries. This interest has resulted in a large number of experimental and theoretical researches which have reacted to create a still greater interest in the subject. The majority of researches on the refractivity of air received their inspiration from the desire to correct astronomical observations for the refraction of the earth's atmosphere, and in this connection much work was done to establish the dependence of such corrections on the state of the air as determined by its temperature, pressure, and constitution. Other investigations were later undertaken more especially to test certain theories and laws proposed to represent the relation of refractivity to density and the connection between refractivity and wave length. These are of primary importance to spectroscopists and other physicists. A brief review of the objects, experimental methods, and results of these earlier investigations will assist in explaining the purpose of this additional investigation on the index of refraction and dispersion of air described in detail in this paper.

1. ASTRONOMICAL OBSERVATIONS

The refractive power of air was first recognized 1900 years ago by Cleomedes, who suspected that it explained the phenomenon of a lunar eclipse when both sun and moon were visible above the earth's horizon. Ptolemy, in the second century, expressed correct ideas of astronomical refraction in his *Optics*. His attempts to measure this refraction and to find the law connecting incidence and refraction angles were unsuccessful. Astronomical refraction was of no practical importance until the observational instruments and methods were sufficiently refined to measure minutes of arc. In fact, Tycho Brahe,¹ in the sixteenth century, was the first to measure this refraction successfully and he constructed the first tables of corrections to astronomical observations. After the discovery in the seventeenth century of Snell's law that the index of refraction is a constant ratio of the sines of the incidence and refraction angles, theoretical researches attempted to find the differential equation of the path of light in the earth's atmosphere.

¹ Tycho Brahe, *Astronomie Instauratae Progymnasmatia*, Francofurt, 1648.

Such an equation appears to have been first derived by Brook Taylor² in 1715. Then followed the work of Cassini, Bouguer, Bernoulli, Simpson, Bradley, Euler, Lambert, Lagrange, Kramp, and others, climaxed by that of Laplace in 1805. Tables constructed on Laplace's formulae were used in astronomy and geodesy for many years. A complete review of these labors of astronomers and physicists upon this complex but interesting subject of astronomical refraction is given in a prize essay by C. Bruhns.³

The earliest absolute value of the refractivity of air was determined by Delambre,⁴ whose result was obtained in 1805 from more than 500 observations of the altitudes of circum polar stars. Later values have been deduced from astronomical observations by Bessel,⁵ Gylden,⁶ Fuss,⁷ and others.

2. PRISM METHOD

The prism method for measuring the refractive power of gases was first used by Biot and Arago⁸ (1806), who observed with a telescope the apparent change of position of an object seen through a hollow prism filled with air when the density of the air was changed by varying the temperature from -1.5 to 25° C and the pressure from 0.005 to 0.8 m of mercury. From these experiments the absolute index

$$n = \frac{\sin i}{\sin r}$$

was obtained for white light and it was concluded that the refractivity of air was proportional to the density. Dulong⁹ (1826) also measured the index of refraction of air for white light by the prism method. Later (1893) Kayser and Runge¹⁰ used a copper prism with quartz windows and placed the prism so as to cover part of a concave grating. Lines in the iron spectrum and in the zinc spectrum were photographed, some of the light passing by the prism and the rest passing through the prism containing air under pressures from 1 to 10 atmospheres. The refractive

² Brook Taylor, *Methodus incrementum*, London, 1715.

³ C. Bruhns, *Die Astronomische Strahlenbrechung*, Leipzig; 1861.

⁴ Delambre, s. Laplace *Mecanique celeste*, 4; 1805.

⁵ Bessel, s. Biot *Compt. rend.*, 40.

⁶ Gylden, *Mem. de l'acad. de St. Petersburg*, 10, No. 1, 1866, and 12, No. 4, 1868.

⁷ Fuss, *ibid.*, 18, No. 4; 1868.

⁸ Biot and Arago, *Memoires de l'Institute*, 7, p. 301; 1806.

⁹ Dulong, *Ann. de Chim. et de Phys.* (3), 31, p. 154; 1826.

¹⁰ Kayser and Runge, *Ann. d. Phys.* 50, p. 293; 1893.

indices for different wave lengths were then determined from the measured shifts of the spectrum lines due to light passing through the prism. Wet air at room temperatures from 11 to 19° C was used for these experiments and the results reduced to represent air at 0° C and 760 mm.

A hollow glass prism with air under pressures from 1 to 5 atmospheres was used by Carnazzi.¹¹ Indices of refraction were determined from measurements of the prism angle and the incidence and emergence angles. These results indicated an increase of about 15 per cent in $\frac{n-1}{d}$ for this pressure range.

Lang¹² investigated the dependence of refractive index of air on temperature by observing beams of light reflected from prism faces through windows of a heating apparatus surrounding the prism and noting the deviations for various temperature differences (2.4 to 95.0° C) between the air inside and outside the apparatus. The observations were represented by the equation $n_t = n_0 - 0.000000905t + 0.0000000235t^2$.

3. INTERFEROMETER METHODS

The interferometer method for the measurement of refractive indices of gases was first applied by Jamin¹³ (1857). Two Fresnel mirrors were used to make two beams of light, traversing tubes 1 m long, interfere and the difference in optical path produced by changing the pressure in one of the tubes was determined with a compensator. Modifications of Jamin's interferometer served later for the work of Ketteler,¹⁴ Lorenz,¹⁵ Mascart,¹⁶ Chappuis and Riviere,¹⁷ Perreau,¹⁸ Walker,¹⁹ Gale,²⁰ Magri,²¹ C. and M. Cuthbertson,²² Kaiser,²³ Gruschke,²⁴ Ahrberg,²⁵ and Koch.²⁶ Ketteler measured the absolute index of ordinary air at 0° C and 760 mm for sodium light and indices for lithium and thallium light

¹¹ Carnazzi, *Il Nuovo Cimento* (6), p. 385; 1897.

¹² Von Lang, *Pogg. Ann.*, 168, p. 448; 1874.

¹³ Jamin, *Ann. de Chim. et de Phys.*, 49, p. 282; 1857.

¹⁴ Ketteler, *Pogg. Ann.*, 124, p. 390; 1865.

¹⁵ Lorenz, *Wied. Ann.*, 11, p. 70; 1880.

¹⁶ Mascart, *Compt. rend.*, 78, p. 617; 1874.

¹⁷ Chappuis et Riviere, *Ann. de Chim. et de Phys.* (6), 14, p. 5; 1888.

¹⁸ Perreau, *Ann. de Chim. et de Phys.*, 7, p. 289; 1896.

¹⁹ Walker, *Phil. Trans.*, 201, p. 435; 1903.

²⁰ Gale, *Phys. Rev.*, 14, p. 1; 1902.

²¹ Magri, *Phys. Zeitschr.*, 6, p. 629; 1905.

²² C. and M. Cuthbertson, *Proc. Roy. Soc.*, 88, p. 151; 1920.

²³ Kaiser, *Ann. d. Phys.* (4), 18, p. 210; 1904.

²⁴ Gruschke, *S. A. Jahresber. Schles. Ges. f. Vaterl. Kultur, Natur.*, Sept., 1909.

²⁵ Ahrberg, *Diss. Halle*; 1909.

²⁶ Koch, *Nova Acta Soc. Upsal* (4), 2; 1909.

relative to that of sodium by observing the differences in pressure required to produce successive coincidences among the fringes due to these different colors. These are the earliest measurements of the dispersive power of air; that is, of refractive indices of air for different wave lengths. Lorenz repeated Ketteler's observations on sodium and lithium light. Mascart divided a beam of white light into two parts, which were sent through two parallel tubes filled with gas and then recombined so as to pass through a spectroscope. When the relative pressures in the two tubes were changed, Talbot bands appeared in the spectrum because of the change in optical path. The index of refraction was determined from a count of the number of bands which passed with a definite change in pressure at points in the spectrum corresponding to four lines in the cadmium arc. Koch used an interference refractometer to measure the indices of air for the residual rays from calcite and gypsum. These waves are longer than those corresponding to an infra-red absorption band due to atmospheric carbon dioxide.

The principle of the Jamin refractometer was used by a number of investigators to find the relation of refractive index to density as affected by pressure or temperature variations.

The experiments of Chappuis and Riviere on air at 21° C indicated a proportionality of refractivity to density over a pressure range from 0 to 20 atmospheres. The same conclusion was arrived at by Perreau who worked with air at 16° C up to about 5 atmospheres pressure. Walker measured the refractivity of air under atmospheric pressure at temperatures of 10 to 100° C and found the coefficient of index variation with temperature to be about 2 per cent less than the expansion coefficient. Gale's experiments showed that the proportionality of refractivity and density held to 0.1 per cent for air under pressures from 1 to 20 atmospheres. Magri investigated the refractivity of air under pressures up to

200 atmospheres and preferred the relation $\frac{n^2 - 1}{(n^2 + 1)d} = \text{Const.}$ The

more simple relation $\frac{n - 1}{d} = \text{Const.}$ was again found by Kaiser for air pressures from 0 to 1 atmosphere.

The Fizeau dilatometer is another type of interferometer which has been used for measurements of the index of refraction of air by Benoit²⁷ and by Scheel.²⁸ In Benoit's experiments a tripod

²⁷ Benoit, *Jour. de Phys.* (2), 8, p. 451; 1889.

²⁸ Scheel, *Verh. d. Deutsch. Phys. Ges.*, 9, p. 24; 1907.

separated two interferometer plates which were illuminated with sodium light to show fringes. The displacement of these fringes when the temperature of the air between the interferometer plates was changed through the range 0 to 80° C showed the coefficient of index variation with temperature to be identical with the coefficient of expansion of the gas, that is, 0.00367. Scheel measured refractive indices for various wave lengths when the air was at temperatures of 16 and -192° C.

Rentschler²⁹ was the first to make use of the Fabry and Perot type of interferometer for the measurement of refractive indices of a gas for different light waves. Silvered plates separated 9.005 mm were used with a concave grating. The interferometer was illuminated with a quartz mercury lamp; the systems of circular interference fringes produced by different radiations were separated by the diffraction grating and photographed when a vacuum existed between the interferometer plates and also when various pressures of air were present. The air had a temperature of 21.1° C, but the fringe shifts from which the indices of refraction were computed were corrected for air at 0° C and 760 mm.

The Fabry and Perot interferometer was also used by Miss Matthews³⁰ and by Miss Howell.³¹ Miss Matthews undertook to test the relation of refractive index to density of air but her results are too meager to warrant any conclusion. Miss Howell used a quartz spectrograph with quartz interferometer plates covered with thin nickel films and measured the index of air for 11 points between 2652 Å and 6678 Å.

4. DIFFRACTION GRATING METHOD

A diffraction grating in a chamber which could be evacuated was used by Runge³² for dispersion measurements in the extreme ultra-violet and very recently by Dickey³³ for refractive index measurements through a range of spectrum from 2493 Å to 6411 Å. Dickey placed a bell jar with a quartz window over the grating and determined the ratios of iron arc wave lengths in air and vacuum by measuring the change in their diffraction angles when the air was removed from the jar. Values of the index of refraction were deduced for 28 points in the above-mentioned spectrum range, the value for each point being determined from the measured shifts of an average of 8 lines in the

²⁹ Rentschler, *Astroph. J.*, 28, p. 345; 1908.

³⁰ Matthews, *Jour. Frank. Inst.*, 177, p. 673; 1914.

³¹ Howell, *Phys. Rev.*, 6, p. 81; 1915.

³² Runge, *Wied Ann.*, 56, p. 44; 1895.

³³ Dickey, *Astroph. J.*, 46, p. 188; 1917.

5. SUMMARY OF PREVIOUS WORK.

TABLE 1.—(n-1)10⁷

[illegible]

TABLE 1.— $(n-1)10^7$ —Continued

	Rentischles, 1906	Ahrberg, 1909	Koch, 1909	Graschke, 1910	Cuthbert- son, 1910	Howell, 1915	Dickey, 1917	Bureau Standards, 1918
λ								
2493							3222.0	3189.3
2567							3182.0	3166.3
2614							3159.0	3152.9
2652						3140.0		3142.8
2665							3138.0	3139.4
2728							3119.0	3124.0
2894						3090.2		3089.7
2967						3073.7		3076.9
3022						3066.2		3067.9
3038							3061.0	3065.3
3188						3043.0		3043.2
3293							3030.0	3031.4
3341	3036.0							3026.0
3450							3012.0	3015.0
3576							3003.0	3003.5
3650	2997.0						2997.0	2997.7
3867							2981.0	2982.3
3941							2977.0	2977.8
3965						2980.0		2975.5
4046	2972.0							2971.9
4119							2967.0	2967.9
4220							2965.0	2963.1
4315							2959.0	2958.7
4358	2956.0	2957.0	2970.9					2956.9
4409							2955.0	2954.9
4471				2978.0		2955.5		2952.3
4518							2951.0	2950.9
4619							2945.0	2947.0
4712				2969.0		2944.0	2942.0	2943.9
4861					2951.1			2939.3
4979							2940.0	2938.7
4922				2962.0				2937.6
5016				2960.0		2935.0		2935.0
5140							2929.0	2932.1
5202							2926.0	2930.5
5270							2925.0	2928.9
5328							2925.0	2927.7
5406							2922.0	2926.1
5460	2930.0	2925.0	2937.0		2936.0			2925.0
5616							2919.0	2922.1
5769	2925.0							2919.4
5780		2919.						2919.3
5790					2929.8			2919.2
5876						2919.6		2917.8
5993			2929.8	2939.0				2917.6
6231							2914.0	2912.8
6412							2910.0	2910.6
6438		2911.0						2910.4
6563					2919.2			2908.9
6678				2930.0		2912.0		2907.7
6706			2918.5					2907.5
67094			2880.6					
86784			2887.5					

With a few exceptions the results in Table 1 are given by the various observers to represent indices of refraction of air at 0° C and 760 mm. In most cases, however, the measurements were not made with air actually at 0° C and 760 mm, but at various temperatures and pressures and then corrected for the density of the air. In some cases water vapor was present in the air, and a few observers have neglected to state the temperature,

pressure, or constitution of the air with which they worked. Although these results are for these reasons not strictly comparable, their disagreement on the whole does not appear to be very large, considering the fact that they represent such a large range of experimental methods, apparatus, and observers. Most of the dispersion measurements, however, have been confined to a comparatively small spectral region, and dispersion formulas given to represent these measurements disagree by more than 10 per cent of the refractivity $(n-1)$ of air for ultra-violet light.

The Cauchy dispersion formulas given by various observers to represent the refractivity of air (0° C and 760 mm) as a function of the wave length in Angstroms are collected here for purposes of comparison, together with some of the values of $(n-1) 10^7$ resulting from the formulas.

Dispersion Formulas for Dry Air (0° C and 760 mm)

$$\begin{aligned} \text{Kayser and Runge} \dots (n-1)10^7 &= 2881.7 + \frac{13.16}{\lambda^2 \times 10^{-8}} + \frac{0.316}{\lambda^4 \times 10^{-16}} \\ \text{Rentschler} \dots \dots \dots (n-1)10^7 &= 2903.1 + \frac{3.80}{\lambda^2 \times 10^{-8}} + \frac{1.23}{\lambda^4 \times 10^{-16}} \\ \text{Cuthbertson} \dots \dots \dots (n-1)10^7 &= 2885.4 + \frac{13.38}{\lambda^2 \times 10^{-8}} + \frac{0.50}{\lambda^4 \times 10^{-16}} \\ \text{Howell} \dots \dots \dots (n-1)10^7 &= 2881.7 + \frac{11.83}{\lambda^2 \times 10^{-8}} + \frac{0.50}{\lambda^4 \times 10^{-16}} \\ \text{Dickey} \dots \dots \dots (n-1)10^7 &= 2878.12 + \frac{11.59}{\lambda^2 \times 10^{-8}} + \frac{0.551}{\lambda^4 \times 10^{-16}} \\ \text{Bureau of Standards} \dots (n-1)10^7 &= 2875.66 + \frac{13.412}{\lambda^2 \times 10^{-8}} + \frac{0.3777}{\lambda^4 \times 10^{-16}} \end{aligned}$$

Observer	$\lambda 2000$	$\lambda 4000$	$\lambda 6000$	$\lambda 8000$
Kayser and Runge	3408.2	2976.3	2920.7	2903.0
Rentschler	3766.9	2974.9	2923.1	2912.0
Cuthbertson	3532.4	2968.6	2926.4	2907.5
Howell	3464.9	2973.6	2918.1	2901.3
Dickey	3512.2	2972.1	2914.6	2897.6
Bureau of Standards	3447.1	2973.2	2915.8	2897.5

Four different expressions have been proposed to represent the relation between the index of refraction of a substance and its density. The earliest one was supported by Newton and indicated $\frac{n^2-1}{d} = \text{constant}$. An empirical relation $\frac{n-1}{d} = \text{constant}$ was found by Dale and Gladstone ³⁴ in measurements of refractive indices of liquids at various temperatures. A new relation $\frac{n^2-1}{(n^2+2)d} = \text{constant}$, was deduced by Lorenz ³⁵ from the elastic

³⁴ Dale and Gladstone, Phil. Trans., 151, p. 317: 1863.

³⁵ Lorenz, Wied Ann. 11, p. 70: 1880.

solid theory and later from the electron theory by Lorentz.²⁶

The fourth relation, $\frac{(n^2+1)d}{n^2-1} = \text{constant}$, was proposed by Magri²⁷

to represent his measurements on air under high pressure. Experiments on gases form the best test of these various relations since the density of gases can be varied more easily than that of solids and liquids. Many of the investigations on air mentioned above were undertaken to find the exact relation of index of refraction to density, but the results are more or less in conflict and point to no definite choice between the different relations proposed. It is evident that this choice depends on extremely accurate measurements, since n^2-1 differs from an integral multiple of $n-1$ only by a term of the order $(n-1)^2$. For air $n-1=0.0003$, approximately, so that $(n-1)^2=9 \times 10^{-8}$. It may be that changes in pressures or temperature used to change the density of the air introduce errors in the observations which are much larger than the difference between the relations under test so that as far as can be judged from measurements on air the proposed relationships of index and density are indistinguishable.

II. PURPOSE

1. WAVE LENGTH CORRECTIONS FOR TEMPERATURE AND PRESSURE OF AIR

The measurements of indices of refraction of air which are described below were inspired by spectroscopic problems connected with the accurate measurement of wave lengths. The fundamental spectroscopic standard has been defined by the International Wave Length Committee as 6438.4696 Å which represents the length of a wave of Cadmium red radiation in dry air at 15° C and 76 cm. Other radiations whose wave lengths are determined relative to the primary standard are called secondary standards and a large number of these are required throughout the entire spectral range to facilitate spectroscopic work. When determinations of secondary standards are made in air at temperatures other than 15° C, and pressures differing from 76 cm. the determinations require corrections which demand a knowledge of the dispersion of the air and the dependence of indices of refraction on density.

²⁶ Lorentz, *Theory of Electrons*, p. 143.

²⁷ Magri, *Phys. Zeitschr.*, 6, p. 629; 1905.

2. WAVE LENGTH CORRECTIONS TO VACUUM

Furthermore, it is very desirable to be able to convert wave lengths measured in air to their values in a vacuum. This is especially important for the discussion of mathematical relations among spectral lines in which it is best to use the vacuum values of oscillation frequencies. For this purpose oscillation frequency is defined as the number of waves contained in 1 cm in vacuum; that is, the reciprocal of the wave length in centimeters reduced to its value in vacuum. To reduce the wave length of any line measured in air to its real value in vacuum it is necessary to multiply by the refractive index of air for light of the particular wave length. For air at 15° C and 76 cm this correction amounts to +2.468 Å for the wave length 9000 Å or -3.05 in the oscillation frequency. At 2000 Å the correction to the wave length is +0.651 Å or -16.28 to the oscillation frequency. In the early work on spectral series the accuracy of the wave length measurements was not very great and did not warrant correction to vacuum value.

For this reason Kayser and Runge used oscillation frequencies deduced from wave lengths measured in air without reduction to vacuum. Recently, the accuracy of wave length measurements has been increased by the use of the interferometer for wave-length comparisons, and also by the use of large diffraction gratings working with the new system of international standards of wave length derived from interferometer experiments. An accuracy of one part in several millions is now striven for in wave length measurements and to maintain this relative accuracy in the values converted to vacuum it is necessary to know the indices of refraction to two or three units in the seventh decimal place.

This Bureau has been interested in the accurate measurement of wave lengths in spectra of the chemical elements and has had considerable success in extending such measurements out into the infra-red region of the spectrum. Interferometer comparisons have been made in the iron spectrum to 9000 Å²⁸ and in the spectra of neon and argon to 8500 Å.²⁸ Grating spectra have been photographed and measured in many cases beyond 9000 Å²⁹ and waves longer than 10 000 Å have been measured in the arc spectra of the ferrous metals.²⁸

To correct these measurements to standard atmospheric conditions, and also to change them to vacuum values, requires extensive and accurate data on the refractive and dispersive powers

²⁸ Unpublished.

²⁹ This Bulletin, 14, p. 372; 1927.

of air. The introductory review and summary of such data show that although a large number of researches on the index of refraction of air have been made, employing many different experimental methods, the observations up to the present time have covered a comparatively small spectral region and are often in rather wide disagreement. Most observers have been content with index determinations for one point, or at most only a few points, in the ultra-violet and short wave visible spectral regions, and no values of the index of air have ever been published for waves in the extreme red and adjacent infra-red regions.

The work described in the present paper was first undertaken for the purpose of measuring the indices of refraction of air for red and infra-red light. When this was completed the observations were carried on through the visible spectrum and pushed out into the ultra-violet until practically the entire spectral region which is accessible to ordinary photography was investigated.

III. EXPERIMENTAL METHOD

1. FABRY AND PEROT INTERFEROMETER FOR MEASUREMENT OF n

For the measurement of absolute indices of refraction of air, convenience and efficiency together with greatest accuracy recommended the method which makes use of the Fabry and Perot interferometer. This apparatus allows the most satisfactory control of temperature and pressure of the air whose optical properties it is desired to determine. It also permits observations to be recorded simultaneously by photography for a large number of different wave lengths. These advantages determined the use of the Fabry and Perot type of interferometer for the present investigations, and a detailed description of the apparatus will be given in Section IV of this paper.

The experimental method was as follows: Diametric sections of the circular (Haidinger) rings produced by the various rays from a source of light illuminating the parallel plates of the interferometer were photographed, first, when a vacuum existed between the interferometer plates, and then when dry air at atmospheric pressure was present. If the distance between the interferometer plates is e , then the number of waves p (order of interference) of length λ existing in this distance when air is absent is $p = e/\lambda$. When dry air at atmospheric pressure is present between the plates the wave length λ is shortened to λ' , and consequently the number of waves contained in the same distance

e is increased to $p' = e/\lambda'$. The total order of interference p at the center of the system of circular fringes in general consists of a whole number P and a fraction p_1 . The fraction p_1 is determined from measurements of the ring diameters and an approximate value of P . If these fractions are obtained for three or more different but known wave lengths, the whole number P and finally the distance e can be found readily and accurately by a method previously described.⁴¹ Similar procedure is required for the determination of p' and P' when air exists between the interferometer plates. When the wave-length values in air were used for finding the order of interference for vacuum between the plates, differences were observed between the computed and measured fractional parts of the order because of the dispersion of the air. These differences were not large enough, however, to introduce any difficulty or ambiguity in the determination of the correct order of interference.

By definition the index of refraction n of air for wave length λ , is $n = \frac{\text{velocity of light of wave length } \lambda \text{ in vacuum}}{\text{velocity of light of wave length } \lambda' \text{ in air}}$. In either case the velocity is the product of frequency and wave length; and since frequency is assumed to remain constant we may write $n = \frac{\lambda \text{ (= wave length in vacuum)}}{\lambda' \text{ (= wave length in air)}}$. Now apply the interferometer to the measurement of n . When the space between the interferometer plates is exhausted of air, the distance e is found to contain p waves of length λ . When air at a definite temperature and pressure is present between the plates, p' waves of length λ' are found to be contained in e . Since e is a constant distance, $e = p\lambda = p'\lambda'$, or $\lambda/\lambda' = p'/p$, and the index of refraction of air under the conditions in which p' is observed can be obtained directly from the observed orders of interference p and p' , that is,

$$n = \frac{p' \text{ (= number of waves in } e \text{ air)}}{p \text{ (= number of waves in } e \text{ vacuum)}}$$

IV. DESCRIPTION OF APPARATUS

1. GENERAL PLAN

The general plan of the apparatus is represented in Figs. 1 and 2, which differ only in the dispersive apparatus.

The image of a source of light O is projected with threefold magnification by a quartz lens L , upon the interferometer plates

⁴¹This Bulletin, 12, p. 198; 1915-1916.

I, which are inclosed in a cylinder *A*, whose ends are closed with quartz windows. An image of the circular fringes produced by the interferometer is projected upon the slit *S* of a spectrograph by means of a quartz fluorite achromatic objective *L*₂. For most of the work a grating spectrograph was used with the grating mounted in parallel light as shown in Fig. 1, where *M* represents a speculum concave mirror, *G* the concave grating, and *P* the photographic plate. The mirror and grating each have a radius of curvature of about 640 cm. The grating was ruled by Dr.

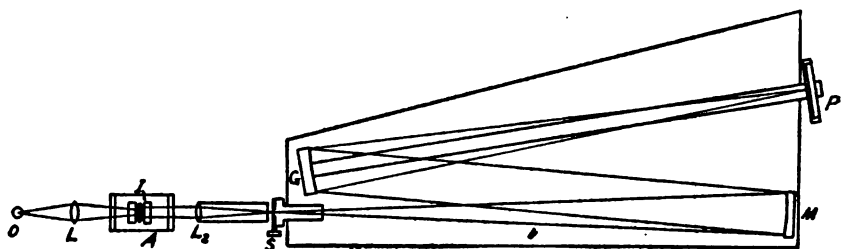


FIG. 1.—Grating spectrograph

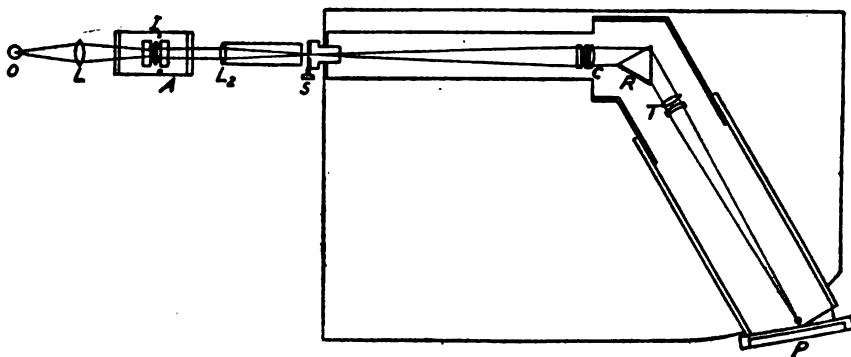


FIG. 2.—Prism spectrograph

Anderson and possesses 39 800 lines on a distance of 13.2 cm. The linear dispersion at *P* with this apparatus is about 10 Å per millimeter in the spectrum of the first order.

In the region of the spectrum where the waves are shorter than 2800 Å the reflecting power of speculum metal becomes very small and too much light was lost in the reflections from the mirror and grating. For these short waves, therefore, a prism spectrograph whose optical parts consist of quartz and rock salt was used to separate the different radiations emanating from the source of light. The prism *R*, Fig. 2, was of rock salt with a 60° angle and

3-inch refracting edge. The collimator *C* is a quartz rock salt objective and the camera lens *T* is made of quartz. This spectrograph was designed for the photography of interference fringes in the spectral region 2000–3000 Å, and gives a linear dispersion of about 1 Å per millimeter (at $\lambda = 2500$ Å) in the focal plane where the photographic plate *P* is placed.

2. SOURCES

The sources of light used in this work were chosen by the following qualifications: (1) The spectra must show a large number of lines of sufficient intensity and regularity of distribution. (2) The lines must be sufficiently sharp or homogeneous to show good interference fringes with comparatively large path differences or orders of interference.

The standard iron arc was used through the range of wave lengths from 2750 Å to 9000 Å and interferometers were used which made the average order of interference about 20 000 waves. In the long wave region these observations were supplemented by the use of electrical discharge tubes containing neon and argon. Observations were made with the spectrum of neon between the wave-length limits 5852 Å and 8495 Å, and with the spectrum of argon from 6500 Å to 8521 Å. The radiations from these gases are quite homogeneous and they were therefore used with a larger order of interference which averaged about 70'000 waves. In the extreme ultra-violet region the most satisfactory source was found to be a copper arc with a positive carbon electrode above. This spectrum was used for observations in the wave length interval 2200 Å–3200 Å with orders of interference averaging 16 000 waves.

3. INTERFEROMETERS

The interferometer plates were glass or quartz disks of 42 mm diameter and 6 or 8 mm thickness. These plates were thinly covered with metallic films by the method of sputtering from a metallic cathode in a vacuum. Four different pairs of plates were used in all, each pair being used in the spectral region where its metallic films showed their highest reflecting power and the least absorption. In the red and infra-red parts of the spectrum glass plates with thin copper films were used. In the visible spectrum glass plates with silver films were used. For the short visible waves and adjacent ultra-violet quartz plates with nickel films were found most efficient while silicon films on quartz plates were found greatly superior to nickel in the shortest ultra-violet.

The interferometer plates *I*, Fig. 3, were held apart and parallel by three invar posts of equal length, held in an invar ring. These invar separators or etalons ranged in length from 2 to 25 mm depending on the source of light and the spectral region which was being used. The system of plates and etalons was placed in a hollow cylinder (5 cm by 10 cm), and held against a flange on one end of this cylinder by screwing a disk *D* down the other end. This disk had a 2 cm hole in its center to allow the passage of light through the interferometer. Three small screws in the flange pressed against springs which bore on the interferometer plate facing the flange and permitted the adjustment of the plates to parallelism by varying the pressure on the separating post.

The disk screw *D* allowed the adaptation of this interferometer mounting to separators or etalons ranging from 2 to 50 mm in length. After the interferometer plates were adjusted so as to be

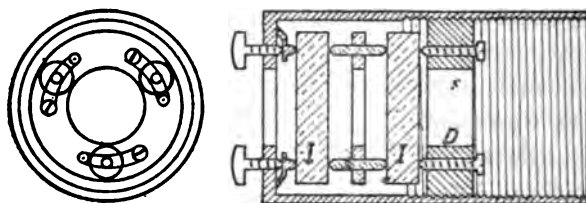


FIG. 3.—Interferometer

strictly parallel, the entire interferometer mounting was slipped into a closely fitting brass tube 40 cm long which was surrounded by a bath whose temperature could be regulated at will as described below.

4. AIR CHAMBER

In order to secure an accuracy in $n-1$ of 1 in the fourth significant figure, it was necessary to control the temperature of the air under investigation and the interferometer to 0.1°C and the pressure of the air to 0.3 mm of mercury. To accomplish this the air chamber *A*, Fig. 4, was surrounded by a bath of alcohol and water. The liquid was circulated through the tubes *D* and *E* by the motor-driven propeller *F*. Cooling was supplied to the liquid by the brine coil *C* and heat by a 5-ohm resistance coil *H*, the current in which was regulated by a relay operated by the thermostat *J*. This apparatus was surrounded with ground-cork insulation and mounted in a wooden box. With this arrangement the liquid around *A* could be held within 0.01°C of the required

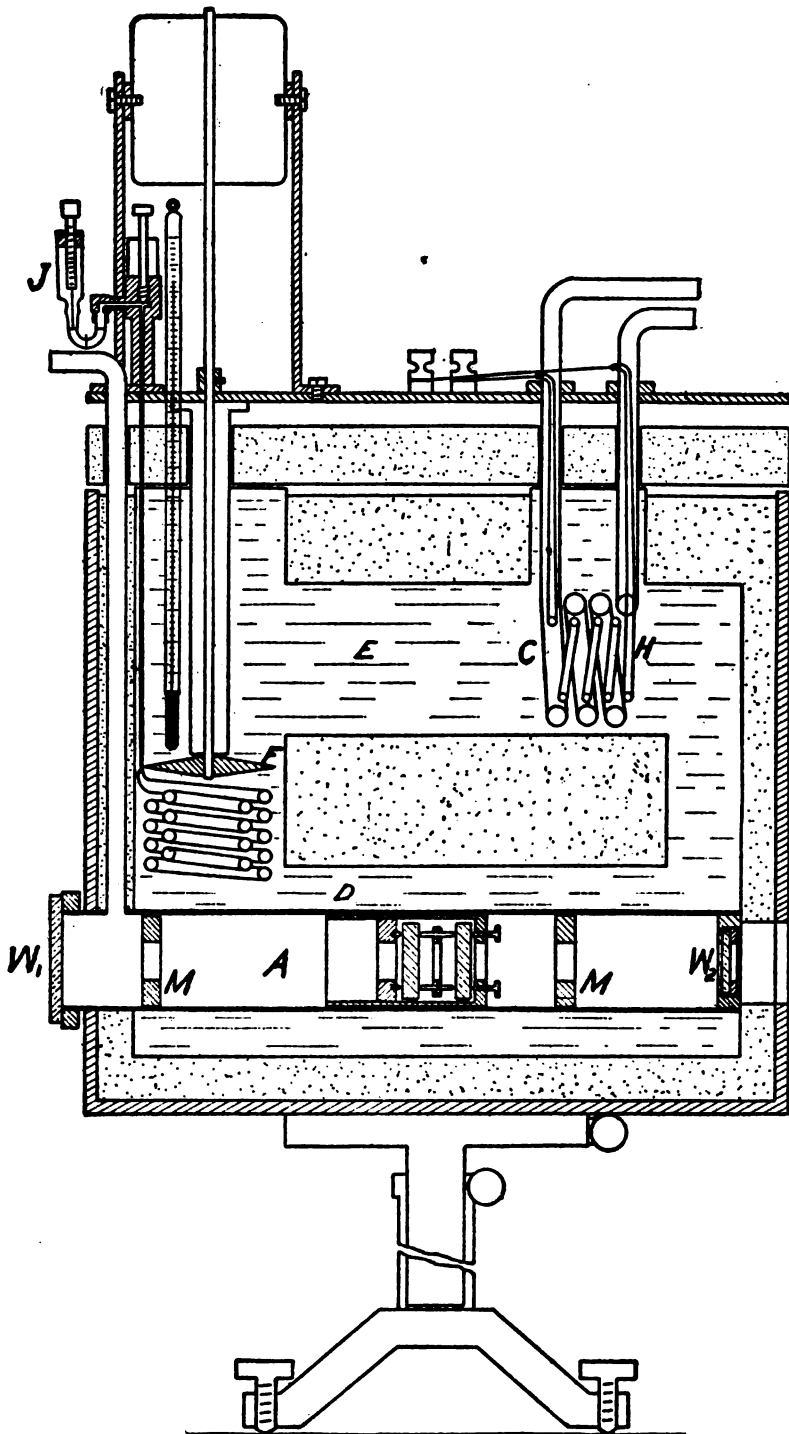


FIG. 4.—Interferometer in temperature control apparatus

temperature for any length of time. Temperatures were read with calibrated thermometers Nos. 8290 C and 14 454 C, and a 5-junction copper-constantin thermocouple calibrated by the heat division of this Bureau.

The chamber *A* consisted of a thin brass tube 40 cm long and 5 cm diameter. The ends were closed with brass rings having 2 cm openings covered with quartz windows, W_1W_2 . The cylinder containing the interferometer *I* fitted closely within the tube *A*, thus making good metallic contact with the bath. Two brass rings, *MM*, 1 cm thick, reduced the radiation from the ends of the tube. A temperature survey was made of *A* with differential thermocouples when the bath was 20° above the room temperature. The interferometer plates remained within 0.01° of constant temperature for one-half hour and differed from the bath temperature by less than 0.02° C. The plates were 0.02° warmer than *M*, and 0.7° above the temperature of the ends of the tube *A*.

5. PRESSURE GAGES

The air was exhausted from the cylinder containing the interferometer by means of a Gaede rotary oil pump. An electrical discharge tube inserted between the pump and the cylinder and operated with a 10 000-volt transformer indicated the degree of exhaustion in the apparatus. For the vacuum observations the apparatus was exhausted of air until the dark space around the electrodes of the electrical discharge tube reached from 5 to 10 mm. This low pressure, corresponding to a few hundredths of a millimeter of mercury was maintained throughout the photographic exposure period.

The pressure observations were made after allowing air at atmospheric pressure to filter through calcium chloride and phosphorus pentoxide and fill the apparatus. These atmospheric pressures were usually read from two different barometers used as working standards in this Bureau, and all the necessary corrections were carefully made so that the final pressure observations are probably correct to one or two hundredths of a millimeter.

6. CONSTITUTION OF THE AIR

The barometer readings represent the pressure of dry air whose constituents at the earth's surface are considered to be present in the following volume percentages:⁴²

Nitrogen.....	78.03
Oxygen.....	20.99
Argon.....	.94
Hydrogen.....	.01
Neon.....	.0012
Helium.....	.0004
Carbon dioxide.....	.03

Krypton, Xenon, and Ozone in very small quantities. When the air was removed from the apparatus the spectrum from the electrical discharge tube showed only a trace of hydrogen remaining.

Next to water vapor the atmospheric constituent in which the largest variation occurs is carbon dioxide. The laboratory in which these experiments were conducted was quite well ventilated, so that the volume percentage of carbon dioxide could not have exceeded two times the normal amount.

7. PHOTOGRAPHIC

The short wave spectral region from the blue to the extreme ultra-violet was recorded on Graflex photographic plates. For longer waves from green to red in the visible spectrum Seed 27 plates were used after staining with pinacyanol. Overlapping spectra were absorbed by a cell of potassium-bichromate solution. The longest waves in the red and adjacent infra-red were photographed on Seed 27 plates after staining with dicyanin. Here the overlapping spectra were removed by placing a screen of Jena red glass No. 4512 in front of the slit of the spectrograph.

The photographic plates measured 5 by 8 inches. Each plate therefore covered an interval of 2000 Å with the grating spectrograph and about 1000 Å with the prism spectrograph in the extreme ultra-violet. The cameras were built so as to allow vertical displacement of the plate holders for a series of exposures on the same plate.

⁴² Hann. Lehrbuch der Meteorologie, 3d ed., p. 5.

V. EXPERIMENTAL PROCEDURE

1. OPTICAL ADJUSTMENTS

After the interferometer plates were made parallel to each other the interferometer mounting was inserted to the middle of the cylinder surrounded by the constant temperature bath. The box containing this apparatus stood on a tripod fitted with leveling screws and rotating top by means of which the interferometer could be leveled and oriented so that the center of the circular fringe system fell exactly on the center of the spectrograph slit.

2. TEMPERATURE REGULATIONS

Spectroscopists are concerned with the index of refraction and dispersion of air at atmospheric pressures and room temperatures since most wave length measurements are made in air under these conditions of pressure and temperature. In these experiments, therefore, extensive observations were made on these optical properties of air at atmospheric pressures and at temperatures of 0, 15, and 30° C, which cover the range of ordinary working conditions.

The temperature of the bath surrounding the interferometer was brought to the desired point and maintained by the thermostat control throughout the experiment. This temperature was held for a period of one to two hours before beginning the photographic exposures in order to eliminate temperature gradients and establish a steady state in the interferometer. Cooling or heating effects due to letting air in or exhausting the air from the apparatus were guarded against by allowing 10 to 15 minutes to elapse after each of these operations for a return to the steady state.

3. PRESSURE OBSERVATIONS

Each set of photographic exposures was begun with a vacuum between the interferometer plates. The apparatus was washed out with dry air once or twice before the low pressure was maintained for the first exposure. This low pressure, as indicated by the electrical discharge tube, corresponded in general to a pressure of only a few hundredths of a millimeter of mercury. After the observations for vacuum between the interferometer plates were recorded a two-way stopcock cut off the pump and admitted air from the room after being dried by passing through calcium chloride and phosphorus pentoxide. The pressure of this air was read from the barometers described above. When the exposures



FIG. 5.—Interference fringes in the spectrum of neon

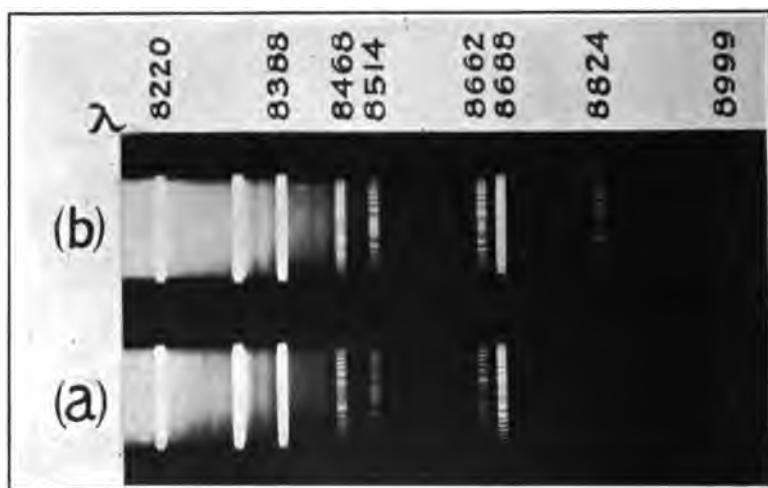


FIG. 6.—Interference fringes in the spectrum of iron

were longer than about 5 minutes the pressure observations were made before and after and the mean value was used in the density reductions.

4. PHOTOGRAPHIC EXPOSURES

In general five photographic exposures were made for each experiment, from which the indices of refraction of air at a definite density could be determined for a certain wave length region. These exposures were made in the following order: (1) Interference rings for vacuum between the interferometer plates; (2) interference rings for dry air between the interferometer plates; (3) interference rings for vacuum between the interferometer plates; (4) interference rings for dry air between the interferometer plates; and (5) gages from which the diameters of the photographed rings could be transformed to their actual size as projected on the slit of the spectrograph. These five exposures were generally made on a 5 by 8 plate, the plate holder being lowered about 1.5 cm between exposures.

The exposure times in the visible and adjacent ultra-violet spectrum ranged from 1 to 10 minutes. The longest infra-red and shortest ultra-violet waves required exposures of 60 minutes. Spectrum regions were chosen in successive experiments to overlap half of the preceding region.

Several photographs which are typical of those made in this work are reproduced in Figs. 5 and 6. Fig. 5 shows the yellow and red portion of the spectrum of neon with interference maxima and minima in the spectrum lines. These fringes were produced by an interferometer whose silvered plates were separated 3.75 mm. The first exposure (a) was made after the air was removed, and the second (b) was made with dry air at atmospheric pressure and 15° C between the interferometer plates. Fig. 6 shows similar interference phenomena in the infra-red spectrum of iron. These fringes were produced by a 10 mm separation of the interferometer plates with copper films. Wave lengths in dry air at atmospheric pressure and 30° C caused (a), and wave lengths in a vacuum gave (b).

5. MEASUREMENT OF RING DIAMETERS

The diameters of the interference rings were measured by means of a micrometer screw with a head graduated to 5 μ . The diameters of the first three rings were measured for most of the lines which were used, and for the remainder the first two rings were

measured. In general, these measurements made it possible to compute the orders of interference within a few thousandths of a wave. The fractional parts of the orders of interference agreed so well for the two vacuum and the two air exposures in each experiment that the means of the two sets were used for index computations in every case. No appreciable systematic differences were observed even in cases where the barometric pressure changed slightly between the two air exposures.

VI. REDUCTION OF OBSERVATIONS

1. CORRECTIONS TO 760 MM

The index of refraction n of air at a measured density was deduced for a particular wave length λ by dividing the number ($p' = P' + p'_1$) of waves in the double distance $2e$, separating the interferometer plates when dry air of determined density was present by the number ($p = P + p_1$) in this same distance when only a negligible quantity of air remained. The number of waves in $2e$ air was from 5 to 25 greater than the number in $2e$ vacuum, depending on the size of the etalon and the wave length.

The density of the air experimented upon depended on its temperature t and the barometric pressure B observed at the time of the experiment. The observed barometric pressures ranged between 730 and 760 mm. The final values of n were deduced for dry air under the standard pressure of 760 mm of Hg, by multiplying the value n_B obtained from the observed barometric pressure B by $\frac{760}{B}$; that is, $n_{760} = n_B \frac{760}{B}$. This assumes that the coefficient of variation of the index of refraction with density is identical with the pressure coefficient of volume change at constant temperature. Although considerable disagreement exists among previous observers as to the exact relation between refractivity and density of air, the deviation from exact proportionality is too small to be appreciable here since the barometric pressures never deviated more than 30 mm from 760 mm in these experiments.

2. FINAL VALUES OF $n-1$

All of the observations are collected in Table 2. Each value of $n-1$ is the mean of the number of observations indicated in the next column, and this is followed by the correction necessary to place this value on the dispersion curve whose equation was determined from all the observations.

TABLE 2.—(n-1) 10⁷ for Dry Air

λA	Source	Observations, 0° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 15° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 30° C, 760 mm	Number of observations	Com- puted minus ob- served
2218	Cu							2982.7	1	-37.3
2246	Cu							2960.4	2	-26.4
2303	Cu							2907.5	2	+4.9
2369	Cu							2879.0	2	+10.9
2392	Cu	3223.4	2	-1.6	3050.0	2	-0.5	2867.1	2	+15.5
2406	Cu	3222.7	1	-2.8				2869.4	2	+9.0
2441	Cu	3209.0	2	-1.6	3046.6	2	-14.0	2862.3	2	+5.2
2492	Cu	3196.6	2	-7.1	3012.2	2	+4.2	2857.7	2	-3.8
2618	Cu	3145.8	2	+6.4	2962.4	2	+19.4	2833.6	2	-10.0
2739	Fe	3101.7	1	+11.0	2932.0	1	+22.4	2804.0	1	-4.0
2766	Cu	3139.4	1	-23.4	2950.5	2	-2.2	2787.1	2	+7.4
2824	Cu	3103.8	2	-3	2940.8	2	-4.2	2779.3	2	+4.8
2851	Fe	3085.8	1	+2.1				2768.3	2	+11.3
2882	Cu	3080.6	1	+11.3	2921.0	2	+4.9	2774.1	1	+6
2918	Fe	3079.8	1	+5.4	2912.3	1	+7.3	2752.1	2	+17.1
2961	Cu	3078.3	2	-6	2914.0	2	-1.2	2746.2	1	+16.6
2987	Fe	3073.7	1	-9	2918.0	1	-9.2	2744.8	1	+14.4
2997	Cu	3074.2	2	-2.5	2915.3	1	-8.0	2742.6	2	+15.3
3010	Cu	3078.2	2	-8.6	2894.8	1	+10.5	2751.9	2	+4.4
3036	Cu	3084.1	1	-18.5	2908.2	1	-6.6	2745.0	2	+7.8
3055	Fe				2910.5	1	-1.7			
3063	Cu	3066.1	2	-4.6	2901.2	1	-3.5	2750.7	1	-1.0
3075	Fe							2747.3	1	+9
3116	Fe	3048.9	1	+5.0	2889.8	1	+8	2746.4	1	-3.1
3175	Fe	3034.3	1	+11.5	2894.0	1	-10.7	2720.2	1	+16.6
3205	Fe	3038.2	1	+3.8	2884.9	1	-5.2	2727.1	1	+6.2
3280	Fe	3028.9	1	+4.0	2872.5	1	-1.1	2732.5	1	-6.6
3347	Fe	3022.7	1	+2.8	2857.0	1	+7.3	2720.8	1	-1.2
3413	Fe	3023.6	1	-5.0	2859.7	1	-1.5	2715.6	1	-1.7
3485	Fe	3004.5	1	+7.2	2857.0	1	-5.1			
3513	Fe	3010.0	1	-9	2858.0	1	-8.5	2716.0	1	-10.1
3594	Fe	2988.9	1	+13.2	2845.0	1	-2.1	2714.9	1	-14.5
3640	Fe	2986.1	1	+12.3	2842.4	1	-3.0	2700.0	1	-2.9
3659	Fe	2998.7	3	-1.8	2836.3	3	+1.9	2699.6	3	-3.8
3701	Fe	2996.4	1	-2.7	2834.2	1	+9	2679.0	1	+13.9
3753	Fe	2988.0	4	+1.9	2831.8	4	-2	2688.0	4	+1.6
3787	Fe				2831.6	2	-2.2			
3790	Fe				2829.1	1	+1			
3843	Fe	2977.5	1	+6.3	2829.2	4	-8.4	2689.5	3	-4.9
3846	Fe							2682.2	1	+1.2
3865	Fe				2826.1	2	-1.6			
3867	Fe	2981.4	3	+8						
3906	Fe	2993.3	1	-13.5	2821.6	3	+6	2676.1	1	+5.0
3935	Fe	2979.8	3	-1.8	2814.8	3	+5.8	2689.1	3	-9.5
3969	Fe				2819.8	2	-1.0			
3977	Fe	2966.2	1	+9.3	2815.9	1	+2.4	2681.4	1	-4.0
3983	Fe	2973.7	3	+1.5	2821.4	3	-3.4	2677.3	3	-2
4005	Fe				2817.5	2	-7			
4021	Fe	2971.7	4	+1.4	2815.7	4	+3	2669.9	4	+5.6
4076	Fe	2970.4	1	-2	2816.4	1	-3.1	2688.7	1	-15.8
4095	Fe	2966.6	3	+2.5	2809.5	3	+2.9	2676.0	3	-4.0
4118	Fe	2973.4	4	-5.5	2809.3	2	+2.0			
4147	Fe				2807.8	4	+2.1	2666.8	4	+2.9
4191	Fe				2807.7	2	+2			
4210	Fe	2970.0	1	-6.6	2806.1	1	-1.1	2668.7	1	-1.6
4213	Fe	2963.6	3	-4	2805.3	3	+1.6	2669.7	3	-2.7
4233	Fe				2807.1	2	-1.1			
4245	Fe	2965.9	3	-4.2	2804.7	3	+8	2668.0	3	-2.3
4282	Fe	2956.1	1	+3.9	2807.6	3	-3.6	2650.7	1	+13.5
4315	Fe	2960.8	3	-2.2	2800.4	3	+2.3	2663.8	3	-8
4352	Fe	2967.5	1	-10.5	2805.9	1	-4.7	2662.8	1	-1.2
4369	Fe	2954.2	3	+2.1	2796.1	3	+4.5	2663.9	3	-2.9
4375	Fe				2798.3	2	+2.1			
4422	Fe	2954.7	3	-6	2792.8	3	+5.7	2664.4	3	-5.5
4427	Fe				2796.7	2	+1.6			

TABLE 2—Continued

λA	Source	Observations, 0° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 15° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 30° C, 760 mm	Number of observations	Com- puted minus ob- served
4484	Fe	2950.9	3	+ 0.9	2796.4	5	- 0.4	2656.2	3	+ 0.5
4494	Fe				2794.4	2	+ .3			
4531	Fe				2789.0	3	+ 5.2	2655.0	3	.0
4547	Fe	2947.3	3	+ 2.1	2791.9	3	- .2	2654.9	3	- 1.2
4592	Fe	2943.4	3	+ 4.4						
4602	Fe				2790.7	2	+ .6			
4647	Fe	2946.3	3	- .4	2787.8	3	+ 3.1	2654.9	3	- 2.8
4691	Fe	2943.0	3	+ 1.4	2785.2	3	+ 4.4	2649.7	3	+ 1.2
4736	Fe	2938.9	3	+ 4.1	2789.2	5	- .9	2653.6	3	- 4.0
4789	Fe	2935.7	3	+ 5.6	2778.4	3	- 1.7	2647.4	3	+ .6
4859	Fe	2938.0	3	+ 1.3	2787.6	5	- 2.8	2646.5	3	- .2
4903	Fe	2937.6	3	+ .4	2783.2	5	+ .5	2647.1	3	- 1.9
4966	Fe	2933.2	3	+ 3.0	2789.1	3	- 7.0	2646.0	3	- 2.3
5001	Fe	2935.5	3	- .2				2642.0	3	+ .8
5012	Fe				2783.0	5	- 2.1			
5051	Fe	2935.9	3	- 1.9	2764.0	3	+ 16.0	2642.0	3	- .3
5110	Fe	2930.5	5	+ 2.1	2784.3	2	- 5.6	2643.3	5	- 3.0
5167	Fe	2926.8	3	+ 4.4	2779.3	5	- 1.8	2639.0	3	+ .2
5171	Fe	2930.9	2	+ .2	2776.9	2	+ .5	2637.3	2	+ 1.8
5232	Fe	2934.2	5	- 4.4	2779.2	5	- 3.1	2636.4	5	+ 1.5
5266	Fe	2938.7	2	- 9.8				2636.5	2	+ .7
5324	Fe	2936.1	2	- 8.4	2775.9	2	- 1.6	2630.8	2	+ 5.4
5397	Fe	2930.1	2	- 3.9	2771.9	2	+ .9	2636.1	2	- 1.3
5455	Fe	2924.4	2	+ .6	2773.4	2	- 1.7	2634.3	2	- .5
5506	Fe	2922.6	2	+ 1.4	2772.8	2	- 2.0	2636.0	2	- 3.1
5569	Fe	2925.9	2	- 3.0	2779.0	2	- 9.2	2631.3	2	+ .5
5624	Fe	2931.3	2	- 9.5	2776.1	2	- 7.3	2632.6	2	- 1.6
5658	Fe	2923.7	2	- 2.5	2770.7	2	- 2.5	2630.0	2	+ .5
5709	Fe	2931.6	2	- 11.1	2771.3	2	- 3.8	2628.2	2	+ 1.6
5763	Fe	2925.1	2	- 5.6	2767.5	2	- .9	2624.1	2	+ 4.9
5852	Ne				2769.2	4	- 3.9	2629.6	3	- 1.9
5881	Ne				2768.1	4	- 3.2	2629.9	3	- 2.6
5883	Fe	2924.6	2	- 7.1	2767.5	2	- 2.6			
5934	Fe				2767.1	2	- 2.9	2638.8	2	- 12.2
5944	Ne				2763.6	4	+ .4	2629.9	3	- 3.4
6003	Fe	2928.5	2	- 12.8	2765.8	2	- 2.6			
6029	Fe							2634.1	1	- 8.8
6065	Fe	2918.9	2	- 4.0	2767.0	2	- 4.5	2624.0	2	+ .9
6074	Ne				2764.2	4	- 1.9	2626.1	3	- 1.3
6096	Ne				2763.6	5	- 1.5	2626.1	3	- 1.6
6136	Fe	2915.2	2	- 1.2	2766.0	2	- 4.4	2623.4	2	+ .7
6143	Ne				2764.9	5	- 3.4	2625.9	3	- 1.9
6163	Ne				2762.1	5	- .8	2626.6	3	- 2.8
6191	Fe	2913.3	2	- .2	2767.0	3	- 6.1	2620.7	2	+ 2.7
6217	Ne				2761.5	5	- .9	2625.7	3	- 2.6
6219	Fe				2762.9	1	- 2.4			
6230	Fe				2767.9	1	- 7.5			
6246	Fe				2767.3	1	- 7.1			
6252	Fe	2912.6	2	- .2	2760.7	2	- .5	2620.7	2	+ 2.0
6266	Ne				2764.9	6	- 5.0	2624.8	3	- 2.3
6297	Fe	2904.0	2	+ 7.9	2746.5	2	+ 13.2			
6304	Ne				2757.7	6	+ 1.9	2623.8	3	- 1.6
6318	Fe				2767.4	1	- 7.9			
6334	Ne				2761.6	7	- 2.3	2624.7	3	- 2.8
6335	Fe	2910.6	2	+ .8	2758.1	2	+ 1.1	2621.0	2	+ .9
6382	Ne				2763.6	7	- 4.9	2624.0	3	- 2.7
6393	Fe	2911.1	2	- .3	2763.7	7	- 5.1	2620.3	4	+ .9
6400	Ne				2765.1	3	- 6.6			
6402	Fe				2763.2	6	- 4.7	2623.9	3	- 2.8
6430	Fe	2910.4	4	- .2	2761.4	7	- 3.2	2615.3	4	+ 5.5
6462	Fe	2905.5	2	+ 4.5	2769.0	1	- 11.0	2621.8	2	- 1.2
6494	Fe	2907.0	2	+ 2.6	2761.8	6	- 4.2	2618.0	2	+ 2.2
6506	Ne				2762.2	6	- 4.7	2621.5	3	- 1.4
6532	Ne				2761.0	7	- 3.8	2622.7	3	- 2.8
6546	Fe	2909.0	4	+ .1	2761.0	7	- 4.0	2619.8	4	.0

TABLE 2—Continued

λA	Source	Observations, 0° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 15° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 30° C, 760 mm	Number of observations	Com- puted minus ob- served
6563	H				2760.2	1	- 3.3			
6598	Ne				2760.2	7	- 3.6	2619.6	3	- 0.3
6609	Fe	2916.9	2	- 8.6	2762.8	1	- 6.4	2612.5	2	+ 6.6
6663	Fe	2913.6	2	- 5.8	2759.6	3	- 3.7	2627.4	2	- 8.8
6678	Ne				2754.6	7	+ 1.2			
6678	Fe	2907.5	2	+ .1	2760.2	6	- 4.4	2618.2	7	+ .3
6717	Ne				2760.2	7	- 4.8	2618.1	3	+ .0
6750	Fe	2901.9	2	- 5.1	2762.4	3	- 7.3	2626.8	2	- 9.0
6752	A				2756.0	2	- .9			
6841	Fe	2907.0	2	- 1.0						
6843	Fe							2609.4	2	+ 7.6
6871	A				2750.9	2	+ 3.1			
6916	Fe							2633.3	2	-16.8
6929	Ne				2757.6	7	- 4.0	2618.4	3	- 2.0
6937	A				2754.5	2	- 1.1			
6945	Fe	2899.7	2	+ 5.4	2749.3	5	+ 4.1	2621.3	2	- 5.2
6965	A				2748.7	2	+ 4.6			
6978	Fe	2903.9	2	+ 1.0	2750.8	3	+ 2.4	2614.5	2	+ 1.5
7016	Fe				2765.4	1	-12.5			
7023	Fe	2909.1	2	- 4.8						
7030	A				2749.0	2	+ 3.7			
7032	Ne				2757.1	8	- 4.4	2617.4	3	- 1.9
7038	Fe							2610.1	2	+ 5.4
7039	Ne				2737.0	1	- 4.5			
7067	A				2748.7	2	+ 3.7			
7068	Fe	2902.4	2	+ 1.6	2767.1	1	-14.7	2624.1	2	- 8.8
7090	Fe	2905.3	2	- 1.5						
7130	Fe	2898.7	2	+ 4.8	2757.6	3	- 5.6	2609.0	2	+ 5.8
7147	A				2749.9	2	+ 2.0			
7164	Fe	2895.0	2	+ 8.3	2761.2	3	- 9.4	2613.6	2	+ 1.0
7173	Ne				2756.7	4	- 5.0			
7187	Fe				2749.3	4	+ 2.3	2626.4	2	-11.9
7207	Fe				2752.9	3	- 1.5			
7223	Fe	2905.7	2	- 3.0				2608.9	2	+ 5.2
7245	Ne				2752.6	6	- 1.5	2616.6	3	- 2.6
7272	A				2748.8	2	+ 2.2			
7293	Fe	2914.5	2	-12.3				2622.4	2	- 8.7
7372	A				2751.2	2	- .9			
7383	A				2746.6	2	+ 3.6			
7389	Fe	2897.7	2	+ 3.8				2611.8	2	+ 1.3
7411	Fe	2900.6	4	+ .7	2751.9	5	- 2.0	2620.2	4	- 7.3
7438	Ne				2742.4	3	+ 7.4			
7445	Fe	2903.6	4	- 2.6	2756.2	3	- 6.5	2609.4	4	+ 3.2
7495	Fe	2906.8	4	- 6.1	2764.5	5	-15.1	2612.7	4	- .3
7503	A				2748.3	2	+ 1.1			
7511	Fe				2749.7	4	- .4			
7514	A				2748.9	2	+ .4			
7531	Fe	2897.2	4	+ 3.3	2740.7	2	+ 8.5	2616.6	4	- 4.4
7568	Fe							2606.8	2	+ 5.1
7586	Fe	2903.0	4	- 3.1				2616.6	2	- 4.7
7620	Fe				2748.1	4	+ .5	2594.7	2	+ 6.9
7635	A				2746.1	2	+ 2.4			
7664	Fe	2895.8	4	+ 3.8	2748.1	3	+ .3	2611.2	4	+ .1
7710	Fe							2611.2	2	- .1
7724	A				2744.9	2	+ 3.1			
7748	Fe	2890.2	4	+ 8.9	2741.7	4	+ 6.2	2603.1	4	+ 7.7
7780	Fe	2897.4	4	+ 1.5	2747.3	3	+ .4	2602.9	4	+ 7.7
7832	Fe	2902.3	4	- 3.6	2750.6	2	- 3.2	2611.2	4	- .8
7937	Fe	2887.8	4	+10.1	2745.5	3	+ 1.4	2615.4	2	- 5.6
7945	Fe	2904.9	2	- 7.0	2747.1	4	- .3	2612.7	2	- 2.9
7948	A									
7998	Fe	2886.7	2	+10.4	2745.4	2	+ 1.4			
8006	A				2748.7	2	- 2.2	2606.6	2	+ 2.9
8014	A				2745.8	2	+ .6			
8046	Fe	2890.3	2	+ 7.2	2743.6	2	+ 2.8			
					2750.1	2	- 3.8	2605.6	2	+ 3.6

TABLE 2—Continued

λA	Source	Observations, 0° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 15° C, 760 mm	Number of observations	Com- puted minus ob- served	Observations, 30° C, 760 mm	Number of observations	Com- puted minus ob- served
8085	Fe	2902.6	2	- 5.5	2739.2	2	+ 6.9	2610.0	2	- 0.9
8103	A				2742.8	1	+ 3.2			
8115	A				2742.0	1	+ 3.9			
8220	Fe	2895.2	3	+ 1.2	2736.1	2	+ 9.3	2612.2	3	- 3.8
8264	A				2742.5	1	+ 2.7			
8327	Fe	2894.3	2	+ 1.5	2738.8	3	+ 6.1	2606.6	3	+ 1.3
8387	Fe	2890.4	2	+ 5.1	2741.2	3	+ 3.4	2599.3	3	+ 8.4
8408	A				2743.6	1	+ .9			
8424	A				2742.7	1	+ 1.7			
8468	Fe	2900.1	2	- 5.0				2606.6	3	+ .7
8514	Fe				2745.9	1	- 1.8	2604.2	3	+ 2.8
8521	A				2746.3	1	- 2.3			
8611	Fe							2599.0	1	+ 7.7
8661	Fe	2881.0	2	+13.2	2740.7	1	+ 2.7	2601.0	4	+ 5.5
8674	Fe							2612.6	1	- 6.1
8688	Fe	2887.3	2	+ 6.7	2734.1	2	+ 9.2	2604.8	4	+ 1.6
8824	Fe				2735.5	1	+ 7.2	2605.4	4	+ 1.2
8999	Fe				2735.6	1	+ 6.5	2591.5	2	+14.8

3. THE CAUCHY DISPERSION FORMULA

The empirical Cauchy formula

$$n - 1 = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4}$$

was used to represent these observations^a of refractivity as a function of the wave length. All the measurements on the refractivity of air at a particular temperature were grouped at intervals of 200 Å with the aid of the dispersion formula given by Miss Howell⁴⁸. This gave 34 observational equations from which were obtained the three normal equations required to solve for the three constants of the Cauchy formula by the method of least squares. This solution was made independently for the observations at each of the three temperatures and gave the following equations:

$$0^\circ \text{ C, } (n - 1) 10^7 = 2875.66 + \frac{13.412}{\lambda^2 \times 10^{-8}} + \frac{0.3777}{\lambda^4 \times 10^{-16}}$$

$$15^\circ \text{ C, } (n - 1) 10^7 = 2726.43 + \frac{12.288}{\lambda^2 \times 10^{-8}} + \frac{0.3555}{\lambda^4 \times 10^{-16}}$$

$$30^\circ \text{ C, } (n - 1) 10^7 = 2589.72 + \frac{12.259}{\lambda^2 \times 10^{-8}} + \frac{0.2576}{\lambda^4 \times 10^{-16}}$$

These equations are represented graphically in Fig. 7.

^a Howell, *Phys. Rev.*, 6, p. 81; 1915.



The algebraic sum of differences of the grouped observations from the equation values is zero, but this is not quite true of the residuals of the individual observations as given in Table 2. The reason for this is that equal weight was given to the 34 values which represented the observations in 200 Å intervals whereas some of them represented slightly more concordant, or a larger number of observations than others.

A total of 1165 index measurements were made:

277 for 122 different wave lengths in air at 0° C,

513 for 185 different wave lengths in air at 15° C,

and 375 for 154 different wave lengths in air at 30° C.

In view of the large number of observations which are fairly well distributed throughout the spectrum and quite concordant it is considered that their collection to 34 points, as explained above, gives dispersion equations which are a sufficiently accurate representation of the observations. The deviations of the individual observations from the computed curves indicate a probable error of 3.8 in a single $(n-1) \cdot 10^7$ for air at 0° C, 3.1 for air at 15° C, and 2.9 for air at 30° C.

4. SOURCES OF ERROR

Although the normal volume percentage of carbon dioxide in the atmosphere at the earth's surface is 0.03, the amount present in even a well-ventilated laboratory with persons present may be twice as large. The index of refraction of carbon dioxide for visible light is about 1.00045 so that the normal amount in the atmosphere contributes little more than one unit in the seventh decimal place of the index of refraction of air. The error introduced in the measurement of the index of refraction of air in a laboratory containing as much as twice the normal amount of carbon dioxide is, therefore, even less than one unit in the seventh decimal place since the increase in carbon dioxide is accompanied by a decrease in oxygen whose index is about 1.00027.

A source of error which has not been mentioned in previous applications of the interferometer to measurement of indices of refraction was called to our attention by Dr. W. F. G. Swann and is due to different dimensions of the apparatus under different pressures. The etalon posts separating the plates of the Fabry and Perot interferometer are larger in a vacuum than they are when surrounded by air at atmospheric pressure. The bulk modulus of invar indicates that the length of these posts is about 2 parts in 3 000 000 less in air than in vacuum. If this compres-

sion is not taken into account in the calculations, the value of the index of refraction will be about one unit too small in the seventh decimal place. However, this correction can not be applied definitely since the effect of similar compression on other parts of the apparatus holding the interferometer plates can not be computed accurately.

It was, at first, thought possible that an error might be introduced by differences of phase change at reflection from the interferometer plates in air and vacuum. If the film of adsorbed gas on the interferometer plates is different at different pressures, it might become evident as a change in the phase correction to wave lengths. Etalons of various sizes from 3 to 25 mm were used in these experiments and the differences in phase change at reflection suffered by different wave lengths were found by a method previously described.⁴⁴ The same relative phase changes were found with low pressures as with atmospheric pressures of air on the plates and the absolute phase changes may be assumed to remain constant also. Furthermore, experiments on contact differences of potential show that films of adsorbed gas are quite stable and only show changes after several hours in very high vacuum.

It appears that the systematic errors which affect these measurements of the refractive index of air are extremely small and more or less compensating. The index is increased slightly by the extra carbon dioxide in laboratory air but this increase is probably of the same order of magnitude as the decrease due to the compression of the etalon. It seems safe to assume, therefore, that these systematic errors in our measurements of the index of refraction of air are entirely negligible.

VII. APPLICATIONS

1. CORRECTION OF WAVE LENGTH MEASUREMENTS MADE IN AIR AT OTHER THAN NORMAL TEMPERATURES AND PRESSURES

The international wave length standards are represented by specified radiations whose wave lengths were measured in dry air at 15° C exerting a barometric pressure equivalent to 760 mm of mercury.⁴⁵ These standards serve as a basis for the precise measurement of all other wave lengths. Variations in the density of the air appreciably affect the absolute values of the wave lengths and when wave length measurements are made in air whose temperature is not 15° C, or whose pressure is not equal to 760 mm,

⁴⁴ This Bulletin, 12, p. 198; 1915-1916.

⁴⁵ *Astroph. J.*, 32, p. 215, 1910; 32, p. 85, 1911; 33, p. 93, 1914.

corrections must be applied to the measured values to reduce them to their values in air under normal conditions. If the index of refraction of air were a constant throughout the spectrum, all the wave lengths would vary in the same ratio, but the dispersion of the air makes this variation a function of the absolute index and of the density of the air at the time the wave length comparisons were made. For example, let a wave length λ be measured in terms of a standard wave length λ' and consider the result if the measurement is made in air whose density is greater than that defined by the normal conditions. The indices of refraction n and n' for these two wave lengths increase proportionately with the density of the air and the wave lengths decrease. But if λ is smaller than λ' , the absolute index of refraction n is larger than n' and the increase in index due to the increased density of air will be a larger amount for n than for n' . Consequently λ is proportionately shorter than λ' in denser air and requires a positive correction to make it comparable with λ' in air at 15° C and 760 mm.

The effect of the temperature and pressure of the air must be taken into account in all accurate comparisons of wave lengths made either by the coincidence method with diffraction gratings⁴⁶ or with interferometers.⁴⁷ The necessary corrections are generally small and may be negligible under certain conditions. In order actually to calculate these corrections let us imagine the wave-length comparisons to be made by means of an interferometer in air whose density, d , is defined by the temperature t and the barometric pressure B . Let λ represent the wave length of a radiation under these conditions, n the index of refraction of this air for this wave length, and λ' and n' analogous quantities for the standard wave length with which λ is to be compared. The same letters with the subscript o may be used to represent the same quantities under standard observing conditions; that is, air at 15° C and 760 mm. It is desired to obtain λ_o . The orders of interference measured under the actual conditions give $p\lambda = p'\lambda'$ ($=2e$). The calculation for λ , however, is made from $\frac{p'\lambda'_o}{p}$ in which the standard wave length is used as if the air had no dispersion, and then a correction is applied.

⁴⁶ Kayser, *Handbuch der Spectroscopie*, 1, p. 719; 1900.

⁴⁷ Buisson and Fabry, *Jour. de Phys.*, 7, p. 169; 1908.

This correction, the difference between the exact value and that calculated, is equal to

$$\delta = \lambda_0 - \frac{p'\lambda'_0}{p} = \lambda_0 \left(1 - \frac{p'\lambda'_0}{p\lambda_0} \right) = \lambda_0 \left(1 - \frac{\lambda\lambda'_0}{\lambda'\lambda_0} \right)$$

But

$$\frac{\lambda}{\lambda_0} = \frac{n_0}{n} \text{ and } \frac{\lambda'}{\lambda'_0} = \frac{n'_0}{n'}$$

Then

$$\delta = \lambda_0 \left(1 - \frac{n_0 n'}{n'_0 n} \right) = \lambda_0 \frac{n'_0 n - n_0 n'}{n'_0 n} = \lambda_0 (n'_0 n - n_0 n')$$

since $n'_0 n$ in the denominator may be called unity. Assuming the refractivity to be proportional to the density

$$\frac{n-1}{d} = \frac{n_0-1}{d_0} \text{ and } \frac{n'-1}{d} = \frac{n'_0-1}{d_0}$$

we have

$$n = \frac{(n_0-1)d}{d_0} + 1 \text{ and } n' = \frac{(n'_0-1)d}{d_0} + 1$$

and the correction reduces to

$$\delta = \lambda_0 (n_0 - n'_0) \frac{d - d_0}{d_0}$$

The factor $\frac{d-d_0}{d_0}$ is easily calculated as a function of temperature and pressure, and is constant for all the waves compared under these conditions.

The quantity $\lambda_0 (n_0 - n'_0)$ may be calculated as a function of the wave length and represented by a curve from which the correction to the wave length can be obtained by multiplying its ordinate by the appropriate density factor $\frac{d-d_0}{d_0}$.

In order to avoid all these troublesome computations hereafter, Tables 3 and 4 have been prepared to give these corrections for the entire range of wave lengths and densities of air in which accurate spectroscopic measurements are ordinarily made. Table 3 gives the density factor $\frac{d-d_0}{d_0}$ in terms of the temperature t of the

air in centigrade degrees and the barometric pressure B of the air in millimeters of mercury.

Table 4 gives the corrections, $\delta = \lambda_0 (n_0 - n_0') \frac{a - d_0}{d_0}$, in ten-thousandths of an Angstrom. These tables are constructed for the correction of wave length measurements in terms of the fundamental spectroscopic standard; that is, the wave length 6438.4696 Å of cadmium red radiation in dry air at 15° C and 760 mm. The density factor (Table 3) is therefore zero for $t = 15^\circ \text{C}$ and $B = 760$ mm., and the correction (Table 4) is always zero for $\lambda = 6438$ Å.

To illustrate the use of these tables, find the correction required by λ when it is measured as 3000.0000 Å in air at 25° C and 720 mm. Table 3 gives $\frac{d - d_0}{d_0} = -0.085$ and this in Table 4 gives a correction of -0.0038 Å to λ . Again, if λ , under the same atmospheric conditions, is measured as 8000.0000 Å in terms of a standard λ' of wave length 4000.0000 Å, say, the measurement will require a correction of $(0.0020 + 0.0008 =) +0.0028$ Å. By easy interpolation in these tables the corrections can be found for all wave lengths between 2000 Å and 10 000 Å when the wave length comparisons are made in air whose density is determined by a temperature range from 9 to 35° C and a pressure range from 600 to 780 mm. of mercury.

2. CORRECTIONS FOR WATER VAPOR IN THE AIR

The refractive index of water vapor has not been fully investigated and the corrections necessary to change wave lengths measured in wet air to their values in dry air are therefore not very accurately known. Lorenz measured the index of refraction of water vapor (specific gravity = 0.0008061) for sodium light as 1.0002500. From this result he recommends a correction of $+0.000041 \frac{m}{760}$ to indices of refraction measured in air containing water vapor of m millimeters pressure.

No measurements on the dispersion of water vapor are known to exist, but if the dispersion is assumed to be comparable with that of air, the corrections to relative wave lengths will not be affected, since they depend on the difference of indices $n_0 - n_0'$.

Two of our experiments on the dispersion of air for waves between 5800 Å and 7500 Å were made with air having a water vapor pressure of 13 mm. The index of refraction was observed to be diminished by approximately the amount given by the Lorenz

TABLE 3. $\frac{d-d_0}{d_0}$

B mm	19° C	111° C	113° C	115° C	117° C	119° C	121° C	123° C	125° C	127° C	129° C	131° C	133° C	135° C
600	-.192	-.200	-.206	-.211	-.216	-.222	-.227	-.232	-.238	-.243	-.248	-.253	-.258	-.262
605	-.186	-.193	-.199	-.205	-.210	-.215	-.221	-.225	-.232	-.236	-.242	-.246	-.252	-.256
610	-.179	-.187	-.192	-.198	-.204	-.208	-.214	-.219	-.225	-.230	-.235	-.240	-.246	-.250
615	-.172	-.180	-.186	-.191	-.197	-.202	-.208	-.212	-.219	-.224	-.229	-.234	-.239	-.244
620	-.166	-.173	-.179	-.185	-.190	-.196	-.202	-.206	-.212	-.217	-.223	-.228	-.233	-.237
625	-.160	-.167	-.172	-.178	-.184	-.189	-.195	-.200	-.206	-.211	-.216	-.222	-.227	-.231
630	-.153	-.160	-.166	-.171	-.177	-.182	-.188	-.194	-.200	-.205	-.210	-.215	-.221	-.225
635	-.146	-.153	-.159	-.165	-.171	-.176	-.182	-.187	-.193	-.199	-.204	-.209	-.214	-.219
640	-.140	-.147	-.152	-.158	-.164	-.170	-.176	-.180	-.187	-.192	-.198	-.203	-.208	-.213
645	-.133	-.140	-.146	-.151	-.158	-.163	-.169	-.174	-.181	-.186	-.191	-.197	-.202	-.206
650	-.126	-.133	-.139	-.145	-.151	-.156	-.163	-.168	-.174	-.179	-.185	-.190	-.196	-.200
655	-.119	-.127	-.132	-.138	-.145	-.150	-.156	-.162	-.168	-.173	-.179	-.184	-.190	-.194
660	-.112	-.120	-.125	-.131	-.138	-.144	-.150	-.155	-.162	-.166	-.172	-.178	-.184	-.188
665	-.105	-.113	-.119	-.125	-.132	-.137	-.144	-.149	-.155	-.160	-.166	-.172	-.177	-.182
670	-.098	-.107	-.113	-.119	-.125	-.131	-.137	-.143	-.149	-.154	-.160	-.165	-.171	-.176
675	-.092	-.100	-.106	-.112	-.118	-.124	-.130	-.136	-.143	-.148	-.154	-.159	-.165	-.170
680	-.086	-.093	-.099	-.105	-.112	-.118	-.124	-.130	-.136	-.142	-.147	-.153	-.159	-.163
685	-.079	-.087	-.093	-.098	-.105	-.111	-.117	-.123	-.130	-.135	-.141	-.147	-.153	-.157
690	-.072	-.080	-.086	-.092	-.099	-.105	-.111	-.116	-.123	-.129	-.135	-.140	-.146	-.151
695	-.065	-.073	-.079	-.085	-.092	-.098	-.105	-.110	-.117	-.122	-.128	-.134	-.140	-.145
700	-.059	-.067	-.073	-.079	-.086	-.092	-.098	-.104	-.111	-.116	-.122	-.128	-.134	-.139
705	-.052	-.060	-.066	-.072	-.080	-.086	-.092	-.097	-.104	-.110	-.115	-.122	-.127	-.133
710	-.046	-.053	-.060	-.066	-.073	-.079	-.085	-.091	-.098	-.104	-.109	-.116	-.121	-.127
715	-.039	-.047	-.053	-.059	-.067	-.073	-.079	-.084	-.091	-.097	-.103	-.109	-.115	-.121
720	-.032	-.040	-.046	-.053	-.060	-.066	-.072	-.078	-.085	-.091	-.097	-.103	-.109	-.114
725	-.026	-.033	-.040	-.046	-.054	-.060	-.066	-.072	-.079	-.085	-.090	-.097	-.103	-.108
730	-.019	-.027	-.033	-.039	-.047	-.053	-.059	-.065	-.072	-.078	-.084	-.091	-.097	-.102
735	-.012	-.020	-.026	-.033	-.040	-.047	-.053	-.059	-.066	-.072	-.078	-.085	-.090	-.096
740	-.005	-.013	-.020	-.026	-.034	-.040	-.046	-.052	-.060	-.066	-.072	-.078	-.084	-.090
745	+.001	-.007	-.013	-.020	-.027	-.034	-.040	-.046	-.053	-.059	-.065	-.072	-.078	-.084
750	+.008	-.000	-.007	-.013	-.021	-.027	-.033	-.040	-.047	-.053	-.059	-.066	-.072	-.077
755	+.015	+.000	-.006	-.014	-.021	-.027	-.033	-.041	-.047	-.053	-.060	-.066	-.072	-.077
760	+.022	+.013	-.006	-.013	-.021	-.027	-.033	-.040	-.047	-.053	-.060	-.066	-.072	-.077
765	+.028	+.020	+.013	+.006	-.001	-.008	-.014	-.020	-.028	-.034	-.040	-.047	-.053	-.059
770	+.035	+.027	+.020	+.013	+.005	-.001	-.008	-.014	-.022	-.028	-.034	-.041	-.047	-.053
775	+.042	+.033	+.026	+.020	+.012	+.005	-.001	-.008	-.015	-.021	-.028	-.035	-.041	-.047
780	+.048	+.040	+.033	+.026	+.019	+.012	+.005	-.001	-.009	-.015	-.021	-.029	-.034	-.041

TABLE 4. $\delta = \lambda_0(n_0 - n_\infty) \frac{d - d_0}{d_0}$ (in ten-thousandths Δ)

$\frac{d-d_0}{d_0}$	λ 2000	λ 2500	λ 3000	λ 3500	λ 4000	λ 4500	λ 5000	λ 5500	λ 6000	λ 6500	λ 7000	λ 7500	λ 8000	λ 8500	λ 9000	λ 9500	λ 10 000
-0.260	-259	-166	-116	-84	-61	-44	-30	-18	-8	+1	+9	+17	+24	+31	+37	+44	+50
-0.250	-249	-160	-112	-81	-59	-43	-29	-18	-8	+1	+9	+16	+23	+30	+36	+42	+48
-0.240	-239	-154	-107	-75	-54	-41	-28	-17	-7	+1	+8	+15	+22	+29	+35	+40	+46
-0.230	-229	-147	-103	-74	-54	-39	-27	-16	-7	+1	+8	+14	+21	+27	+33	+39	+44
-0.220	-219	-141	-98	-71	-52	-37	-26	-15	-7	+1	+8	+14	+20	+26	+32	+37	+42
-0.210	-209	-134	-94	-68	-50	-35	-24	-15	-7	+1	+8	+14	+20	+25	+30	+35	+40
-0.200	-199	-128	-89	-65	-47	-34	-23	-14	-6	+1	+7	+13	+19	+24	+29	+34	+38
-0.190	-189	-122	-85	-61	-45	-33	-22	-13	-6	+1	+6	+12	+18	+23	+27	+32	+36
-0.180	-179	-115	-80	-58	-42	-30	-21	-12	-5	+1	+6	+11	+17	+21	+26	+30	+34
-0.170	-169	-109	-76	-55	-40	-29	-20	-12	-5	+1	+6	+11	+16	+20	+24	+29	+32
-0.160	-159	-102	-71	-52	-38	-27	-19	-11	-5	+1	+6	+10	+15	+19	+23	+27	+31
-0.150	-149	-96	-67	-48	-35	-25	-17	-10	-4	+1	+5	+9	+14	+18	+22	+25	+29
-0.140	-139	-90	-62	-45	-33	-24	-16	-10	-4	+1	+5	+9	+13	+17	+20	+24	+27
-0.130	-129	-83	-58	-42	-31	-22	-15	-9	-4	+1	+5	+9	+12	+16	+19	+22	+25
-0.120	-119	-77	-54	-39	-28	-20	-14	-8	-4	+0	+4	+8	+11	+14	+17	+20	+23
-0.110	-109	-70	-49	-36	-26	-19	-13	-8	-3	+0	+4	+7	+10	+13	+16	+18	+21
-0.100	-100	-64	-45	-32	-24	-17	-12	-7	-3	+0	+4	+7	+9	+12	+14	+17	+19
-0.090	-90	-58	-40	-29	-21	-15	-10	-6	-3	+0	+3	+6	+8	+11	+13	+15	+17
-0.080	-80	-51	-36	-26	-19	-14	-9	-6	-2	+0	+3	+5	+7	+10	+12	+13	+15
-0.070	-70	-45	-31	-23	-17	-12	-8	-5	-2	+0	+3	+5	+7	+9	+10	+12	+13
-0.060	-60	-38	-27	-19	-14	-10	-7	-4	-2	+0	+2	+4	+6	+7	+9	+10	+11
-0.050	-50	-32	-22	-16	-12	-8	-6	-4	-2	+0	+2	+3	+5	+6	+7	+8	+10
-0.040	-40	-26	-18	-13	-9	-7	-5	-3	-1	+0	+1	+3	+4	+5	+6	+7	+8
-0.030	-30	-19	-13	-10	-7	-5	-3	-2	-1	+0	+1	+2	+3	+4	+5	+6	+7
-0.020	-20	-13	-9	-6	-5	-3	-2	-1	-1	+0	+1	+1	+2	+2	+3	+3	+4
-0.010	-10	-6	-4	-3	-2	-2	-1	-1	-0	+0	+0	+0	+1	+1	+1	+2	+2
± 0.000	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0	± 0
+0.010	+10	+6	+4	+3	+2	+2	+1	+1	+1	+0	+0	+1	+1	+1	+1	+2	+2
+0.020	+20	+13	+9	+6	+5	+3	+2	+1	+1	+0	+1	+1	+2	+2	+3	+3	+4
+0.030	+30	+19	+13	+10	+7	+5	+3	+2	+1	+0	+1	+1	+2	+2	+3	+5	+6
+0.040	+40	+26	+18	+13	+9	+7	+5	+3	+1	+0	+0	+1	+2	+3	+4	+5	+7
+0.050	+50	+32	+22	+16	+12	+8	+6	+4	+2	+0	+0	+1	+2	+3	+4	+7	+10
+0.060	+60	+38	+27	+19	+14	+10	+7	+4	+2	+0	+2	+3	+4	+5	+6	+8	+11
+0.070	+70	+45	+31	+23	+17	+12	+8	+5	+2	+0	+2	+4	+5	+6	+7	+9	+13
+0.080	+80	+51	+36	+26	+19	+14	+9	+6	+2	+0	+2	+4	+5	+6	+7	+9	+13
+0.090	+90	+58	+40	+29	+21	+15	+10	+6	+3	+0	+3	+5	+7	+8	+9	+11	+15
+0.100	+100	+64	+45	+32	+24	+17	+12	+7	+3	+0	+4	+7	+9	+10	+11	+13	+17
+0.110	+109	+70	+49	+36	+26	+19	+13	+8	+3	+0	+4	+7	+9	+11	+12	+14	+19
+0.120	+119	+77	+54	+39	+28	+20	+14	+8	+4	+0	+4	+8	+11	+14	+17	+20	+25
+0.130	+129	+83	+58	+42	+31	+22	+15	+9	+4	+1	+5	+9	+12	+15	+18	+22	+27
+0.140	+139	+90	+62	+45	+33	+24	+16	+10	+4	+1	+5	+9	+13	+17	+20	+24	+29
+0.150	+149	+96	+67	+48	+35	+25	+17	+10	+4	+1	+5	+9	+13	+17	+20	+24	+29
+0.160	+159	+102	+71	+52	+38	+27	+19	+11	+5	+1	+6	+10	+15	+19	+23	+27	+31
+0.170	+169	+109	+76	+55	+40	+29	+20	+12	+5	+1	+6	+11	+16	+20	+24	+29	+34
+0.180	+179	+115	+80	+58	+42	+30	+21	+12	+6	+1	+6	+11	+17	+21	+26	+30	+35
+0.190	+189	+122	+85	+61	+45	+33	+22	+13	+6	+1	+6	+12	+18	+23	+27	+32	+36
+0.200	+199	+128	+89	+65	+47	+34	+23	+14	+7	+1	+8	+13	+19	+24	+29	+34	+38
+0.210	+209	+134	+94	+68	+50	+35	+24	+15	+7	+1	+8	+14	+20	+25	+30	+35	+40
+0.220	+219	+141	+98	+71	+52	+37	+26	+15	+7	+1	+8	+14	+20	+25	+30	+35	+40
+0.230	+229	+147	+103	+74	+54	+39	+27	+16	+7	+1	+8	+14	+20	+25	+30	+35	+40
+0.240	+239	+154	+107	+75	+54	+41	+28	+17	+7	+1	+8	+14	+20	+25	+30	+35	+40
+0.250	+249	+160	+112	+81	+59	+43	+29	+18	+8	+1	+9	+16	+23	+29	+36	+42	+48
+0.260	+259	+166	+116	+84	+61	+44	+30	+18	+8	+1	+9	+17	+24	+31	+37	+44	+50

expression. Furthermore, this decrease was about the same for different wave lengths. These results on wet air are not complete enough to add much to the subject of dispersion of light by water vapor, but they show that the corrections to index of refraction as recommended above for water vapor in the air are quite proper.

3. CORRECTIONS TO CONVERT TO VACUUM VALUES

Corrections required to convert wave lengths or oscillation frequencies measured in air at 20° C and 760 mm to their value in vacuum are found in Watts' Index of Spectra, appendix E, page 51, and in Kayser's Handbuch der Spectroscopie, II, page 513. These are applicable to Rowland's standard wave lengths and all wave lengths determined from them, but must be revised to apply to the international system of wave lengths which represent values measured in air at 15° C and 760 mm.

The dispersion equation representing our observations on the indices of refraction of air at 15° C and 760 mm was used to construct a new table of corrections. Table 5 gives the values of wave length in air, (λ), refractivity, $(n-1)10^7$, vacuum correction to λ , $(n\lambda-\lambda)$, oscillation frequency in air, $\left(\frac{1}{\lambda}\right)$, and vacuum correction to frequency, $\left(\frac{1}{n\lambda}-\frac{1}{\lambda}\right)$, from 2000 Å to 10 000 Å at intervals of 50 Å. This table shows, for example, that a light wave of length 5000.000 Å in dry air at 15° C and 760 mm is increased in length by 1.391 Å in a vacuum. The frequency (number of waves per centimeter) in air is 20 000.00, but in a vacuum it is diminished by 5.56. The vacuum frequencies corresponding to any wave lengths in the international scale can be readily determined by the use of Table 5 together with a table of reciprocals with seven place arguments.

4. ASTRONOMICAL REFRACTION

Newcomb states in his Compendium of Spherical Astronomy, page 223:

There is, perhaps, no branch of practical astronomy on which so much has been written as on this (astronomical refraction) and which is still in so unsatisfactory a state. The difficulties connected with it are both theoretical and practical. The theoretical difficulties arise from the uncertainty and variability of the law of diminution of the density of the atmosphere with height, and also from the mathematical difficulty of integrating the equations of the refraction for altitudes near the horizon, after the best law of diminution has been adopted.

TABLE 5.—Table of corrections to be Applied to Wave Lengths in Air at 15° C and 760 mm to Reduce Wave Lengths and Frequencies to Vacuum Values

λ	$(n-1)10^7$	$(n\lambda-\lambda)$ add	$\frac{1}{\lambda}$	$\left(\frac{1}{n\lambda}-\frac{1}{\lambda}\right)$ subtract
2000	3255.82	0.6512	50 000	16.274
2050	3220.12	.6601	48 780	15.703
2100	3187.86	.6695	47 619	15.175
2150	3158.63	.6791	46 511	14.687
2200	3132.07	.6891	45 454	14.232
2250	3107.87	.6993	44 444	13.808
2300	3085.75	.7097	43 478	13.412
2350	3065.50	.7204	42 553	13.041
2400	3046.91	.7313	41 666	12.692
2450	3029.81	.7423	40 816	12.363
2500	3014.05	.7535	40 000	12.053
2550	2999.48	.7649	39 215	11.759
2600	2986.00	.7764	38 461	11.481
2650	2973.50	.7880	37 735	11.217
2700	2961.88	.7997	37 037	10.967
2750	2951.08	.8115	36 368	10.728
2800	2941.00	.8235	35 714	10.500
2850	2931.60	.8355	35 087	10.283
2900	2922.80	.8476	34 482	10.076
2950	2914.57	.8598	33 898	9.877
3000	2906.85	.8721	33 333	9.687
3050	2899.60	.8844	32 786	9.504
3100	2892.79	.8968	32 258	9.329
3150	2886.38	.9092	31 746	9.160
3200	2880.33	.9217	31 250	8.998
3250	2874.63	.9343	30 769	8.842
3300	2869.24	.9469	30 303	8.692
3350	2864.15	.9595	29 850	8.547
3400	2859.33	.9722	29 411	8.407
3450	2854.76	.9849	28 985	8.272
3500	2850.43	.9977	28 571	8.142
3550	2846.32	1.0104	28 169	8.016
3600	2842.41	1.0233	27 777	7.893
3650	2838.69	1.0361	27 397	7.774
3700	2835.16	1.0490	27 027	7.660
3750	2831.79	1.0619	26 666	7.550
3800	2828.58	1.0749	26 315	7.442
3850	2825.51	1.0878	25 974	7.337
3900	2822.59	1.1008	25 641	7.235
3950	2819.79	1.1138	25 316	7.137
4000	2817.12	1.1268	25 000	7.041
4050	2814.56	1.1399	24 691	6.948
4100	2812.11	1.1530	24 390	6.857
4150	2809.76	1.1661	24 096	6.769
4200	2807.51	1.1792	23 809	6.683
4250	2805.36	1.1923	23 529	6.599
4300	2803.29	1.2054	23 255	6.517
4350	2801.30	1.2186	22 988	6.437
4400	2799.39	1.2317	22 727	6.360
4450	2797.55	1.2449	22 471	6.285
4500	2795.78	1.2581	22 222	6.211
4550	2794.08	1.2713	21 978	6.139
4600	2792.44	1.2845	21 739	6.069
4650	2790.86	1.2978	21 505	6.000
4700	2789.34	1.3110	21 276	5.933
4750	2787.88	1.3242	21 052	5.868
4800	2786.46	1.3375	20 833	5.804
4850	2785.09	1.3508	20 618	5.741
4900	2783.78	1.3640	20 406	5.680
4950	2782.50	1.3773	20 202	5.620
5000	2781.27	1.3906	20 000	5.561
5050	2780.08	1.4039	19 801	5.503
5100	2778.93	1.4173	19 607	5.447
5150	2777.81	1.4306	19 417	5.392
5200	2776.74	1.4439	19 230	5.336

TABLE 5—Continued

λ	$(n-1)10^7$	$(n\lambda-\lambda)$ add	$\frac{1}{\lambda}$	$\left(\frac{1}{n\lambda} - \frac{1}{\lambda}\right)$ subtract
5250	2775.69	1.4572	19 047	5.285
5300	2774.68	1.4706	18 867	5.234
5350	2773.70	1.4839	18 691	5.183
5400	2772.75	1.4973	18 518	5.133
5450	2771.83	1.5106	18 348	5.084
5500	2770.94	1.5240	18 181	5.037
5550	2770.07	1.5374	18 018	4.990
5600	2769.23	1.5508	17 857	4.944
5650	2768.41	1.5642	17 699	4.899
5700	2767.62	1.5775	17 543	4.854
5750	2766.85	1.5909	17 391	4.811
5800	2766.10	1.6043	17 241	4.768
5850	2765.37	1.6177	17 094	4.726
5900	2764.66	1.6311	16 949	4.685
5950	2763.98	1.6446	16 806	4.644
6000	2763.31	1.6580	16 666	4.604
6050	2762.65	1.6714	16 528	4.565
6100	2762.02	1.6848	16 393	4.527
6150	2761.40	1.6983	16 260	4.489
6200	2760.80	1.7117	16 129	4.452
6250	2760.22	1.7251	16 000	4.415
6300	2759.65	1.7386	15 873	4.379
6350	2759.09	1.7520	15 748	4.344
6400	2758.55	1.7655	15 625	4.309
6450	2758.02	1.7789	15 503	4.275
6500	2757.51	1.7924	15 384	4.241
6550	2757.00	1.8058	15 267	4.208
6600	2756.51	1.8193	15 151	4.175
6650	2756.04	1.8328	15 037	4.143
6700	2755.57	1.8462	14 925	4.112
6750	2755.11	1.8597	14 814	4.081
6800	2754.67	1.8732	14 705	4.050
6850	2754.23	1.8866	14 598	4.020
6900	2753.81	1.9001	14 492	3.990
6950	2753.39	1.9136	14 388	3.961
7000	2752.99	1.9271	14 285	3.932
7050	2752.59	1.9406	14 184	3.903
7100	2752.21	1.9541	14 084	3.875
7150	2751.83	1.9676	13 986	3.847
7200	2751.46	1.9811	13 888	3.820
7250	2751.10	1.9945	13 793	3.793
7300	2750.74	2.0080	13 698	3.767
7350	2750.39	2.0215	13 605	3.741
7400	2750.06	2.0350	13 513	3.715
7450	2749.72	2.0485	13 422	3.690
7500	2749.40	2.0620	13 333	3.665
7550	2749.08	2.0756	13 245	3.640
7600	2748.77	2.0891	13 157	3.616
7650	2748.46	2.1026	13 071	3.592
7700	2748.17	2.1161	12 987	3.568
7750	2747.88	2.1296	12 903	3.545
7800	2747.59	2.1431	12 820	3.522
7850	2747.31	2.1566	12 738	3.499
7900	2747.03	2.1702	12 658	3.476
7950	2746.76	2.1837	12 578	3.454
8000	2746.50	2.1972	12 500	3.432
8050	2746.24	2.2107	12 422	3.410
8100	2745.99	2.2243	12 345	3.389
8150	2745.74	2.2378	12 269	3.368
8200	2745.49	2.2513	12 195	3.347
8250	2745.25	2.2648	12 121	3.326
8300	2745.02	2.2784	12 048	3.306
8350	2744.78	2.2919	11 976	3.286
8400	2744.56	2.3054	11 904	3.266
8450	2744.34	2.3190	11 834	3.246

TABLE 5—Continued

λ	$(n-1)10^6$	$(n\lambda-\lambda)$ add	$\frac{1}{\lambda}$	$\left(\frac{1}{n\lambda}-\frac{1}{\lambda}\right)$ subtract
8500	2744.12	2.3325	11 764	3.227
8550	2743.90	2.3460	11 695	3.208
8600	2743.69	2.3596	11 627	3.189
8650	2743.49	2.3731	11 560	3.170
8700	2743.28	2.3867	11 494	3.152
8750	2743.09	2.4002	11 428	3.134
8800	2742.89	2.4137	11 363	3.116
8850	2742.70	2.4273	11 299	3.099
8900	2742.51	2.4408	11 235	3.080
8950	2742.32	2.4544	11 173	3.063
9000	2742.14	2.4679	11 111	3.046
9050	2741.96	2.4815	11 049	3.029
9100	2741.79	2.4950	10 989	3.012
9150	2741.61	2.5086	10 928	2.995
9200	2741.44	2.5221	10 869	2.979
9250	2741.28	2.5357	10 810	2.963
9300	2741.11	2.5492	10 752	2.947
9350	2740.95	2.5628	10 695	2.931
9400	2740.79	2.5763	10 638	2.915
9450	2740.64	2.5899	10 582	2.899
9500	2740.48	2.6035	10 526	2.884
9550	2740.33	2.6170	10 471	2.869
9600	2740.18	2.6306	10 416	2.854
9650	2740.04	2.6441	10 362	2.839
9700	2739.89	2.6577	10 309	2.824
9750	2739.75	2.6713	10 256	2.809
9800	2739.61	2.6848	10 204	2.795
9850	2739.47	2.6984	10 152	2.780
9900	2739.34	2.7119	10 101	2.766
9950	2739.20	2.7255	10 050	2.752
10 000	2739.07	2.7391	10 000	2.738

The practical difficulties involve the relation of refractive index with density of air as depending upon pressure, temperature, and humidity, and on the necessary inattention to the dispersion of light. In practice the astronomer usually disregards all dependence of refractive index on color. This must involve errors in zenith distance observations on stars whose colors are intrinsically different and especially in observations of stars near the horizon where the short waves are largely absorbed and scattered by our atmosphere causing the stars to be represented only by longer waves. Because of these difficulties, tables of corrections for astronomical refraction are probably better determined from astronomical observations than from laboratory measures. The refraction tables which have been applied to the major portion of astronomical observations are those of Bessel founded upon the observations of Bradley and published in *Tabulae Regiomontanae*. In 1870 the Pulkowa tables were published under the title *Tabulae Refractionum in usum speculae Pulcovensis Congestae*. These are based upon the researches of Gylden⁴⁸ and are supplemented

⁴⁸ Gylden, *Mém. de l'Acad. de St. Petersburg*, 10, No. 1, 1866, and 12, No. 4, 1868.

by those of Fuss.⁴⁰ The Pulkowa tables are founded upon a smaller refraction constant than Bessel's, and Newcomb (loc. cit.) says the most recent discussions and comparisons indicate a still greater diminution in the constant. The constants under discussion represent the white light index of refraction of air at 0° C, 760 mm, as follows:⁴¹

Bessel.....	1.00029257, from Bradley.
Gylden.....	29232, from Pulkowa, 1842-1849
Fuss.....	29161, from Pulkowa, 1867-1869
Newcomb.....	29195, from Greenwich, 1877-1886
Bauschinger.....	29152, from Munchen, 1891-1893
Courvoisier.....	29180, from Heidelberg, 1900-1903

Under the same conditions our observations give about 1.0002923 assuming the wave length 5560 Å of maximum visibility for white light.

For photographic work a larger refraction constant must be used than for visual. If the photographic maximum is at 4200 Å the refraction constant should be 1.0002968 and the photographic corrections are therefore 1.0154 larger than the visual. The ratio of these corrections has been found experimentally by Wilsing⁴² to be 1.01539.

5. OPTICAL TEMPERATURE COEFFICIENT

If the Dale and Gladstone law, $\frac{n-1}{d} = \text{constant}$, holds for air, the temperature coefficient of index variation should be identical with the temperature coefficient of volume change at constant pressure and any refractivity of air at temperature t° C could be expressed in terms of the refractivity of air at 0° C by the relation

$$(n-1)_t = \frac{(n-1)_0}{1 + \alpha t}$$

in which $\alpha = 0.00367$.

The following values of the optical temperature coefficient of air show considerable disagreement: Lorenz, 0.00367; Von Lang, 0.00310; Mascart, 0.00383; Benoit, 0.00367; and Walker, 0.00360.

Our observations on the refractivity of air at temperatures of 0, 15, and 30° C show that α is a function of the wave length and increases rapidly as the absorption band in the ultra-violet is approached. This is shown in Table 6.

⁴⁰ Fuss, *ibid.*, 18, No. 4; 1866.

⁴¹ Winklemann, *Handbuch der Physik*, 6 (2), p. 534.

⁴² Wilsing, *Astronom. Nachr.*, 145, p. 293; 1898.

TABLE 6

λ	α
8500	0.003672
7500	3674
6500	3678
5500	3685
4500	3700
3500	3738
2500	3872

A decrease in the density of air due to an increase in its temperature either apparently or actually changes the position of the ultra-violet absorption band. As the temperature of the air rises the absorption and dispersion of short waves of light diminish and this effect is probably explained by the increased kinetic energy of the gas molecules rather than by the decreased density of the gas. It would be interesting to test this by ultra-violet dispersion measurements of air with density reductions at constant temperature.

Within the range of our observations the refractivity of air at any temperature for any particular wave length λ can be obtained from measurements on air at 0°C by the relation mentioned above if a correction be applied to α which is a function of λ . A very simple correction to α is contained in the expression

$$(n-1)_t = \frac{(n-1)_0}{1+t \frac{(\alpha + 3 \times 10^6)}{\lambda^2}}$$

in which $\alpha = 0.00367$; which allows the calculation of $(n-1)_t$ within the limits of probable error of observation.

6. DISPERSION FORMULAS

A large amount of theoretical work on refraction and dispersion of transparent materials has been done in the past, and various theories have been tested by measurements of indices of refraction of solids and liquids over a wide range of spectrum from the ultra-violet through the visible and far out into the infra-red. It is especially important for this purpose to measure indices of refraction in the neighborhood of absorption bands. Measurements of refractivities of gases have heretofore been confined to a comparatively small portion of the spectrum, and although more work has been done with air than with other gases the review of previous

work showed how incomplete it was. It was hoped that the present investigation of the dispersion of air for red and infra-red light would give an indication of an absorption band in the infra-red so that the Sellmeier or Ketteler-Helmholtz dispersion formula could be tested in its representation of the observations. The empirical Cauchy equation with three constants was first used to represent the measurements, and although the representation was quite satisfactory, there seemed to be a slight deviation in the red which might be explained by an absorption band in the infra-red. An effort was made to locate such a band from our dispersion measurements, but without success.

Water vapor has a very complicated absorption spectrum in the infra-red, but dry air is known to be quite transparent to long waves. The only constituent of air which seems to absorb much infra-red light is carbon dioxide. Although carbon dioxide constitutes only 0.03 per cent of the volume of ordinary air, the absorption is quite marked for wave lengths $42\,700\text{ A}^{52,53}$ and $147\,000\text{ A}^{54}$ when these waves are received through several meters of air. Rubens and Wartenburg⁵⁵ have found dry air to be quite transparent for wave lengths $230\,000\text{ A}$, $520\,000\text{ A}$, $1\,100\,000\text{ A}$, and $3\,140\,000\text{ A}$. The absorption bands due to the carbon dioxide in the air are narrow and weak compared to the absorption band of air for ultra-violet light and it seems doubtful that they can exert any marked influence on dispersion. The effect of these bands is probably similar to that of the A band due to atmospheric oxygen which strongly absorbs light of wave length 7600 A , but relatively small thicknesses of air freely transmit these waves.

Koch⁵⁶ measured the index of refraction of air as 1.00028806 for residual rays from gypsum ($\lambda = 67\,094\text{ A}$) and 1.00028875 for residual rays from calcite ($\lambda = 86\,784\text{ A}$). If no absorption band exists between 9000 A and $67\,000\text{ A}$, and if it is permissible to extrapolate the formula representing our observations, we obtain $n = 1.0002876$ for the longest waves.

It is probably necessary, therefore, to conclude that if there is any marked absorption of infra-red waves by air, the band of absorption must lie at extremely long waves.

Moreover, the electromagnetic theory has connected the index of refraction n of a medium to its dielectric constant D by the

⁵² Paschen, Wied. Ann., 58, p. 334; 1894.

⁵³ Stulescu, Phil. Mag., 80, p. 737; 1915.

⁵⁴ Rubens, Wied. Ann., 64, p. 584; 1898.

⁵⁵ Rubens and Wartenburg, Phys. Zeitschr., 12, p. 1080; 1911.

⁵⁶ Koch, Nova Acta Soc. Upsal (4), 2; 1909.

relation $n^2 = D$. This relation was derived for long waves and is not generally satisfied in the visible spectrum where all substances have more or less dispersion. For certain gases, such as hydrogen, nitrogen, and dry air, which show only small dispersion and no strong absorption in the infra-red, $n^2 = D$ approximately, even when n is taken for waves in the visible spectrum.⁶⁷ For example,⁶⁸ D for dry air is 1.000590 and n^2 for sodium light is 1.000584. The dispersion for such substances is well represented by the simple empirical formula of Cauchy

$$n = a + \frac{b}{\lambda^2} + \frac{c}{\lambda^4}$$

If n^2 is not equal to D for visible rays, the dispersion can not be represented by Cauchy's formula, but the Sellmeier or Ketteler-Helmholtz formula which takes into account the free periods corresponding to absorption bands of wave length λ_k must be used. Neglecting absorption for wave lengths λ this formula is generally written as follows:

$$n^2 = 1 + \sum \frac{m_k \lambda^2}{\lambda^2 - \lambda_k^2}$$

The effect on n of each species of ions K is represented by a term in the sum Σ in which m_k is the dielectric constant of the K ions. Then $1 + \Sigma m_k$ is the observed dielectric constant D of the substance for infinitely long waves. If there are two families of resonance ions, one producing the ultra-violet absorption band of wave length λ_v and the other responsible for the infra-red absorption band of wave length λ_r , the dispersion is represented by

$$n^2 = 1 + \frac{m_v \lambda^2}{\lambda^2 - \lambda_v^2} + \frac{m_r \lambda^2}{\lambda^2 - \lambda_r^2}$$

Miss Howell⁶⁹ gave a Sellmeier formula for air having an infra-red absorption band of wave length 31 800 Å, "probably due to traces of water vapor," but this seems very doubtful.

If there is only one family of resonance ions, the formula reduces to

$$n^2 = 1 + \frac{m_v \lambda^2}{\lambda^2 - \lambda_v^2}$$

⁶⁷ Topley, *Ann. d. Phys.* (4), 26, p. 59; 1906.

⁶⁸ Schmidt, *Ann. d. Phys.* (4), 11, p. 121; 1903.

⁶⁹ Howell, *Phys. Rev.*, 6, p. 81; 1915.

and this first term of Sellmeier's formula gives a satisfactory representation of the dispersion of all transparent substances in which the infra-red absorption is negligible.

The observational equations previously described were solved by the method of least squares and gave the following expression for the dispersion of air at 0° C and 760 mm

$$n^2 = 1 + \frac{0.00057378\lambda^2}{\lambda^2 - 595,260}$$

This simple Sellmeier dispersion formula represents the observations nearly as well as the Cauchy formula given above.

VIII. SUMMARY

A survey of previous researches on refraction of air shows that many investigators have worked either with white light or with one monochromatic radiation, and dispersion measurements have been limited to a small interval of the spectrum. No index measurements exist for waves longer than those corresponding to orange light, and in the ultra-violet the dispersion formulae disagree by more than 10 per cent of the refractivity.

Recent work in spectroscopy makes it very desirable to have more accurate and extensive data on the index of refraction and dispersion of air. The international system of standard wave lengths expresses the lengths of waves in air at 15° C and 760 mm, and all wave-length measurements made under other conditions require small corrections because of the effect of temperature and pressure of the air upon its optical dispersion. Furthermore, it is often desirable to multiply wave lengths measured in air by the indices of refraction of air for these wave lengths and thus convert them to their value in a vacuum. An accuracy of one part in several millions is now striven for in the measurement of wave lengths, and to maintain their relative accuracy in the values reduced to vacuum it is necessary to know the indices of refraction within a few units in the seventh decimal place.

For several years this Bureau has been engaged in the accurate measurement of wave lengths. Interferometer comparisons of standard wave lengths have been made throughout a large range of spectrum and the grating spectra of more than 50 of the chemical elements have been photographed and measured in the red and infra-red spectral regions. In connection with these accu-

rate measurements of wave lengths it was thought advisable to measure the absolute indices of refraction of air for the entire spectral region which is accessible to photography.

Accuracy and efficiency recommended the use of the Fabry and Perot interferometer for this work, since this apparatus can be conveniently inclosed in a chamber in which the temperature and pressure of the air can be regulated as desired, and it also permits simultaneous observations for a large number of different wave lengths. Sections of the circular fringes, produced by various radiations from a source of light illuminating the parallel plates of the interferometer, were photographed either with a grating or a rock-salt prism spectrograph, first when the space between the plates was evacuated, and then when dry air at measured temperature and pressure was present.

The index of refraction for a particular wave length was obtained directly from measurements of the photographed interference fringes which allowed the ratio of lengths of this wave in vacuum and in air to be calculated. Observations were made at spectrum intervals of about 40 Å from the extreme ultra-violet at 2200 Å, through the visible spectrum and in the infra-red to 9000 Å.

Complete sets of observations were made on dry air at atmospheric pressure and at temperatures of 0, 15, and 30° C. These are quite closely represented by the following dispersion formulas of the Cauchy form:

$$(n-1)_0 \times 10^7 = 2875.66 + \frac{13.412}{\lambda^2 \times 10^{-8}} + \frac{0.3777}{\lambda^4 \times 10^{-16}}$$

$$(n-1)_{15} \times 10^7 = 2726.43 + \frac{12.288}{\lambda^2 \times 10^{-8}} + \frac{0.3555}{\lambda^4 \times 10^{-16}}$$

$$(n-1)_{30} \times 10^7 = 2589.72 + \frac{12.259}{\lambda^2 \times 10^{-8}} + \frac{0.2576}{\lambda^4 \times 10^{-16}}$$

These observations are used in the construction of a table giving the corrections which must be applied to wave lengths measured in air whose density is not normal. A table of corrections to convert wave lengths or frequencies measured in air to their values in a vacuum is also given.

The coefficient of index variation with temperature was found from these measurements to be a function of the wave length. For long waves this optical temperature coefficient is identical with the density temperature coefficient—that is, $\frac{1}{273}$ —but as the

ultra-violet absorption band is approached it increased rapidly, becoming $\frac{1}{258}$ at 2500 Å.

There seems to be no definite evidence of any strong absorption of infra-red light by dry air and it is, therefore, possible to represent the optical dispersion of air by the first term of Sellmeier's formula quite satisfactorily.

WASHINGTON, March 13, 1918.

VARIANCE OF MEASURING INSTRUMENTS AND ITS RELATION TO ACCURACY AND SENSITIVITY

By Frederick J. Schlink

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I. INTRODUCTION: FUNDAMENTAL DEFINITIONS

The distinction between accuracy and sensitivity in engineering and physical measuring instruments is not always carefully maintained in the discussion and calibration of such instruments. In fact certain phases of the meaning of these two terms have not been clearly expounded. The fundamental concepts are, of course, clear:

(1) An instrument is accurate when its indications accord with the true values of the quantity being measured.

(2) An instrument is sensitive when a change in the quantity being measured is accompanied by a change in the indication of the instrument.

II. ACCURACY

Neither of the above definitions is of much utility until amplified and expressed quantitatively. In the first place, instrumental accuracy as the term is defined above, is only an ideal,

since it is impossible to obtain perfect exactitude in the reading of any measuring instrument on account of divers unavoidable mechanical limitations imposed by the coaction of the working parts; the existence of friction; alterations in the properties of measuring elements with time and with changes in the environment of temperature, atmospheric pressure, humidity, etc.; and normally uncontrollable variations in the manner, frequency, and rate of operation.

In the common measurements of industry and commerce sufficient accuracy is sought to permit of dispensing with the use of corrections, allowing the employment of observed readings directly, partly to eliminate errors of calculation and partly to enable such indispensable instruments as pressure gages, measuring rules, micrometer calipers, etc., to be used by persons not possessed of special technical training. In the laboratory, on the other hand, the numerical equality of the instrumental reading to the true value of the quantity being measured is not essential (except for the arithmetical convenience which results from having small values for the corrections) since the exactness of the work will usually require that the corrections of the instrument be determined numerically at relatively close intervals throughout its reading range, unless those corrections are very small. The readings of the instrument, therefore, will be corrected before conclusions are finally drawn from them. By so doing the convenience requisite in industrial or commercial instruments is sacrificed, while exactness is served.

Instrumental accuracy is usually expressed numerically, in a negative sense, by defining the error or correction for various values of the quantity being measured. The error arising from whatever source, observed in an indication of the instrument, divided by the true value of the measured quantity may be termed the relative or *specific inaccuracy* at a given reading, the negative term being justifiable on the basis of custom and ease in application. With this in mind it is seen that *specific accuracy* may be expressed as the ratio of the value of the quantity being measured to the error of the instrumental indication at that value (this of course being the reciprocal of the quantity defined above). The ratio expressing instrumental accuracy, while not of value in the ordinary use of instruments, will be of service in rating the performance of an instrument.

III. SENSITIVITY

1. DEFINITION

By definition, any instrument which shows a change of reading for any change, however great, in the quantity being measured, is sensitive or has sensitivity. Clearly, then, this term has little significance until means are available for expressing it numerically. For particular instruments, especially those read by null methods, sensitivity has often been expressed in terms of the smallest change of value of the quantity being measured capable of causing a perceptible response or change in the reading of the instrument. Expressed in this manner, the term is necessarily unscientific and loose in its significance, since differences in the observers' acuteness of perception, or personal sensitivity, as it might be called, would result in widely varying estimates of the sensitivity of the instrument; moreover, using this criterion, we should fail to distinguish between insensitiveness and sluggishness, a distinction which, as later considerations will show, is an important one.

For example, the sensitivity of scales and balances was long defined in terms of the smallest added load which would effect a perceptible change in the position of equilibrium of the beam or pointer. The current method of expressing sensitivity in terms of measured motion at the pointer tip is an improvement, and the best current practice is to define the sensitivity of a balance as the number of linear units by which the rest point is displaced—when measured at the pointer tip—for a unit change in the load on the load pan or platform. Balance makers have not commonly adopted this practice, however, and it is still customary with them in their catalogues to refer to balances as being sensitive to a given number of milligrams when no definite value of pointer motion is meant, but merely a visually perceptible one (probably somewhat less than a millimeter). For instruments of this character, even the method of basing the sensitivity on motion at the pointer tip does not seem satisfactory, since it depends upon the accident of pointer length, and, on the basis of this definition, a given balance can be endowed with almost any desired sensitivity by the use of an optical lever or other means of high magnification.¹

A better basis would be the angular deflection of the rest position of the beam per unit addition of load to the load pan, the first

¹ The author is not unmindful of the practical limitations upon the use of a high magnification with a balance or other measuring instrument of mediocre quality. This question will be discussed later in considering the factor of variance.

expressed in radians, and the second in grams. True, this manner of expression is open to an objection from the strictly practical point of view, in that no balance, and in fact few measuring instruments commonly read by the null method, are ever subjected to a deflection of so much as a radian; the unit of angle suggested is simply larger than the practical deflection, instead of being smaller, as custom might seem to demand. All actual measurements of sensitivity, however, must for other reasons be restricted to small deflections, principally because the sensitivity itself is subject to more or less regular variation over the scale of deflections; the deflection used in determining sensitivity must be small enough, therefore, that variations over that range of deflection are quite negligible.

2. PASSIVENESS DISTINGUISHED FROM INSENSITIVENESS

Certain aspects of the action of friction in introducing error into instrumental readings have been discussed before, notably in Gramberg's *Technische Messungen* and in Weinstein's *Physikalische Maasbestimmungen*.² The variance error to be discussed in the next section has its rise in friction, in that the divagation of reference points or axes from the paths or positions determined under the assumption of a perfect and ideally fitted mechanism, is an effect which would not obtain except for the resistance which the friction offers against the taking up of the minimum or most stable positions by the journals. That action, dependent though it is upon friction, is of such definite importance from another viewpoint that it will be treated separately.

Simple frictional resistance to turning or sliding, however, has a bearing upon the ordinary concepts of sensitiveness which should be mentioned in the present connection. Its effect is to retard or delay the motion of the indicating element for both increasing and decreasing values of the quantity being measured. The amount of the static friction will be roughly proportional to the restoring force in the system, so that in the usual case, as the reading increases, there will be a tendency for proportionate increase of mechanical hysteresis due to friction, and the error which it causes will be greater, the less the forces available to effect motion of the mechanism, for a unit change in the measured quantity. •

² See also the author's paper: *A Stabilized-Platform Weighing Scale of Novel Design*, Technologic Paper No. 106 of the Bureau of Standards, 1918, pp. 9-12.

The immediate effect of turning or sliding friction is to prevent response of the instrument reading to certain small changes in the measured quantity. On this account it becomes necessary to draw a distinction between sensitivity and immediacy of response. It is clear that if our first and commonly accepted definition of sensitivity be strictly adhered to, we will come upon the peculiar circumstance of an instrument having zero sensitivity whenever any finite change in the quantity being measured is accompanied by no deflection whatever of the indicating element. Nevertheless, exactly this occurs in an instrument having friction or lost motion. In fact, it is often true in instruments of less precise character that a considerable change in the measured quantity can be effected without the occurrence of any motion whatever of the indicator; this period of inaction of the indicator persists during the taking up of slack and the overcoming of the static friction of the operating parts, much as the first pull of a locomotive on a train fails to produce motion of the caboose until the slack between all the cars has been taken up, and until the static tractive resistance of all the cars has been overcome.

With this in mind, it will be admitted that the usual definition of sensitivity leads to an absurdity, in that the determination of the sensitivity of an instrument would depend upon the absolute rather than the relative magnitudes of the quantities entering into the observation, and in that it would, moreover, involve an abrupt changing over from zero sensitivity to finite sensitivity. In view of the irrational nature of any such assumption, the definition of sensitivity must be modified, thus: Sensitivity in an instrument is the rate of change in the indication of such instrument with respect to change in the quantity being measured, it being necessarily assumed for the purposes of this definition that friction and lost motion in the mechanism have been eliminated or are negligible. (A similar postulate applies to the (analogous) determination of the scale value in instruments graduated directly in the units of the quantity being measured.) We thus distinguish between instrumental passiveness (or sluggishness) and instrumental insensitiveness, a distinction which, so far as known to the writer, has not hitherto been set forth.

The factor of passiveness may then be determined by noting the smallest alteration in the quantity to be measured which will produce any change whatever in the indication of the instrument. The amount of the least alteration in the value of the measured

quantity producing instrumental response, divided by the initial value of the measured quantity, may be called the passivity of the instrument at that point.

For the present purpose, then, we must revert to the term "perceptible movement," which has properly been rejected in so far as it concerns definitions of sensitivity. This is logical on the ground that the passiveness evidenced by delayed response has been overcome so soon as any motion whatever of the indicating element has taken place. This factor when present can be observed in the complete hysteresis loop of the instrument, as measured by the length of the horizontal line which appears in the *measured quantity—indication* curve of the instrument at points of reversal of reading. (See under Backlash, *infra*.) Passiveness then is a special case of the phenomenon of variance, the factor discussed in the section immediately following. The present section has been introduced in this order on account of the intimate relationship of passiveness to the determination of sensitivity, observations of which are peculiarly subject to error in the presence of high instrumental passivity.

IV. VARIANCE

1. DEFINITION

The third important factor to determine in the calibration of a measuring instrument is that of variance, which is defined as the range, at any given value of the measured quantity, of variation in reading which may be exhibited by the instrument under repeated application of the same value of the quantity being measured, after a steady reading has been attained, the environment remaining unchanged. This quantity, which overlaps the passiveness factor defined above, may also be called the range of uncertainty of indication, in that it represents the range within which the readings of the instrument may be expected to lie when all causes of variation save those inherent in the instrument are eliminated. The specific variance or *variancy* (the same etymological distinction being maintained as heretofore) may be defined as the ratio of the range, at any given value of the measured quantity, of variation in reading which may be exhibited by the instrument under repeated application of the same value of the quantity being measured, divided by the value of the measured quantity itself, the same assumptions applying as above as to the attainment of a steady state of indication

(see Drift, *infra*) and as to the maintenance of unchanged environment. This factor has rarely been determined in tests of measuring instruments; ignoring it in their use, as is commonly done, may cause appreciable error, and it is therefore important that it be recognized or expressed.

In the case of the usual direct-reading instrument, the variance is disclosed as the displacement observed between the upward and downward branches of the hysteresis loop when the instrument is subjected to a complete cycle of operation from a lower to a higher indication, returning again to the lower indication, while plotting point by point the instrumental readings against actual values of the quantity being measured. It is obvious from the nature of the hysteresis loop and the causes underlying it that the amount of the variance will depend upon the previous history of the instrument and, specifically, upon the immediately precedent cycles of operation through which the mechanism has moved.³

The hysteresis loop for a measuring instrument exhibits many of the characteristics of that of a structural material under stress, in that the loop is the narrower the less the range of operation (corresponding, in the case of the stressed specimen, to the range of stress); and in that it is the narrower the more delicate and workmanlike the construction, somewhat as the hysteresis loop of a stressed specimen is reduced in area with homogeneity and fineness of structure.

This curve, showing the readings (or errors in the readings) of the instrument over its whole scale range, plotted against corresponding values of the quantity being measured, for increasing and decreasing values of that quantity, is a valuable and in fact indispensable index to the operating characteristics of an instrument and affords distinctive and easily interpreted information regarding defects of design and workmanship discoverable with certainty in no other way.

³ Much of the discussion of this paper relates directly only to mechanical instruments whose indications are of a reversible character; of another class are time-measuring instruments and some types of integrating instruments in which the readings can not be made to repeat or recur at will; these are not completely amenable to the present treatment, but require special consideration. That this distinction is fundamental is clear from the fact that a chronometer can not be said to have sensitivity, since time is in its very essence epochal and irreversible. These methods do, however, apply with complete validity to many devices and mechanisms not always considered to be comprised in the term "instrument," e. g., telephone receivers and transmitters and phonograph reproducers, and a large class of controlling or value-limiting mechanisms, such as thermostats, barostats, hygrometers, voltage regulators, engine governors, and carburetors for internal-combustion engines, all of which are in fact in large measure subject to perturbations arising from effects to be classed as irreversibility. This is true, moreover, of liquid-column instruments such as mercurial barometers, for even in such instruments resistance to motion does not actually vanish at zero velocity of the liquid. Imperfect reversibility in measurable amount is known to exist in mercurial barometers and other liquid manometers.

Consider the curve shown in Fig. 1, which is plotted from observations on an automatic or self-indicating weighing scale. This particular scale is one in which the load is equilibrated by

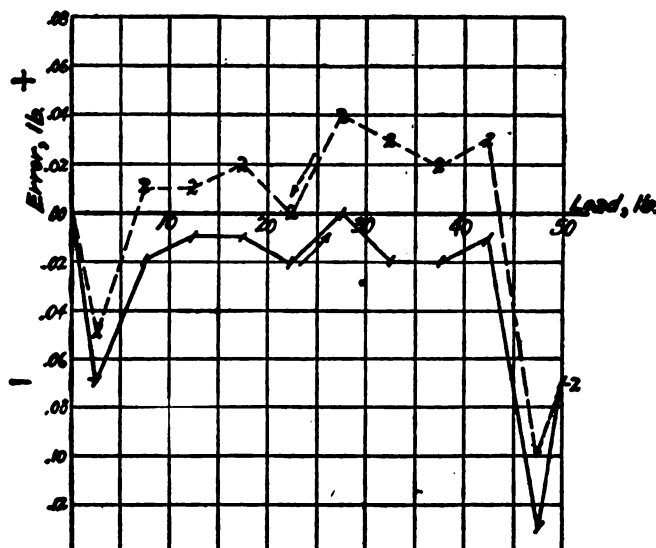


FIG. 1.—Hysteresis loop of automatic weighing scale of the cam-pendulum type (comprising no elastic force-measuring elements)

Note the manner in which the curve of decreasing readings reflects the aberrations of the curve of increasing readings, with a distinct tendency toward wider separation at the middle of the load range, so that the hysteresis loop would have a distinctly lenticular form, if the median line or mean error curve were rectilinear.

the variable turning moment of a pair of oppositely rotating pendulums mounted on ball bearings. The effect shown, therefore, is certainly not the hysteresis of inelasticity, such as would

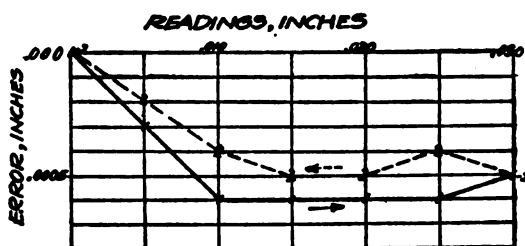


FIG. 2.—Hysteresis loop of dial indicator of the screw-train type, having high multiplication, an instrument much used for direct reading of small displacements and other linear magnitudes

be observed in the test of a spring scale, although, admittedly, elastic hysteresis is present in very small and negligible amount in every instrument, due to variations in the working stresses (in this case, in the load-sustaining parts of the scale which

transmit the forces from the platform to the pendulum). The sources of the hysteresis exhibited in the figure are manifold; though diverse in apparent character, however, all are expressible as backlash or are closely analogous to it.

Figs. 2 and 3 show other typical examples of hysteresis effects, the instrument of Fig. 2 having no source of hysteresis or inelasticity, while that of Fig. 3 does.

In one case so important was the recording of the backlash characteristic that the writer, in the inspection on a contract comprising 1200 automatic scales, found it advisable to note for each scale complete sets of readings taken at suitable intervals over the whole range of graduation for increasing and decreasing

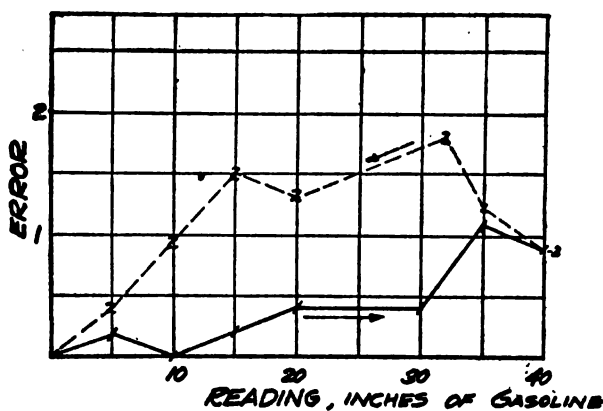


FIG. 3.—Hysteresis loop of depth gage for gasoline tank, a diaphragm type of pressure gage graduated in inches of gasoline. (Ordinates in inches of gasoline)

loads, the results of the test observations being plotted directly into graphs like that of Fig. 1. The median line of the loop so obtained gave the characteristic error of the scale, correctable by suitable alteration of the weights of the pendulums and of the contour of the cams upon which were wrapped thin metallic tapes transmitting the load from the platform system to the pendulums. The width of the hysteresis loop, moreover, gave a good index of the workmanship of pendulum bearings and connections.

It will readily be appreciated how valuable is the information which such a graph affords in evaluating the fundamental characteristics of an instrument and in indicating defects in workmanship and design.

2. TYPES OF VARIANCE

In all instruments which use a spring element for equilibrating the changes in the measured quantity directly or through conversion from displacements to forces, a source of variation exists in the imperfect elasticity of such a spring, whose changes in shape or dimensions are always to some extent irreversible. This imperfect reversibility of the stress-strain relation implies a hysteresis effect in the calibration of the instrument. Apart from this, we have the effects of imperfections in the mechanism itself, next to be discussed, as evidenced in such parts as journals, bearings, and gearing.⁴

(a) *Backlash*.—The function of turning pairs in a measuring instrument is distinctly different from that which they fulfill in a machine. In a machine the exact maintenance of the distances between centers of rotation of its elements is not ordinarily a fundamental requisite. Except in rare instances, only approximate invariability of distance relationships in the linkwork is required. In a measuring instrument, however, the whole result to be obtained depends upon the maintenance of constant or at least determinate intervals between the parts of the mechanism which transfer the forces or motions involved from the point of their reception to the point of registration or indication. This, expressed kinematically, requires that links must have a constant virtual length (if comprising only lower pairs) or determinate virtual length, constant and definite for any given configuration of the linkwork (if comprising higher pairs).

It will readily be seen that the presence of slack or backlash in the mechanism of an instrument will have the same effect on the indication as an equivalent advertent displacement of those parts of the instrument receiving the force or displacement. The term backlash is here used to imply only looseness of fit resulting in play of the coacting parts. Such irregular and uncontrolled defects in the linkwork itself are usually, in fact, magnified at the point of registration, owing to the multiplication of the motion from the receiving to the registering element.

The effect of backlash is usually assumed to be that shown in the hysteresis loop at *B* in Fig. 4, viz, a rhombus of which the two

⁴ There are, of course, other sources of lag that are evidenced as loops in the calibration curves of instruments. As typical examples may be mentioned the irreversibility in the thermal expansion of some materials, and effects more or less closely a function of time, such as temperature variations occasioned by changes in the state of stress, lag in the drainage of liquid from the walls of a tube, and the thermal lag noticeable in certain types of electrical instruments and resulting from temperature changes accompanying the flow of current.

parallel horizontal sides (magnified, it may be, by the multiplication of the mechanism) correspond exactly with the geometrical clearance in the bearing. The portion *ab* represents the phase of load addition during which no change of indication occurs, pending the taking up of the lost motion. *cd* is the corresponding phase at the beginning of the returning portion of the curve. During this phase the pin or journal is assumed to be returning to contact with the opposite contact face of the bearing.

As a matter of fact, the only defect of workmanship which could produce this type of error is typified in the view at the left in Fig. 4, in which a round pin or journal coacts with an oval bearing, the lesser diameter of the bearing being exactly equal to the diameter of the journal. In this case the clearance of the journal in its

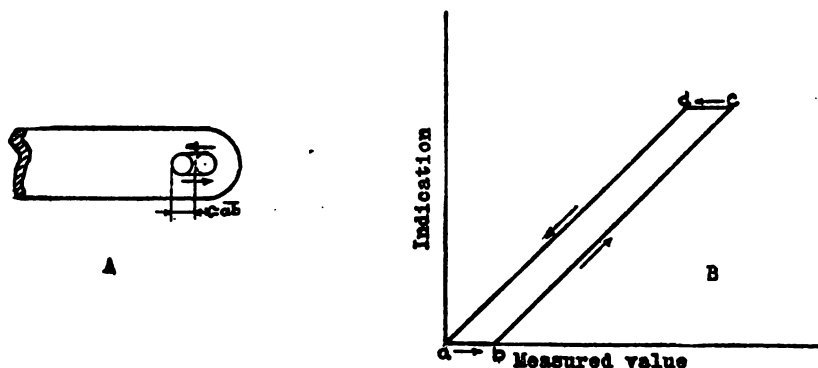


FIG. 4.—The rhomboidal type of hysteresis loop, rarely encountered in instruments. The unusual type of action of journal and bearing shown at A, or of toothed gears mating with circumferential clearance, will produce this form of calibration curve.

bearing occurs in one direction only. Such a bearing detail could occur but rarely and then only in instruments of which the bearings were badly worn; moreover, a hysteresis loop of the form shown in B could hardly occur in any rationally designed instrument since some portion of the mechanism is usually restrained by a counterweight or spring (often a hairspring at the indicator spindle) to the end of keeping the journals in contact with the same general faces of the bearing, thus providing what is known as force-closure of the linkwork. Such a hairspring can not, however, eliminate all backlash effects so long as a finite "running" clearance exists between the components of the turning pairs, as will be shown hereafter.

Were an instrument to comprise only the form of backlash just defined, a constant displacement would exist between the upward

and downward branches of its hysteresis loops, so that a fairly definite and useful correction could be applied for the backlash error, if only the phase of the instrument movement were known.

As a matter of fact, the shape of the hysteresis loop for an instrument is lenticular, and not dissimilar to that of an imperfectly elastic structural specimen under stress. (To be sure, such a specimen with an applied extensometer is itself a measuring instrument, being a spring-controlled weighing scale with, ordinarily, a stiff spring and high magnification.) The reason for this shape of the hysteresis loop will be noted by reference to Fig. 5.

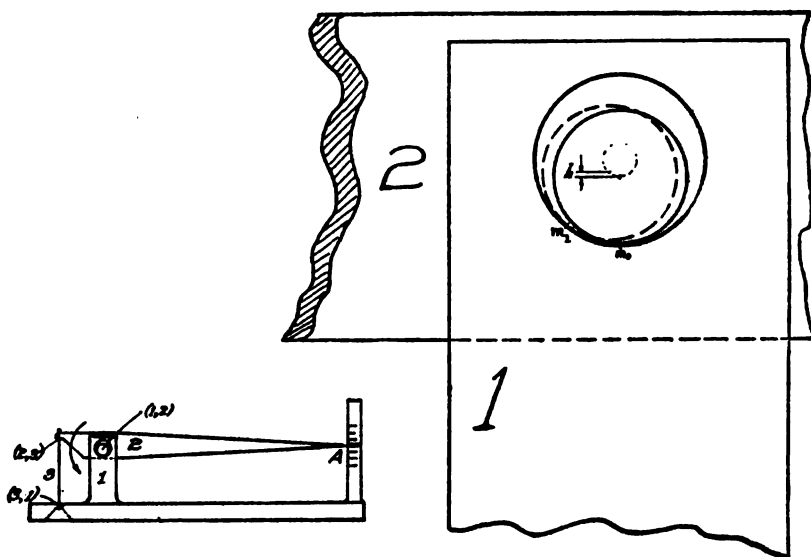


FIG. 5.—Illustrating the operation of instrumental "backlash" in introducing aberrations in the geometrical relation of journal to bearing

Let 1 and 2 be two conjoined links of the mechanism of a measuring instrument. For the sake of concreteness, we may consider them comprised in the simplified hair hygrometer shown in the diagram at the left of the figure. During the operation of the hygrometer the links 1 and 2 rotate relatively to each other about some more or less definite center $(1, 2)$. The kinematic consideration of such a jointure would, of course, be to assume perfectly circular rotation about coincident geometrical centers of the journal and bearing. For most kinematic problems this approximation is amply close. In instruments, however, as has already been pointed out, the whole performance of the device depends upon the definiteness of the geometrical configuration at a given value

of the quantity being measured. Let us therefore examine into the action taking place in a single bearing having a loose journal. As z rotates relatively to r —assuming that force-closure exists to keep the bearing and journal in contact at some point, and the hair 3 taut, as would be the case in any well-designed instrument—the journal in part rolls and in part slides around the interior of the bearing, its center describing the arc of a circle whose radius is equal to the difference in the radii of the journal and the bearing. The point $(1, 2)$ then for ordinary motions of the instrument will have a motion in this arc, which motion is superposed on the pure self-rotation which it derives from the intended operation of the mechanism. It is seen, therefore, that the motion of the pointer tip over the graduated reading scale does not bear a simple geometric relation to the lengthening or shortening of the link 3 , being in fact distorted by the complex motion at $(1, 2)$.

In the case of a horizontal spindle turning in ball bearings, the exact performance may be somewhat different from the above, although the effect is the same. In cases which have been examined, the ring of balls surrounding the spindle appears to roll as a unit with the turning of the spindle, while the spindle itself remains in contact with, and has its weight supported by, one pair of balls near the bottom of each ball race. As the ring of balls turns, due to the rolling of the individual balls in the race, the spindle is carried upward an amount depending roughly upon the radial clearance between the spindle and the interior of the ring of balls.

The effect of this divagation of the point $(1, 2)$ in introducing error in the instrument reading may be simply expressed as the variation which it produces in the leverage ratio (in this case, in the two arms of the link 2). As the pin shifts to the left and upward in its bearing, a point which is the instantaneous center of relative motion of the two links and which lies at a determinate point situate between the center of the bearing in r and the contact point m_1 , shifts in such a way as to decrease the ratio of the two arms $\frac{(3, 2) - (1, 2)}{(1, 2) - A}$. On this account point A moves increasingly faster in relation to the point $(3, 2)$. When link 2 rotates in the opposite direction, the contact point rises along the right-hand face of the bearing, increasing the ratio $\frac{(3, 2) - (1, 2)}{(1, 2) - A}$, so that A will move increasingly slower in relation to the point $(3, 2)$. This mode of action will clearly account for the general lenticular

form of calibration curve exhibited by an instrument having backlash. In the case of instruments comprising spring force-resisting elements, such as occur in pressure gages, heat-engine indicators, aneroids, and the like, a similar loop will arise from inelastic action of the spring. This effect will be combined in the calibration curve with that due to backlash.

The limiting amount by which the point of contact may be displaced from the lowermost or most stable point of the bearing could, if necessary, be approximated from a knowledge of the weights of the parts, the direction and magnitude of the other forces involved, and the coefficient of friction of the materials of the journal and the bearing.

When the maximum sidewise displacement of the center of the journal is reached, a more or less steady⁸ state of relative slipping between journal and bearing takes place, so that insofar as concerns hysteresis from journal action, it is to be expected that the backlash loop corresponding to this portion of the operating range of the instrument will be flat. The loop will, therefore, be made up of three phases: A lower lenticular portion; a middle oblong, approximately straight-sided portion; and an upper lenticular portion. This state of affairs has been approximately confirmed by experiment. It may easily be that the calibration curve will not form a completely closed loop since local roughness and variations in surface conditions of bearing and journal may operate to prevent complete reversion of the parts to their initial positions. Specifically, this may occur when the force designed to effect force-closure of the system is insufficient in amount. The amount by which the loop fails of closure may be termed the set. The relative or specific set is the ratio of this residual deflection divided by the deflection which occasioned it.

In cases in which a link typified by 3, functions as a means of transferring or modifying a motion from 2 to 4, the effect of the interior rolling of journals in bearings is to lengthen link 2 and shorten link 4 for rotation of 2 and 4 in one direction, and vice versa for rotation in the other direction, thus again acting to widen the hysteresis loop.

Clearance between the engaging teeth of gears and toothed racks which appear in many instruments introduces backlash

⁸ It may even be that a step-by-step motion of the linkwork and indicator, due to the discontinuities in the relative motion, referred to above, can be distinctly observed, typifying in a sense the point-by-point manifestation of passiveness in preventing continuous response of the instrument mechanism; and reflecting the discontinuity between the values of the static and kinetic friction of the linkwork.

effects of the same general nature as those outlined above. In many cases the backlash effect in a gear train will be superposed upon that of the journals and bearings of the train. In every case there are certain phases of motion in which such effects will be additive. In gear trains, moreover, there is especial likelihood of occurrence of the true backlash type of loop shown in Fig. 4 at *B*, since clearance at the pitch line will tend to result in actual discontinuity in the transference of motion from one part of the train to another, whenever the direction of rotation is reversed.

(b) *Irregular Variance.*—In the case of instruments which are characterized by poor workmanship or are in a state of ill repair, the hysteresis loops obtained on successive runs may be far from concordant in either shape or magnitude, this condition being of course a result of inaccurate fitting of such serious order that variations in the friction and journal displacement, even for a par-

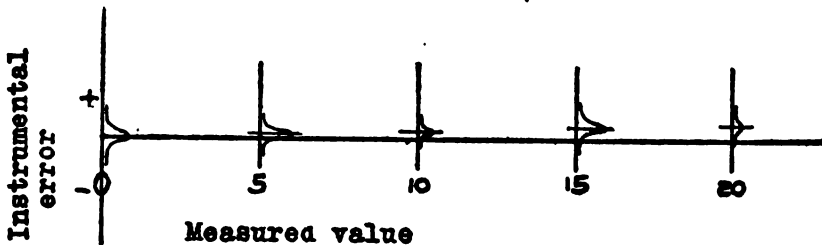


FIG. 6.—Illustrating the representation of irregular variance by reference to a family of probability curves developed at various points along the instrumental scale

ticular indication, become significant. In the case of instruments of this kind, the variance may be well expressed and defined by plotting frequency curves of the readings or errors obtained for a given value (or for a series of definite values) of the measured quantity repeatedly applied.

One series of points will be plotted for readings taken at varying rates and amounts of increase of the measured quantity terminating in the value corresponding to the particular point of the scale under investigation, and another set for decreasing values terminating at the same point. Such a curve, in which frequency of occurrence of a particular reading or error is plotted against the true value of the measured quantity at that reading, gives the probability of occurrence of any amount of variation from, say, the mean instrumental reading.

A series of such frequency curves taken at various parts of the reading range may be plotted as shown in Fig. 6, in which case

the surface which may be conceived of as enveloping the family of probability curves when they are thrown into three dimensions, will enable one to determine the probability of a given error at any point along the reading scale. The principal utility of such a graph, however, will be in the criterion which it affords as to the comparative performance of instruments of diverse design or workmanship.

Lest it be felt that the importance of the hysteresis error due to bearing clearance and backlash has in the foregoing been overestimated, the following should be given consideration. It may well be true that in some instruments, especially those of the least mechanical complexity, the error is small; in other cases it may have an appreciable value and yet be of minor importance as compared with the hysteresis due to imperfect elasticity of elastic resistor elements and as compared with the drift or elastic after-effects, which also frequently accompany spring force-resisting elements. Nevertheless, in many instruments the backlash type of error arising in purely mechanistic sources is the one which governs the practicability of the desired accuracy and sensitivity. As examples, automatic or self-indicating weighing scales and balances, especially those using the principle of pendulum inclination, may again be called to mind. Another important class of instruments, in which the backlash effect is the limiting and preponderant factor, is that of length-measuring instruments commonly employed for measuring small differences of displacement. Examples of these are found in the dial-and-plunger instruments which use wheel or screw trains of high multiplication to convert rectilinear displacements at the receiving point into rotations of an indicator over a graduated dial. Were it not for the variance effects of mechanical hysteresis in these instruments they would obviously be applicable to the most refined length comparisons. In point of fact, one of the types most used, due to unavoidable deficiencies in the fit of the wheelwork spindles and meshing gears, shows variance differences amounting to several graduations, at a given displacement of the receiving point, the dial of this instrument being divided to $\frac{1}{10\,000}$ inch of plunger displacement. This, then, is an example of an instrument in which variance errors have been the limiting factor; the extremely high magnification and considerable range of reading, with the necessarily large number of coacting parts required thereby, makes their minimization a very difficult matter.

Almost without exception, the complete calibration of measuring instruments of nearly every type, not excepting those using a liquid meniscus as indicating element, should include a determination of the variance characteristics. The magnitude of the variation shown in this determination will indicate whether or not it will be worth while to analyze the hysteresis effect into the factors which compose it, viz, the backlash effect, the imperfect elasticity effect, and the drift or elastic aftereffect.

(c) *Drift*.—Another phenomenon of considerable importance in some types of instrument is that of drift, which is a time effect characterized by a more or less gradual movement of the indicator, asymptotically to a definite reading, after all conditions external to the instrument have become constant. It would seem that very little of this effect can arise from causes originating in the kinematic relations of the linkwork; it must be due chiefly to elastic aftereffect or the hysteresis of inelasticity in stressed portions of the measuring elements. If we accept, as applying to stresses other than torsional, the conclusions of J. J. Guest and F. C. Lea in their recent paper ⁶ on hysteresis phenomena, we may ascribe all such effects to conditions of actual overstrain in imperfectly elastic force-measuring or force-sustaining parts. An example is seen in the aneroid barometer, where drift takes place as a result of slow settling of the diaphragm or spring to a steady state of stress. It is quite reasonable to suppose that in corrugated diaphragms, such as are used for this purpose, zones of decided overstrain do exist.

3. MEANS FOR REDUCING VARIANCE

(a) *Details of Design*.—Several possible remedies are to be considered for minimizing the hysteresis loop of measuring instruments arising from the causes described. The most obvious one is, of course, to improve the fit of journals in their bearings to such an extent that the bearing clearance is negligible. This procedure is oftentimes mechanically impracticable for several reasons: First, that when an ideally close fit is had, the journals and their bearings must have cross sections which are perfectly circular, since otherwise any relative rotation would result in binding; second, slight differences in the temperature of the parts or in the condition of the lubricant, or the introduction of dust in the use of the instrument, would increase the friction to an

⁶ J. J. Guest and F. C. Lea, Torsional Hysteresis of Mild Steel, Proc. Royal Soc., June 1, 1917.

amount fatal to proper operation. Perhaps the best type of bearing to reduce these difficulties is the conical pivot, consisting of two oppositely directed cones engaging in suitably supported conical hollows, both the cone and its mating depression being quite sharp, and suited to each other as to included angle.* This requires that the conical depression shall include a somewhat larger angle than the cone, in order that complete contact occur only at a point or infinitesimal surface in the axis of rotation. This is the type of bearing commonly used for the balance wheels of alarm clocks and low-priced watches. While having relatively low friction and maintaining a practically invariable relation of bearing to journal, it can not, of course, withstand any considerable load.

Another method, which looks very promising in the limited application it has had in weighing scales and a very few other

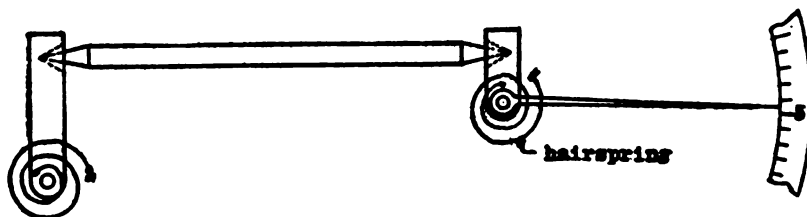


FIG. 7.—A simple type of connector having low frictional resistance and affording satisfactory constancy in distance relationships

measuring instruments, is the use of a flexible or ribbon-like elastic connector, secured to the conjoined members by means of clamps or equivalent device. With such a connector, the constancy of distance between centers of relative rotation for a given configuration of the system is assured in a practically perfect manner, neglecting temperature and similar effects which are common to all types of bearings. The only contribution of this type of connector to the total hysteresis of the instrument lies in its own elastic hysteresis, which should be very small indeed for very thin bands or ribbons of steel or similarly favorable material, operated as they commonly are, through but small or moderate arcs of motion.

Knife-edges in V-shaped bearings or single cone points and cups arranged as shown in Fig. 7 may be used if the shape and position of the bearing and edges are such as to insure the contact remaining always at the intended point. It is difficult to see, however, where this arrangement can offer any important advan-

tage over the flexible connector just described, except in the case of precision instruments for use in the laboratory, where the most careful operation and handling can be assured.

The flexible connector has another important property which has been utilized to a considerable extent in the design of automatic scales, viz, that a correction of the motion of the parts to obtain a linear or uniform scale of graduations can readily be secured by arranging that one or both ends of the tape wind upon a cam of suitable contour, a feature whose importance will readily be appreciated. It does not seem that the use of the flexible connector in such instruments as pressure gages (including aneroids), recording thermometers, hygrometers, tachometers, and the like has been as extensive as the favorable properties of the device warrant, and a marked improvement in the constancy or reproducibility of reading of such instruments should be secured by employment of this arrangement. Critical examination of the usual pressure gages and tachometers, as well as complex recording instruments generally, will convince one that the importance of special care in the design of the turning pairs in the linkwork has not been commonly appreciated, and it does not seem that the best practicable performance of such mechanisms has been approximated.

In connection with the pressure-measuring element familiar in the aneroid, an important development has been noted in certain recording instruments, and is now being manufactured for use in aviation instruments of the indicating type, its object being to eliminate gearing and jointed linkwork. Instead of the common diaphragm or monoplanar Bourdon tube of small movement, a helically coiled Bourdon tube is used, connected at one end to the fixed frame of the instrument, and at the other, through a temperature compensator which need not be described, to the indicator spindle, which latter is concentric with the helix. As pressure is applied, the Bourdon tube tends to uncoil, and drives the coaxial pointer directly, without the need for gear or link trains of any kind. A considerable arc of pointer movement can easily be obtained in this way without complication, and it would seem, with very satisfactory reliability. The importance of improvement in this field is appreciated when it is known how extensively the compact and inexpensive pressure-diaphragm construction has recently been applied in the development of every variety of aviation instrument, in which use, of course, the direct, pointer-reading type is to all intents and purposes a necessity as against

the liquid-column manometer. The diaphragm capsule and the Bourdon tube are now employed in thermometers, air-speed indicators, altimeters, liquid-depth gauges, and in many other applications apart from the simple determination of static pressure.

(b) *Effects of Vibration.*—With the considerations developed in the foregoing in mind, it is easy to see how the effects of vibration treatment suitably applied to an instrument at the time of reading will tend to correct the errors introduced by friction and backlash. It is a well-known fact that with certain instruments, such as Bourdon tube or diaphragm pressure gauges, the reading tends to settle toward a more reliable and reproducible value if the instrument is tapped or jarred. The interaction of parts which brings this about will be perfectly in line with the facts developed above. Briefly the effect is of two related sorts: (1) The minimization of static friction by the momentary disengagement or separation of coacting bearings and journals, and (2) the opportunity which the same disengagement affords, for journals to settle to the "minimum" or base point of contact in their bearings. On the basis of this reasoning it would seem that the best results from vibration treatment before a reading is taken will ensue when the vibration starts with a relatively large amplitude and decreases finally to practically zero amplitude, since the slight shifting about the optimum indication, due to the amplitude of the vibration itself, is thus eliminated. In brief, the effect of vibration is to permit the forces which tend toward a definite, unique, equilibrium point and are inherent within the instrument, to operate against the minimum possible resistances in moving the parts toward that equilibrium; in another sense, energy is applied from without the system to replace that unavoidably lost in the deficiencies of the mechanism. On this point an analogy with magnetic hysteresis exists. A substance is most accurately brought to a given state of magnetization by subjecting it to a field whose intensity oscillates with decreasing amplitude about that (or corresponding) value of magnetization.

A priori, it would seem that by vibration treatment judiciously applied at the time a reading is taken, errors of result due to the mechanical sources of variance can be almost completely eliminated.

4. RELATION OF VARIANCE TO USEFUL SENSITIVITY

It is often found that particular measuring instruments are given a sensitivity far higher than warranted in the face of the error obtainable in reading and resulting from the variance present.

Similarly, the graduation of instruments is often found to be far closer than the large amount of the variance justifies. Care should be taken in the design of measuring instruments that the units of graduation and the openness of the scale are not out of all proportion to the effective reproducibility of reading possible. For testing or laboratory instruments, the mean interval of graduation should not be less than five times the mean variance, since it is to be expected that observations will be noted accurately to one-fifth the smallest graduation or less. For commercial instruments the unit of graduation and the variance may be more nearly equal, say in the ratio of 2 to 1.

It is obviously misleading and absurd to graduate a tachometer to a single mile per hour or revolution per minute when the reading at a given true speed may vary as much as 5 miles per hour or revolutions per minute. Such inconsistencies, however, are very frequent in practice, as is the related but less serious one of having a needlessly open scale on a very variant instrument.

Unnecessary time and care in taking readings are required when the sensitivity is ill-proportioned to the variance. Instruments are not common which possess such refinement of workmanship that the characteristics of successive hysteresis loops are uniform enough to permit of practical correction for the variance of observed readings. For these reasons the sensitivity may easily be disadvantageously high, in tending to induce an erroneous estimate of the precision of results, and if this mistake does not occur, in requiring careful investigation of the particular instrument involved to determine the portion of its maximum sensitivity which is really and practically available in service.

Other factors may well have a part in determining the sensitivity to be aimed at in a given instance, for example: In an equal arm balance it is preferable to use a relatively low inherent sensitivity, enhancing the visibility of changes in the deflection of the balance beam by the use of an optical lever, microscope and scale, or similar external means. In this particular case the advantages gained are (1) decreased period of oscillation, which reduces the drift and other errors flowing from the protraction of the observations over a long period of time; and (2) the greater ease of manipulation and control of a balance having low inherent sensitivity.

The factors of inaccuracy (or accuracy), sensitivity, specific set and variance may, as a convenient means of arriving at and expressing the "figure of merit" of an instrument, be referred to

the total range of graduation instead of to the value of the measured quantity under observation, as presented in the definitions of the foregoing pages. For example, it will be of service to express the maximum set or maximum variance observed in the reading of an instrument in terms of its ratio to the total range of values represented in the graduation or use of the instrument, in order to arrive at a single significant number representing a measure of utility or merit of the instrument with regard to the particular property in question. In like manner the reciprocal of any of these quantities (that is, range of graduation divided by maximum set or maximum variance, etc.) may be similarly employed.

V. SUMMARY

The terms accuracy, sensitiveness, and variance in reference to the characteristics of a measuring instrument are most useful when quantitatively defined. In practice the first is best expressed in terms of its reciprocal, as the ratio of error, arising from whatever cause, observed in an indication of the instrument, divided by the true value of the quantity being measured, this value being the *specific inaccuracy* at that reading. This factor is useful in rating the performance of an instrument, while the absolute error observed is used as a means of correcting observations for use in calculation.

Sensitivity is the rate of displacement of the indicating element with respect to change of the measured quantity. The mode of expressing sensitivity should be definite and not dependent upon the observer's personal judgment; moreover, when possible it should be so chosen as to be independent of the accident of dimensions in the indicating element where such dimensions or the final magnification of indicator movement are within reasonable limits alterable at will.

The effect of passiveness or sluggishness in the action of instruments is to be clearly distinguished from insensitiveness. Emphasis is laid upon the fact that sensitivity can not be determined by direct measurement unless friction and lost motion are sensibly eliminated or are negligible. The amount of the passiveness at any point of the instrumental scale is measured by the smallest alteration in the quantity to be measured which will produce any change whatever in the indication of the instrument. The relative sluggishness or *passivity* at any point is that change of value of the measured quantity which effects the first perceptible re-

sponse in the indication of the instrument, divided by the initial value of the measured quantity.

The term variance is to express all changes of indication intrinsic within the instrument itself, and not per se indicative of change in the measured quantity. It is defined as the range, at any given value of the measured quantity, of variation in reading which may be exhibited by the instrument under repeated application of the same value of the quantity being measured, after a steady reading has been attained, the environment remaining unchanged. The specific variance or *variancy* is the ratio of the range of variation above defined divided by the value of the measured quantity itself.

Variance arises in three causes of mechanistic character: (1) Backlash, the operation of which in producing variance effects is fully discussed in the complete paper, and (2) friction, the primary manifestation of the latter being passiveness, discussed above. Moreover, (3) in all instruments using any of the various forms of springs as the force-resisting or restoring element, variance arises in the imperfect elasticity of that spring; this source of instrumental hysteresis is a direct reflection of the hysteresis loop in the stress-strain relation of the material composing the spring. The variance, which embodies all of the foregoing sources of variation from unique reading for a given value of the quantity to be measured, represents the range of uncertainty of indication exclusive of the factor of drift, which, being a time effect, demands special consideration. The latter effect, however, is peculiar to instruments in which a considerable range of stress in the parts occurs during their operation, or in which the elastic properties of the elements of the mechanism are unfavorable. The set of an instrument is the amount by which the indicator fails to return to its initial position after a deflection has occurred. The relative or *specific set* is the ratio of this residual deflection divided by the deflection which occasioned it.

The errors manifested as variance are often of great importance and should be examined into before they are assumed to be negligible, as they occur in all types of instruments and often have the effect of delimiting the field of useful application of the instruments.

The variance of an instrument of good characteristics can be defined by reference to the hysteresis loop obtained through cyclic variation of the measured quantity, against the values of which quantity are plotted the corresponding readings (or errors) of the

instrument. In other cases the variance can be expressed by reference to a family of probability curves giving the frequency of occurrence of each particular reading (or error) for a given value of the measured quantity (or, conversely, giving the frequency of particular deviations of the measured quantity for a given reading).

Calibration curves of typical instruments are presented to show the character of the hysteresis loops and the nature and amount of variance errors as actually determined.

Instrumental variance may be reduced by simple changes in details or simplification of design, the former relating especially to the points of jointure in the link work. Important factors in the design of such connections have failed of recognition in the development of many common instruments. Vibration of some kinds of instruments at the time the reading is taken will reduce the variance on account of its effect in minimizing the static friction opposing the motion of the parts of the mechanism to the position of equilibrium.

The amount of variance determined for an instrument should establish the optimum sensitivity to be sought in its adjustment, and also, roughly, the scale interval. High sensitivity or needlessly minute graduation, when accompanied by high variance, are likely to be both uneconomical and misleading. A working basis for determining the sensitiveness to be sought in the adjustment of the instrument, and the closeness of graduation can readily be established.¹

WASHINGTON, February 9, 1918.

¹ Apr. 15, 1919. Other papers by the present author discussing the subject of instrument variability, which have appeared since the foregoing was printed as a separate, are: *The Determinateness of the Hysteresis of Indicating Instruments* (Journ. Wash. Acad., Vol. 9, No. 2, Jan. 19, 1919) and *The Concept of Resilience with Respect to Indicating Instruments* (Journ. Frank. Inst., February, 1919).

The principal modifications and extensions of the treatment of the present paper are with respect to the exact specification of the conditions under which hysteresis loops in instrument calibration are to be taken in order to obtain concordant results; the demonstration of the high order of reproducibility of the hysteresis loops observed under such conditions; and the principle of rating an indicating instrument on the basis of the smallness of area of the hysteresis loop corresponding to a specified range of operation, which affords a numerical measure of the perfection of energy restoration or resiliency of the instrument, and, hence, of the invariability or consistency of its indications when it is applied to unregularized, acyclic use.

On page 747 after the word "for" in the third line of the last paragraph, understand the words "slowly and aperiodically." For reasons, consult second reference above. Referring to page 755, "(b) Irregular Variance," this method now appears particularly applicable to integrating instruments, since indicating instruments even of the crudest sorts, recent results show, can be so operated as to give very concordant successive calibrations. The probability curves of Fig. 6, instead of being slowly asymptotic to the several vertical axes, should meet those axes a short and definite distance from the respective normals of the curves.

MEASUREMENTS OF WAVE LENGTHS IN THE SPECTRUM OF NEON

By Keivin Burns, W. F. Meggers, and Paul W. Merrill

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I. INTRODUCTION

For several years past the Bureau of Standards has made use of the neon spectrum in various researches with the Fabry and Perot interferometer. As a result a considerable number of direct comparisons of neon wave lengths with the primary cadmium standard, as well as a large number of relative determinations of neon wave lengths, have accumulated. Although much additional work on the neon spectrum as a whole is desirable, our results with the interferometer are fairly definitive for the wave lengths of 55 of the stronger lines, and it is thought advisable to publish these without further delay.

Wave-length measurements in the spectrum of neon are of special interest and importance for several reasons. The ordinary or primary spectrum of neon consists, roughly, of three groups of strong lines in addition to a larger number of more or less evenly distributed faint lines. Two of these groups of strong lines lie in the ultra-violet, one between 2900 A and 3100 A, and the other between 3300 A and 3700 A. The third group lies in the visible yellow and red between 5800 A and 7500 A. This group contains more than 30 strong lines whose intensities are much greater than those of lines in the other two groups. The great strength of these lines gives the neon spectrum importance as a source of standard wave-lengths. The radiations are quite homogeneous and the wave lengths can therefore be measured with considerable accuracy by means of the interferometer. These radiations are so homo-

geneous that they produce interference fringes even when the retardation or path difference of the interfering beams exceeds 300 000 waves.¹

The great accuracy in wave-length determinations, to which these radiations are susceptible, adds interest to the subject of regularities in the spectrum of neon. Watson² was the first to observe that most of the strong lines in the neon spectrum could be placed in groups of three or four whose frequency differences seemed to be constant. Accurate measurements of wave lengths permit a test of the actual constancy of these frequency differences and bring out a remarkable set of spectral regularities.

The first neon wave-length measurements by means of an interferometer were made by Priest,³ who compared the wave length 5852 Å with the fundamental spectroscopic standard, the red radiation of cadmium 6438 Å. Later he⁴ determined the values of 9 other neon wave lengths relative to that of the yellow line 5852 Å of neon. Other determinations relative to Priest's value of 5852 Å were made by Meggers⁵ and by Perard.⁶ A direct comparison of the neon line 6402 Å with the cadmium line 6438 Å was made by Takamine.⁷ Further measurements were made by Meissner⁸ but it has been impossible for us to obtain a copy of the publication in which the details of this work appear. Some of Meissner's results are given in another paper,⁹ in which frequency differences are discussed.

II. APPARATUS

Two fused quartz tubes containing neon gas at a pressure equivalent to that of several millimeters of mercury were used in these investigations. One of the tubes has been described elsewhere in this Bulletin.¹⁰ This identical tube has done much service during the past three years and is still in excellent condition.

The cadmium tubes were of the H type used by Michelson and others. A few of these tubes were made of fused quartz and the remainder were made of glass. They contained a few milligrams of metallic cadmium which was vaporized when the tubes were heated to 320° C in a small brass oven.

Both neon and cadmium tubes were caused to emit light by passing a high-tension alternating current through them. The

¹ Fabry and Buisson, *Jour. de Phys.*, (5), 8 p. 442; 1912.

² Watson, *Astroph. J.*, 88, p. 399; 1912.

³ Priest, this Bulletin, 6, p. 573; 1912.

⁴ *Ibid.*, 8, p. 539; 1912.

⁵ Meggers, this Bulletin, 12, p. 302; 1915.

⁶ Perard, *Comptes Rendus* 154, p. 1798; 1912.

⁷ Takamine, *Proc. Tokio Math.-Phys. Soc.*, (2) 8, p. 9; 1915.

⁸ Meissner, *Ann. d. Phys.*, 51, p. 115; 1926.

⁹ Meissner, *Phys. Zeitschr.*, 17, p. 549; 1926.

¹⁰ This Bulletin, 13, p. 302; 1915.

60-cycle commercial current was transformed from 110 volts to about 10 000 volts. From 0.5 to 3.0 amperes was used on the primary causing about 3 to 20 milliamperes to pass through the tubes.

The Fabry and Perot type of interferometer, consisting of two partially reflecting plane surfaces held a few millimeters apart and exactly parallel by an invar separator or etalon, was used for all of our interferometer measurements of wave lengths. Each radiation of the neon light passing through the interferometer formed circular fringes which were brought to a focus in the same plane by an achromatic lens. In order to separate these superposed images, they were projected on the slit of a prism or grating spectrograph, portions of them accordingly being photographed as transverse bars across the spectral lines. The instruments employed have already been described in this Bulletin.¹¹

III. MEASUREMENT OF WAVE LENGTHS IN THE SPECTRUM OF NEON

1. INTERFEROMETER MEASUREMENTS

The neon wave lengths which were measured with the aid of interferometers are given in Table 1. This table contains the results of six different sets of wave-length comparisons. The first three sets comprise measurements of neon wave lengths in terms of the cadmium standard $\lambda = 6438.4696$ Å, and the last three sets represent determinations of neon wave lengths relative to each other. Each set of measurements will be referred to by column number in the following summary of important particulars relating to each:

Column 1 represents a large number of neon waves whose lengths were measured directly in terms of the fundamental standard $\lambda = 6438.4696$ Å. The interferometer plates used for these measurements were the identical nickered quartz plates used by Fabry and Buisson¹² in their measurement of secondary standards in the spectrum of the iron arc, and were also used by Burns¹³ for the same purpose. Separations of 2, 7.5, 10, 15, 20, and 25 mm were used for the interferometer plates in this work on the neon spectrum.

The results in column 2 were obtained with interferometers having quartz plates with thin nickel films and those in column 3

¹¹ This Bulletin, 18, p. 245, 1916; and 14, p. 159, 1917.

¹² Fabry and Buisson, *Astroph. J.*, 28, p. 169; 1908.

¹³ Burns, this Bulletin, 18, p. 179; 1915.

were derived from interferometers whose plates were of glass covered with thin copper films. These plates and films were the same ones used by Merrill¹⁴ in measurements of wave lengths in the helium spectrum. Etalons of 5, 10, 25, and 40 mm were used for column 2 and the same etalons were again used for column 3.

For column 1, the practice was to take photographs of the neon spectrum both before and after an exposure to the cadmium spectrum. The interferometers were used at room temperatures and in many cases slight changes in the etalons were shown by small differences in the diameters of the fringes photographed in the first and last exposures. The mean of both neon measurements was compared with the cadmium measurements and it was hoped that the effect of changes in the interferometer would be eliminated by this procedure. In columns 2 and 3 the neon and cadmium spectra were photographed simultaneously upon the same plate to avoid the difficulty of temperature changes in the interferometer.

Column 4 contains measurements of neon wave lengths originally made in terms of Priest's value for the yellow neon line, viz, 5852.4862 Å. This work was done with the same plates and films used for the measurements in column 1, and was in fact undertaken to determine the phase change correction of these films for Burns's determinations of secondary standards in the iron spectrum mentioned above. The values given in column 4 were recalculated in terms of 5852.488 Å as the standard.

The neon spectrum was recently made use of at this Bureau in some work on the refractive index of air, and the relative wave-length determinations in column 5 are incidental to this work. For the wave-length reductions the values 6096.1629 Å, 6334.4280 Å, 6929.4677 Å, and 7032.4129 Å were assumed as standards and the correction for dispersion of phase change at reflection was avoided by using differences of large and small orders of interference, as explained elsewhere in this Bulletin.¹⁵ The four assumed standards above, and those mentioned later, are derived by taking the weighted means of columns 2 and 3 in Table 1. Etalons of 3.75, 7.5, and 25 mm were used. The relative phase change corrections for the other five columns were actually computed from wave-length measurements with large and small interferometers, as previously described.¹⁶

¹⁴ Merrill, this Bulletin, 14, p. 159; 1917.

¹⁵ This Bulletin, 14, p. 163; 1917.

¹⁶ This Bulletin, 18, p. 245; 1916.

The values for red and infra-red lines contained in column 6 were determined relative to the following values assumed as standards:

6506.5281
6532.8826
6598.9528
6678.2762
6717.0428

The glass plates with copper films which were used for 3 and 5, were also used for 6. Separations of 3, 10, and 25 mm were used in the interferometers.

TABLE 1.—Interferometer Measurements of Wave Lengths in the Neon Spectrum

I	λ Adopted	1	2	3	4	5	6	Meissner	Priest
5	3369.904	904			904				
6	3417.906	9059			907				
6	3447.705	7053			707				
6	3454.197	1970			199				
5	3460.526	526			534				
4	3464.340	340							
5	3466.581	5814			588				
6	3472.578	5783			574				
4	3498.067	0668							
4	3501.218	2182			221				
5	3515.192	1921			197				
8	3520.474	4737			475				
4	3593.526	526							
4	3593.634	634							
5	3600.170	170							
5	3638.664	664							
8	5330.779	779							
7	5341.096	096							
6	5400.562	5620							
4	5764.419	419							
2	5820.155	155							
10	5852.488	4879	4885	4877	488	4882		488	4862
6	5881.895	8952	8960	8951	895			896	8958
8	5944.834	8344	8347	8339	835	8940		834	8944
4	5975.534	5345		5333	534			534	
4	6029.997	9972		9969				999	
7	6074.336	3872	3884	3374	337	3387		387	3383
8	6096.163	1629	1634	1627	161	1630		163	1608
9	6143.062	0618	0632	0621	068	0626		061	0600
5	6163.594	5944	596	5929				594	
4	6217.280	2818	280	2804	280			279	
7	6266.495	4951	4950	4949	496	4964		495	4948
4	6304.789	7888	790	7890	789			788	7929
8	6334.428	4279	4280	4282		4280		428	
8	6382.991	9918	9908	9914	991	9912		991	9882
10	6402.245	2441	2452	2471		2456		246	2392
9	6506.528	5270	5285	5279	534	5272	5281	527	
4	6532.883	8829	883	8823	885	8844	8810	881	
5	6598.953	9528	953	9527	953	9540	9525	953	
8	6678.276	2758	2755	2766	276	2752	2767	275	
5	6717.043	0428	0423	0430	044	0464	0444	042	
8	6929.468	4675	4683	4677	470	4675	4678	465	
3	7024.049						0486		
9	7032.413	4132	4136	4128	415	4111	4118	410	
3	7059.111						111		

TABLE 1.—Interferometer Measurements of Wave Lengths in the Neon Spectrum—Continued

I	λ Adopted	1	2	3	4	5	6	Meissner	Priest
5	7173.939					937	9400	938	
8	7245.167	169				162	1688	165	
6	7438.902					903	9016	885	
5	7488.885						885		
5	7535.784						784		
3	7544.050						050		
4	8136.408						408		
4	8300.369						369		
8	8377.606						606		
4	8495.380						380		

The column λ in Table 1 gives, for the most part, the mean of our interferometer comparisons of neon wave lengths with the cadmium standard. The values of 12 long waves are given relative to shorter neon waves. These final results are given to three decimal places and are preceded by a column indicating the relative intensities of the spectral lines.

The relative accuracy of the values in this column may be judged by considering columns 1 and 6. If the values in these columns are given to four decimal places, the line has been well observed and the wave length is probably correct to one or two thousandths of an angstrom. When the value is given to only three places in column 1 or 6, the line is not so well observed, either on account of its being too faint or too close to another line for accurate measurement. The constant differences, discussed below, indicate that the accuracy obtained in measuring of the very faint lines is still considerable. The line 3369 is a very close double; this line and the pair at 3593 Å may be in error by as much as 0.01 Å. All the other lines are probably correct to within four thousandths of one angstrom.

Meissner's values are in remarkably close accord with ours except for one or two lines. The same can be said of Priest's values. The largest discrepancy occurs in the wave length 6402 Å, which is the strongest line in the neon spectrum. Priest's value, 6402.2392 Å, was nearly reproduced by Takamine¹⁷ who, by direct comparison with the cadmium standard 6438.4696 Å, found 6402.2395 Å for the neon wave length. Our value is about one part in a million larger and a similar result is also given by Meissner. The reason for this disagreement is not apparent.

¹⁷ Takamine, Proc. Tokyo Math.-Phys. Soc., 8, p. 9; 1915.

2. GRATING MEASUREMENTS

In addition to the stronger lines in the neon spectrum which were measured by interferometer methods, many faint lines in the red and infra-red spectral regions were measured from photographs obtained with a concave diffraction grating. A description of the grating appears elsewhere in this Bulletin.¹⁸

Plates stained with pinacyanol or dicyanin were used in this spectrum photography. Some of the wave lengths were measured in terms of the interferometer values of neon lines and the remainder were obtained from secondary standards in the spectrum of the iron arc. More work of this kind is desirable but it is not feasible for us at present. The results thus far secured are therefore presented in Table 2 in this paper. The wave lengths are given to hundredths of an angstrom and are probably correct in most cases to 2 or 3 units in the last place. Most of the lines in Table 2 are so faint compared with those in Table 1 that their intensities have been indicated by zero.

TABLE 2.—Grating Measurements of Wave Lengths in the Neon Spectrum

λ	I	λ	I	λ	I	λ	I
5343.22	4	5974.65	1	6272.97	1	7472.36	2
5689.60	1	5982.50	0	6275.98	0	7937.02	1
5719.21	1	5987.90	2	6293.68	1	7943.18	3
5746.30	2	5991.63	1	6313.61	1	8082.45	3
5760.58	1	6000.90	0	6328.14	2	8118.51	1
5804.44	2	6046.08	0	6330.87	1	8128.84	0
5811.42	1	6064.50	0	6351.75	0	8259.32	1
5816.57	0	6117.96	1	6364.92	0	8265.96	2
5828.92	0	6128.44	3	6409.67	1	8267.20	0
5868.35	0	6150.24	1	6421.66	1	8365.72	0
5872.19	0	6156.06	0	6444.68	1	8376.99	0
5872.85	2	6174.93	1	6602.75	0	8418.38	2
5902.48	2	6182.13	1	6639.94	0	8591.24	2
5906.44	1	6189.02	0	6652.06	2	8634.60	2
5913.62	1	6198.04	0	6666.81	0	8654.38	2
5918.93	1	6206.73	1	6737.96	0	8679.55	1
5934.45	0	6213.84	2	6759.54	0	8681.80	1
5939.38	0	6225.72	0	7112.29	0	8780.64	2
5961.63	1	6246.70	1	7242.68	0	8783.71	1
5965.48	2	6258.77	1	7304.90	0		

¹⁸ This Bulletin, 14, p. 371: 1917.

^a There is a ghost near to each of these lines and the value given for its wave length may be in error on this account. The line at 6401.0 is very faint on our plates. The fact that Watson finds this line to be strong suggests that it may be due to an impurity.

3. CONSTANT FREQUENCY DIFFERENCES

It is well known that all of the lines in the ordinary helium spectrum can be placed in six series. The spectra of the other rare gases, however, seem to be differently constructed. Rossi¹⁹ has found series relationships among some faint neon lines, but he was unable to find them among the stronger lines. In a search for series among the stronger lines of neon Watson²⁰ found that most of these lines belonged to either triplets or quadruplets in which the frequency differences between corresponding members were constant. These constant frequency differences are very interesting in the light of accurate measurements. Meissner²¹ has already described some of these frequency differences resulting from his wave-length measurements of the strong lines in the red part of the neon spectrum. Our wave-length measurements are somewhat more extensive and include determinations of wave-lengths for most of the strong lines in one of the ultra-violet groups. All cases which seem to present constant frequency differences are collected in Table 3. This table is arranged on the principle that each group is a quadruplet. Fifteen groups are shown, six of which are complete quadruplets. Six of the groups lack the second member, one lacks the third, and two lack the second and fourth members. Watson found eight or possibly nine additional groups among shorter wave lengths (2911 Å to 3168 Å). The values given to three places of decimals in Table 3 are derived from interferometer measurements of wave lengths given in Table 1, while those given to two decimal places are obtained from grating measurements. Thirty-seven of the 55 strong neon lines represented in Table 1 seem to be connected by frequency differences which justify their appearance in Table 3. This accounts for all the strong lines in the yellow and red except 5852 Å and 6402 Å. The line 6074 Å was heretofore also considered an exception and its relation to 6652 Å may be fortuitous. Of the 24 strong lines in the ultra-violet spectrum between 3369 Å and 3754 Å all find a place in Table 3 except 3472 Å and 3520 Å.

¹⁹ Rossi, *Phil. Mag.*, 26, p. 981; 1913.

²⁰ Watson, *Astroph. J.*, 63, p. 399; 1911.

²¹ Meissner, *Phys. Zeitschr.*, 17, p. 549; 1916.

TABLE 3.—Frequency Differences in the Neon Spectrum

No.	Intensity	λ L. A. in air	λ L. A. in vacuum	Frequency in vacuum	$\nu_2 - \nu_1$	$\nu_3 - \nu_1$
				$\nu_1 \quad \nu_2 \quad \nu_3$	$\nu_2 - \nu_1$	$\nu_3 - \nu_1$
1	3	8082.45	8084.67	12 369.09	1070.06	1070.06
2	6	7438.902	7440.948	13 439.149	359.358	1429.42
3	8	7245.167	7247.161	13 798.507	417.450	1846.87
4	9	7032.413	7034.349	14 215.957		
1	5	7173.939	7175.913	13 935.309		
2						1429.434
3	9	6506.528	6508.322	15 364.943	417.447	1846.881
4	8	6334.428	6336.176	15 782.390		
1	3	7024.049	7025.983	14 232.885	1070.073	1070.073
2	4	6332.883	6334.684	15 302.958	359.358	1429.431
3	8	6382.991	6384.752	15 662.316	417.449	1846.880
4	4	6217.280	6218.996	16 079.765		
1	8	6929.468	6931.376	14 427.150		
2						1429.433
3	4	6304.789	6306.529	15 856.583	417.450	1846.883
4	9	6143.062	6144.758	16 274.033		
1	5	6717.043	6718.894	14 983.402	1070.079	1070.079
2	7	6266.495	6268.225	15 953.481	359.39	1429.47
3	3	6128.44	6130.13	16 312.87	417.41	1846.880
4	4	5975.534	5977.185	16 730.282		
1	8	6678.276	6680.116	14 969.799		
2						1429.452
3	8	6096.163	6097.847	16 399.231	417.448	1846.880
4	8	5944.834	5946.477	18 816.679		
1	3	6652.06	6653.89	15 028.80		
2						1429.35
3	7	6074.338	6076.016	16 458.153		
4						
1	5	6598.953	6600.772	15 149.743	1070.079	1070.079
2	5	6163.594	6165.296	16 219.822	359.354	1429.433
3	4	6029.997	6031.663	16 579.176	417.448	1846.881
4	6	5881.895	5883.521	16 996.624		
1	2	3754.17	3755.23	26 629.51	1069.74	
2	2	3609.18	3610.21	27 699.25		
3						1846.48
4	3	3510.73	3511.73	28 475.99		
1	4	3701.16	3702.21	27 010.90		
2						1428.95
3	5	3515.192	3516.194	28 439.846	417.446	1846.89
4	4	3464.340	3465.329	28 857.292		
1	3	3685.71	3686.76	27 124.12		
2						1429.23
3	4	3501.218	3502.216	28 553.250	417.67	1846.90
4	3	3450.74	3451.72	28 971.02		
1	3	3682.22	3683.26	27 149.83		
2						1429.24
3	4	3408.067	3409.064	28 579.070	417.450	1846.69
4	6	3447.705	3448.689	28 996.523		
1	5	3633.664	3634.696	27 512.618		
2						1429.409
3	6	3454.197	3455.183	28 942.027		
4						
1	5	3600.170	3601.193	27 768.573	1070.065	1070.065
2	5	3466.581	3467.570	28 838.638	359.14	1429.21
3	2	3423.94	3424.92	29 197.78	418.11	1847.32
4	3	3375.60	3376.57	29 615.89		
1	4	3393.634	3394.656	27 819.077	1070.019	1070.019
2	5	3460.526	3461.514	28 889.096	360.225	1430.244
3	6	3417.906	3418.883	29 249.321	416.607	1846.851
4	5	3369.906	3370.870	29 665.928		

The agreement between different sets of the constant frequency differences is remarkable, especially for the red lines, whose wave lengths have been more thoroughly investigated than those of

the ultra-violet group. 3593 A is a close double line and could not be accurately measured. This no doubt accounts for the low value of the difference (1070.019). 3369 A is also double, but the components could not be separated. Although the other two lines of this quadruplet never appeared to be double, it is possible that they also are very close pairs.

In terms of the frequency or numbers of waves per centimeter these differences are constant to one part in four or five millions and this limit to the constancy is imposed by the uncertainty in the wave-length measurements. If the frequency differences are assumed to be in reality strictly constant, our values afford a test for the accuracy of our wave-length determinations and show that the interferometer measurements are rarely in error by more than one one-thousandth of an angstrom.

By averaging the frequency differences derived from the interferometer measurements of the long waves we arrive at the following frequency relations

$$\nu_2 = \nu_1 + 1070.077$$

$$\nu_3 = \nu_1 + 1429.434 = \nu_2 + 359.356$$

$$\nu_4 = \nu_1 + 1846.881 = \nu_2 + 776.805 = \nu_3 + 417.449$$

The significance of these wonderful constant differences among frequencies is very little understood at the present time.

Considering only the strong lines the spectrum of neon falls naturally into three groups of lines which diminish in average intensity from the red to the ultra-violet end. Each of these three large groups contains smaller groups in which the frequencies of homologous members differ by a constant as shown above, but there appears to be no definite relation between the intensities of members of the same or of different groups. In some cases the members of a quadruplet have, roughly, the same intensities, while in others strong lines are associated with very faint ones. It may be that all of the groups are really quadruple and that the missing lines have not yet been observed on account of their small intensities.

Certain regularities have also been observed among the intense red neon lines from the point of view of magnetic separations. The Zeeman effect on the spectrum lines of neon was first studied by Lohmann²² and more extensively by Takamine and Yamada.²³

²² Lohmann, Diss., Halle; 1907.

²³ Takamine and Yamada, Proc. Tokyo Math.-Phys. Soc., 7, p. 278; 1914.

The magnetic separations of the corresponding members of Watson's quadruplets and triplets were found to be quite similar except in a few cases. Omitting the first members of these groups, the remaining members show magnetic components which increase in number by three from one to the next. In quadruplets, all lines designated by λ_1 become triplets when observed transverse to a magnetic field, λ_2 become sextets, and λ_3 nonets. In the case of triplet groups, λ_1 show as nonets and λ_2 give 12 components.

Furthermore, not only the magnetic separations, but also the intensity distribution in nonets is the same throughout each type of line. The physical significance and interpretation of all these regularities in the spectrum of neon is one of the attractive problems in physical science at the present time.

IV. SUMMARY

The wave lengths of 55 lines in the neon spectrum have been measured by means of the interferometer. These lines lie in the region 3369 Å to 8495 Å. The strong lines in the visible region of the spectrum have been observed with great accuracy, the probable error being one part in several millions, or less than one-tenth the width of the line. These strong lines were observed by means of three different pairs of interferometer plates which were each used on several interferometers. The ultra-violet lines and all the strong lines in the visible were compared directly with the fundamental standard 6438 Å. Some of the deep-red and infra-red lines were compared with well-determined lines in the visible neon spectrum.

Seventy-nine faint lines in the visible and infra-red neon spectrum have been measured by means of a concave grating. The probable error of these grating measurements is one or two hundredths angstrom. The region covered by the grating observations extends from 5343 Å to 8783 Å.

The constant differences discovered by Watson are found to hold with remarkable exactness in the case of lines which are strong enough to be measured with the highest accuracy. In fact, the differences are exactly constant within the limits set by the accuracy of the wave lengths.

WASHINGTON, April 6, 1918.

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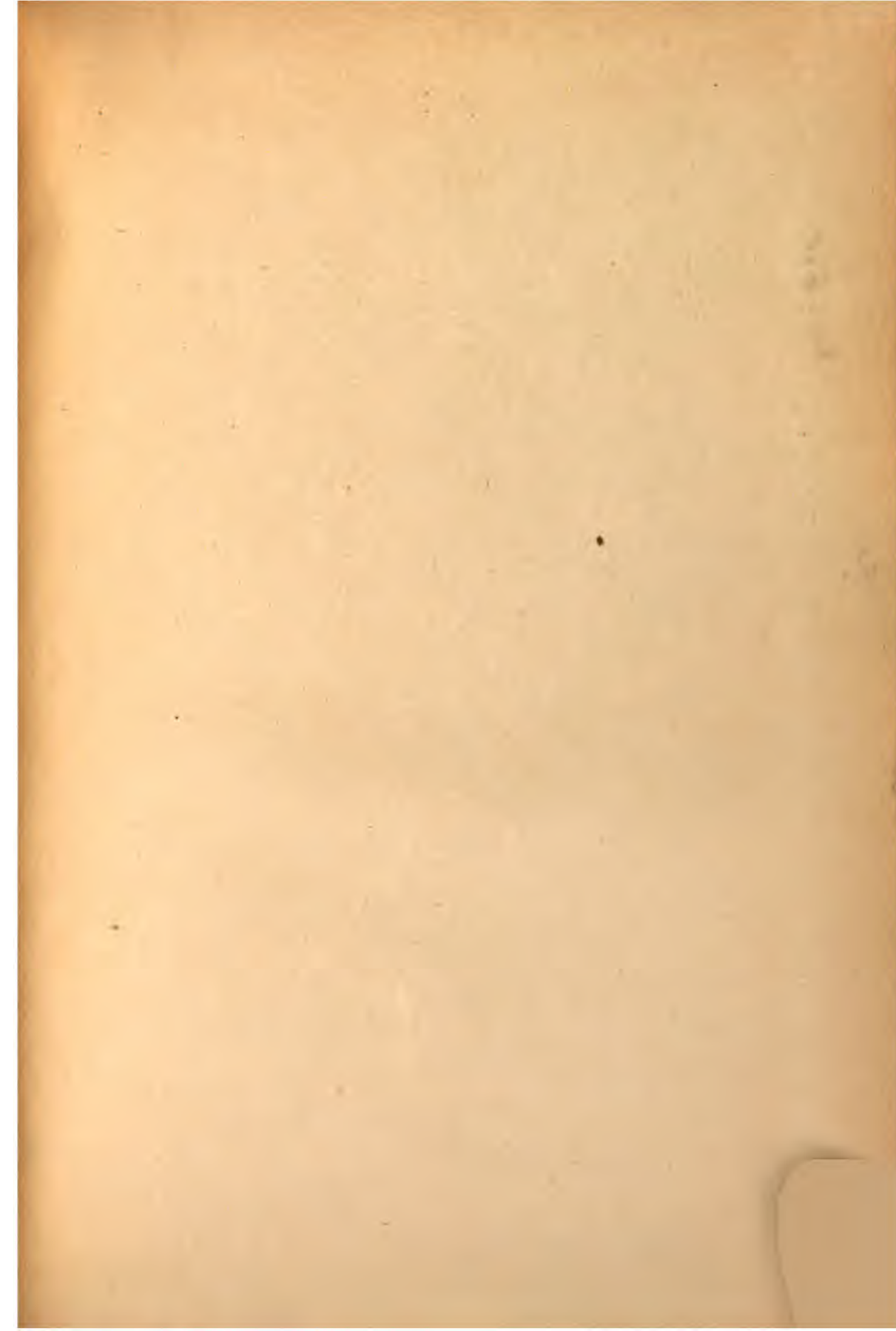
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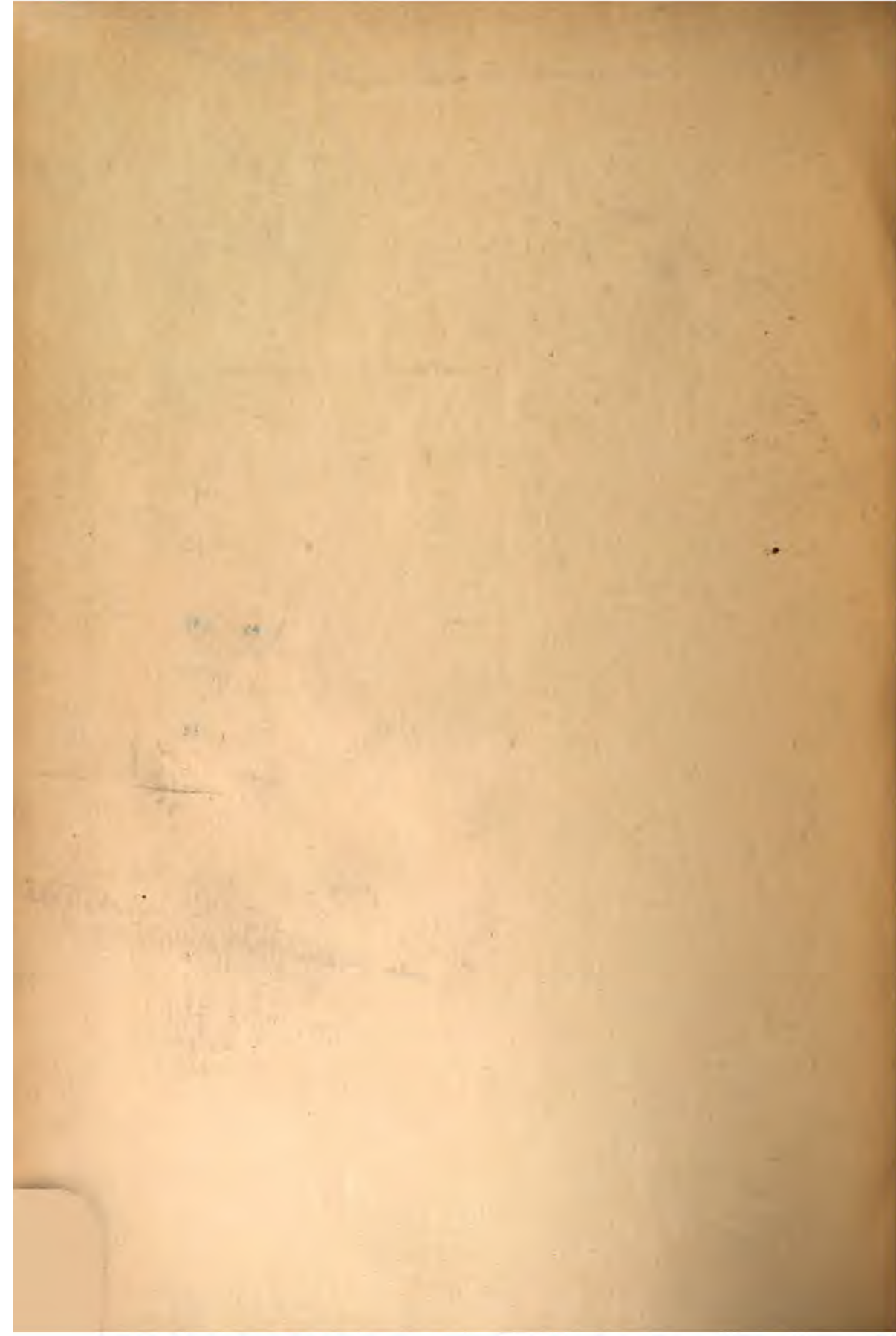
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